Statistical mechanics of random two-player games

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Using methods from the statistical mechanics of disordered systems, we analyze the properties of bimatrix games with random payoffs in the limit where the number of pure strategies of each player tends to infinity. We analytically calculate quantities such as the number of equilibrium points, the expected payoff, and the fraction of strategies played with nonzero probability as a function of the correlation between the payoff matrices of both players, and compare the results with numerical simulations.

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The adaptation to the behavior of others and to a complex environment is a process central to economics, sociology, international relations, and politics. Game theory aims to model problems of strategic decision making in mathematical terms: Two or more interacting participants, called players, make decisions in a competitive situation. Each player receives a reward, called the payoff, which not only depends on his own decision, but also on those of the other players. In the generic setup a number of players choose between different strategies, the combination of which determines the outcome of the game specified by the payoff to each player. Each player strives to achieve as large a payoff as possible. One of the cornerstones of modern economics and game theory is the concept of a Nash equilibrium [1]; for an introduction, also see Ref. [2]. A Nash equilibrium (NE) describes a situation where no player can unilaterally improve his payoff by changing his individual strategy given that the other players all stick to their strategies. However, this concept is thought to suffer from the serious drawback that in most games there is a large number of Nash equilibria with different characteristics but no means of telling which one will be chosen by the players, as would be required of a predictive theory.

This conceptual problem already shows up in the paradigmatic model of a bimatrix game between two players X and Y, where player X chooses a so-called *pure strategy* $X_i \in (1,\dots,N)$ with probability $x_i \geqslant 0$, and player Y chooses strategy $Y_j \in (1,\dots,N)$ with probability $y_j \geqslant 0$. The vectors $\mathbf{x}=(x_1,\dots,x_N)$ and $\mathbf{y}=(y_1,\dots,y_N)$ are called *mixed strategies*, and are constrained to the $(N-1)$-dimensional simplex by normalization. For a pair of pure strategies (i,j) the payoff to player X is given by the corresponding entry in his payoff matrix a_{ij}, whereas the payoff to player Y is given by b_{ij}. The *expected payoff* to player X is thus given by $\nu_x(\mathbf{x},\mathbf{y})=\Sigma_{i,j}x_i a_{ij} y_j$ and analogously for player Y. A Nash equilibrium $(\mathbf{x}^*,\mathbf{y}^*)$ is defined by

$$\begin{aligned}\nu^x(\mathbf{x}^*,\mathbf{y}^*)&=\max_{\mathbf{x}}\nu^x(\mathbf{x},\mathbf{y}^*)\\ \nu^y(\mathbf{x}^*,\mathbf{y}^*)&=\max_{\mathbf{y}}\nu^y(\mathbf{x}^*,\mathbf{y}).\end{aligned}\tag{1}$$

The condition for a NE with a *given set* of expected payoffs ν^x and ν^y may be written as

$$\begin{aligned}&\sum_j a_{ij}y_j-\nu^x\leqslant 0,\quad x_i\geqslant 0,\quad x_i\left(\sum_j a_{ij}y_j-\nu^x\right)=0\quad\forall\; i,\\ &\sum_i x_i b_{ij}-\nu^y\leqslant 0,\quad y_j\geqslant 0,\quad y_j\left(\sum_i x_i b_{ij}-\nu^y\right)=0\quad\forall\; j,\end{aligned}\tag{2}$$

where we have dropped the * indices for simplicity. The first column ensures that there are no pure strategies (and thus also no mixed strategies) which will yield a payoff larger than ν^x to player X and ν^y to player Y. Thus no player will have a reason to deviate from his mixed strategy. The second column ensures that the mixed strategies may be interpreted as probabilities and the third ensures that $\nu^x=\Sigma_{i,j}x_i a_{ij} y_j$ and analogously for player Y. In this situation there exists no mixed strategy which will increase the expected payoff to X if Y does not alter his strategy, and vice versa for Y. Nash's theorem [1] states that for any bimatrix game there is at least one NE.

The third column in Eq. (2) states that whenever x_i is strictly positive, $\Sigma_j a_{ij}y_j=\nu^x$ and whenever $\Sigma_j a_{ij}y_j-\nu^x$ is strictly negative, x_i is zero. Thus for a given set of strategies played with nonzero probability (out of 4^N possible choices), the values of all nonzero components of a mixed strategy can be determined by solving the resulting linear equations $\Sigma_j a_{ij}y_j=\nu^x\ \forall\ i{:}x_i>0$ and $\Sigma_i x_i b_{ij}=\nu^y\ \forall\ j{:}y_j>0$ subject to the normalization condition.

Apart from applications in economics, politics, sociology, and mathematical biology, there exists a wide body of mathematical literature on bimatrix games concerned with fundamental topics such as exact bounds for, e.g., the number of NE [3] and efficient algorithms for locating them [4]. For games even of moderate size a large number of NE are found, forming a set of disconnected points. In general the different NE all yield different expected payoffs to the players.

However, many situations of interest are characterized by a large number of possible strategies and complicated relations between the strategic choices of the players and the resulting payoffs. In such cases it is tempting to model the payoffs by random matrices in order to calculate *typical* properties of the game. This idea is frequently used in the statistical mechanics approach to complex systems such as

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spin glasses [5,6], neural networks [7], evolutionary models [8,9], or hard optimization problems [10,11]. Recently this approach has been used to investigate the typical properties of zero-sum games [12] obeying $a_{ij}=-b_{ij}$. A partial analysis of bimatrix games using the so-called annealed approximation was given in Ref. [13].

In this paper we investigate the properties of Nash equilibria in bimatrix games with a large number of pure strategies and random entries of the payoff matrices. In this approach characteristics of the game are encoded in the distribution of payoff matrices—with only a few parameters—instead of the payoff matrices themselves. Using techniques from the statistical mechanics of disordered systems such as the replica trick we calculate the typical number of NE with a given payoff.

The paper is organized as follows: Having set up the probability distribution of payoffs to be considered, we construct an indicator function for NE which will allow us to count the number of NE. Then the average of the logarithm of the number of NE over the disorder will be calculated. The solution is discussed both in game theoretic and geometric terms, and is compared with the results of numerical simulations. Finally, we give a summary and an outlook to future developments.

I. DISTRIBUTION OF PAYOFF MATRICES

We consider bimatrix games with square payoff matrices $\{a_{ij},b_{ij}\}$ where $i,j=1,\ldots,N$, where the thermodynamic limit consists of $N\to\infty$. We assume that the entries of the payoff matrices at different sites are identically and independently distributed. Since the two payoff matrices may be multiplied by any constant or have any constant added to them without changing the properties of the game in any material way, there is no loss of generality involved in considering payoffs of order $N^{-1/2}$ and of zero mean. In the thermodynamic limit one finds that only the first two moments of the payoff distribution are relevant, as is generally the case in fully connected disordered systems described by mean-field theories. Hence the entries of the payoff matrices may be considered to be Gaussian distributed. Then the only property of the distribution of payoffs which is not fixed by these specifications is the correlation κ between entries at the same site of the two payoff matrices.

We thus choose the entries of the payoff matrices to be drawn randomly according to the probability distribution

$$p(\{a_{ij}\},\{b_{ij}\})=\prod_{ij}\frac{N}{2\pi\sqrt{1-\kappa^2}}\times\exp\left\{-\frac{N(a_{ij}^2-2\kappa a_{ij}b_{ij}+b_{ij}^2)}{2(1-\kappa^2)}\right\}, \quad (3)$$

i.e., a Gaussian distribution with zero mean, variance $1/N$ and correlation $\langle\langle a_{ij}b_{kl}\rangle\rangle=\kappa\delta_{ik}\delta_{jl}/N$ for all pairs (i,j) and (k,l). Here and in the following, the double angles denote the average over the payoff distribution (3). For $\kappa=-1$, Eq. (3) includes a Dirac δ $\delta(a_{ij}+b_{ij})$ corresponding to a zero-sum game and we recover the situation of Ref. [12] as a special case. $\kappa=0$ corresponds to uncorrelated payoff matrices, and $\kappa=1$ is the so-called symmetric case $a_{ij}=b_{ij}$, where the two players always receive identical payoffs.

Thus the parameter κ describes the degree of similarity between the payoffs to either player, and can be used to continuously tune the game from a zero-sum game to a purely symmetric game. In the former case, the gain of one player is the loss of the other, so generally negative κ corresponds to a competitive situation, whereas for positive κ there are many pairs of strategies which are beneficial to both players.

II. ENTROPY OF NASH EQUILIBRIA AND THE INDICATOR FUNCTION

In this section we construct an indicator function which is zero at a NE with payoffs ν^x and ν^y to players X and Y, respectively, and nonzero everywhere else. This function will be made the argument of a properly normalized Dirac δ function. Integrating the Dirac δ function over the mixed strategies of both players, we are effectively counting the number of NE with the specified payoffs. From the resulting spectrum of NE the statistical properties of NE in bimatrix games may be deduced. Since we expect the number of NE to scale exponentially with the size of the game our central tool of investigation will be the *entropy of Nash equilibria* defined by $S(\nu^x,\nu^y)=(1/N)\ln\mathcal{N}(\nu^x,\nu^y)$, where $\mathcal{N}(\nu^x,\nu^y)$ is the number of NE with the specified payoffs per unit interval within a small interval around ν^x and ν^y. Since $NS(\nu^x,\nu^y)$ is expected to be an extensive quantity, we may assume that $S(\nu^x,\nu^y)$ is self-averaging, and in the thermodynamic limit the average value of the entropy will be realized with probability 1. Hence the central goal of our calculation will be to evaluate $\langle\langle S(\nu^x,\nu^y)\rangle\rangle$.

In this framework the total number of NE is given by

$$\mathcal{N}=\int d\nu^x d\nu^y e^{NS(\nu^x,\nu^y)}, \quad (4)$$

so in the thermodynamic limit the NE will be exponentially dominated by the maximum of the curve $S:=\max S(\nu^x,\nu^y)$. This implies that a randomly chosen NE will yield the payoffs where the maximum occurs with probability 1. On the other hand, the line where $S(\nu^x,\nu^y)=0$ delimits the smallest and the largest values of ν^x,ν^y for which there is still an exponential number of NE.

The three expressions may be encoded in a single condition [14] by introducing the variables $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with

$$\tilde{x}_i=\begin{cases} x_i, & x_i>0 \\ \sum_j a_{ij}y_j-\nu^x, & x_i=0, \end{cases}$$
$$\tilde{y}_j=\begin{cases} y_j, & y_j>0 \\ \sum_i x_i b_{ij}-\nu^y, & y_j=0. \end{cases} \quad (5)$$

Condition (2) may be written as

$$I_i^x=\tilde{x}_i\Theta(-\tilde{x}_i)-\left(\sum_j a_{ij}\tilde{y}_j\Theta(\tilde{y}_j)-\nu^x\right)=0,$$
$$I_j^y=\tilde{y}_j\Theta(-\tilde{y}_j)-\left(\sum_j \tilde{x}_i\Theta(\tilde{x}_i)b_{ij}-\nu^y\right)=0, \tag{6}$$

so for positive $\tilde{x}_i$, $\tilde{x}_i=x_i$, whereas for negative $\tilde{x}_i$ we have $\tilde{x}_i=\Sigma_j a_{ij}\tilde{y}_j\Theta(\tilde{y}_j)-\nu^x$. Furthermore we have $x_i=0$ if $\tilde{x}_i<0$ and $\Sigma_j a_{ij}\tilde{y}_j\Theta(\tilde{y}_j)-\nu^x=0$ for $\tilde{x}_i>0$. The condition $x_i(\Sigma_j a_{ij}y_j-\nu^x)=0$ is thus satisfied automatically. Analogous relations hold for player Y. The new variables therefore serve as a convenient tool to encode the ''complementary'' quantities x_i and $\Sigma_j a_{ij}y_j-\nu^x$ in a single variable. Analogous relations hold for player Y. The density of NE with payoffs ν^x and ν^y may thus be written as

$$\mathcal{N}(\nu^x,\nu^y)=\int d\mu(\tilde{\mathbf{x}})d\mu(\tilde{\mathbf{y}})\prod_i \delta(I_i^x)\prod_j \delta(I_j^y)\left\|\frac{\partial(\mathbf{I}^x,\mathbf{I}^y)}{\partial(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}\right\|, \tag{7}$$

where the mixed strategies are rescaled to $\Sigma_i x_i=\Sigma_j y_j=N$ so we define the measure $d\mu$ as

$$d\mu(\tilde{\mathbf{x}})=\prod_i d\tilde{x}_i\delta\left(\sum_i \tilde{x}_i\Theta(\tilde{x}_i)-N\right). \tag{8}$$

This scaling of the mixed strategies assumes that the extensive number of strategies are played with nonzero probability, so the individual terms x_i and y_j are all of order $O(1)$. The integrals over $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ effectively amount to choosing a set of strategies with $\tilde{x}_i>0$ and $\tilde{y}_j>0$, solving the resulting linear equations for the components played with nonzero probability, and checking if this candidate for a NE fulfills conditions (2). It may thus be viewed as performing the so-called support enumeration algorithm analytically [4].

III. CALCULATION OF THE TYPICAL NUMBER OF NASH EQUILIBRIA

In this section we calculate the average of $S(\nu_x,\nu_y)$ over disorder (3). In order to represent the logarithm of Eq. (7) we use the replica-trick $\ln\mathcal{N}=\lim_{n\to 0}(d/dn)\mathcal{N}^n$ and compute the average over $\mathcal{N}^n$ for integer n taking the limit $n\to 0$ by analytic continuation at the end. Using integral representations of the Dirac δ function we obtain

$$\mathcal{N}^n(\nu^x,\nu^y)=\prod_{a,i}^{n,N}\int\frac{d\mu(\tilde{\mathbf{x}}^a)d\hat{x}_i^a}{2\pi}\prod_{a,j}\int\frac{d\mu(\tilde{\mathbf{y}}^a)d\hat{y}_j^a}{2\pi}$$
$$\times\exp\Bigg\{-i\sum_{a,i}\tilde{x}_i^a\Theta(-\tilde{x}_i^a)\hat{x}_i^a$$
$$+i\sum_{a,i,j}\hat{x}_i^a a_{ij}\tilde{y}_j^a\Theta(\tilde{y}_j^a)-i\nu^x\sum_{a,i}\hat{x}_i^a$$
$$-i\sum_{a,j}\tilde{y}_j^a\Theta(-\tilde{y}_j^a)\hat{y}_j^a+i\sum_{a,i,j}\tilde{x}_i^a\Theta(\tilde{x}_i^a)b_{ij}\hat{y}_j^a$$
$$-i\nu^y\sum_{a,j}\hat{y}_j^a\Bigg\}(\|\det(D)\|)^n, \tag{9}$$

where a runs from 1 to n. The most awkward term of this expression is the absolute value of the normalizing determinant, i.e., the Jacobian matrix of $\mathbf{I}^x$ and $\mathbf{I}^y$ given by

$$D:=\frac{\partial(\mathbf{I}^x,\mathbf{I}^y)}{\partial(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}=\begin{pmatrix}\delta_{ii'}\Theta(-\tilde{x}_i) & -a_{ij}\Theta(\tilde{y}_j)\\ -b_{ij}\Theta(\tilde{x}_i) & \delta_{jj'}\Theta(-\tilde{y}_j)\end{pmatrix}, \tag{10}$$

which arises from the coefficients of $\tilde{x}$ and $\tilde{y}$ in $\mathbf{I}^x$ and $\mathbf{I}^y$. Since we are only interested in the absolute value of the determinant, we are free to interchange rows and columns of this matrix. Rearranging the rows and columns of D such that the p_xN strategies with $\tilde{x}_i>0$ and the p_yN strategies with $\tilde{y}_j>0$ are grouped together, one finds that only the resulting quadratic submatrix of size $N(p_x+p_y)$ by $N(p_x+p_y)$ contributes to the determinant of D. From Eq. (2) one finds that the distinction between p_x and p_y is immaterial, since the number of strategies played by player X at any NE always equals that played by Y and the determinant of D is zero for $p_x\neq p_y$. In the following we will assume that $(1/N)\ln(\det D)$ is a self-averaging quantity depending on $p_x=p_y$. Splitting off the normalizing determinant, the average over the payoffs may now easily be performed, details of the calculations are given in Appendix A. The average over the disorder introduces a coupling between the replicas, and one introduces the symmetric matrix of the overlaps between mixed strategies of each player as order parameters,

$$q_{ab}^x=\frac{1}{N}\sum_i \tilde{x}_i^a\Theta(\tilde{x}_i^a)\tilde{x}_i^b\Theta(\tilde{x}_i^b),$$
$$q_{ab}^y=\frac{1}{N}\sum_j \tilde{y}_j^a\Theta(\tilde{y}_j^a)\tilde{y}_j^b\Theta(\tilde{y}_j^b), \tag{11}$$

as well as their conjugates $\hat{q}_{ab}^{x,y}$. At nonzero values of κ we also obtain terms which couple the phase-space variables $\tilde{x}_i$ to the auxiliary variables $\hat{x}_i$ and similarly for player Y, so we also introduce the order parameters

$$R_{ab}^x=\frac{1}{N}\sum_i i\hat{x}_i^a\tilde{x}_i^b\Theta(\tilde{x}_i^b),\quad R_{ab}^y=\frac{1}{N}\sum_j i\hat{y}_j^b\tilde{y}_j^a\Theta(\tilde{y}_j^a). \tag{12}$$

Similarly, in order to include the normalizing determinant we introduce the order parameters

$$p_a^x=\frac{1}{N}\sum_i \Theta(\tilde{x}_i^a),\quad p_a^y=\frac{1}{N}\sum_j \Theta(\tilde{y}_j^a), \tag{13}$$

giving the fraction of strategies played at a NE.

Anticipating the limit $n\to 0$, the quenched average of the normalizing determinant may be computed using results from the theory of random matrices as outlined in Appendix A, giving $\langle\langle\ln(\|\det(D)\|)\rangle\rangle=Np(\ln p-1)$.

We finally obtain

$$\langle\langle \mathcal{N}^n(\nu^x,\nu^y)\rangle\rangle = \prod_{a\geq b}\int \frac{dq_{ab}^{x,y}d\hat{q}_{ab}^{x,y}}{2i\pi/N}\prod_{a,b}\int\frac{dR_{ab}^x dR_{ab}^y}{2i\pi/(\kappa N)}\prod_a \times\int\frac{dp_a^{x,y}d\hat{p}_a^{x,y}}{2i\pi/N}\delta(p_a^x-p_a^y)\prod_a\int\frac{dE_a^{x,y}}{2i\pi/N}$$
$$\times\exp\left\{-N\sum_{a\geq b}q_{ab}^{x,y}\hat{q}_{ab}^{x,y}-\kappa N\sum_{a,b}R_{ab}^x R_{ab}^y + N\sum_a p_a^{x,y}\hat{p}_a^{x,y}+N\sum_a E_a^{x,y}\right\}$$
$$\times\exp\{N[G^x+G^y]\}\langle\langle\|\det(D)\|^n\rangle\rangle, \qquad (14)$$

where

$$G^x=\ln\prod_a\int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}\exp\{\mathcal{L}^x(\{\tilde{x}^a,\hat{x}^a\})\}$$
$$:=\ln\prod_a\int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}\exp\left\{\sum_{a\geq b}\hat{q}_{ab}^x\tilde{x}^a\Theta(\tilde{x}^a)\tilde{x}^b\Theta(\tilde{x}^b)+\kappa\sum_{a,b}R_{ab}^y i\hat{x}^a\tilde{x}^b\Theta(\tilde{x}^b)-\frac{1}{2}\sum_{a,b}q_{ab}^y\hat{x}^a\hat{x}^b\right.$$
$$\left.-i\sum_a\tilde{x}^a\Theta(-\tilde{x}^a)\hat{x}^a-i\nu^x\sum_a\hat{x}^a-\sum_a E_a^x\tilde{x}^a\Theta(\tilde{x}^a)-\sum_a\hat{p}_a^x\Theta(\tilde{x}^a)\right\}, \qquad (15)$$

$$G^y=\ln\prod_a\int\frac{d\tilde{y}^a d\hat{y}^a}{2\pi}\exp\{\mathcal{L}^y(\{\tilde{y}^a,\hat{y}^a\})\}$$
$$:=\ln\prod_a\int\frac{d\tilde{y}^a d\hat{y}^a}{2\pi}\exp\left\{\sum_{a\geq b}\hat{q}_{ab}^y\tilde{y}^a\Theta(\tilde{y}^a)\tilde{y}^b\Theta(\tilde{y}^b)+\kappa\sum_{a,b}R_{ab}^x\tilde{y}^a\Theta(\tilde{y}^a)i\hat{y}^b-\frac{1}{2}\sum_{a,b}q_{ab}^x\hat{y}^a\hat{y}^b\right.$$
$$\left.-i\sum_a\tilde{y}^a\Theta(-\tilde{y}^a)\hat{y}^a-i\nu^y\sum_a\hat{y}^a-\sum_a E_a^y\tilde{y}^a\Theta(\tilde{y}^a)-\sum_a\hat{p}_a^y\Theta(\tilde{y}^a)\right\}.$$

In the thermodynamic limit $N\to\infty$ the integrals over order parameters in Eq. (14) may be performed by saddle point integration. In order to be able to analytically continue the saddle point to $n\to 0$ we choose the replica-symmetric (RS) ansatz for the order parameters

$$q_{aa}^{x,y}=q_1^{x,y},\quad \hat{q}_{aa}^{x,y}=-\frac{1}{2}\hat{q}_1^{x,y}\quad\forall\ a,$$
$$q_{ab}^{x,y}=q_0^{x,y},\quad \hat{q}_{ab}^{x,y}=\hat{q}_0^{x,y}\quad\forall\ a>b,$$
$$R_{aa}^x=R_1^x,\quad R_{aa}^y=R_1^y\quad\forall\ a,$$
$$R_{ab}^x=R_0^x,\quad R_{ab}^y=R_0^y\quad\forall\ a\neq b, \qquad (16)$$
$$p_a^{x,y}=p^{x,y},\quad \hat{p}_a^{x,y}=\hat{p}^{x,y}\quad\forall\ a,$$
$$E_a^{x,y}=E^{x,y},\quad\forall\ a.$$

$q_1^x=(1/N)\Sigma_i x_i x_i$ denotes the self-overlap of the mixed strategies of player X, whereas $q_0^x=(1/N)\Sigma_i x_i^1 x_i^2$ characterizes the overlap between the mixed strategies corresponding to two distinct NE, and analogously for player Y.

The integrals over $\tilde{x}^a$, $\hat{x}^a$, $\tilde{y}^a$, and $\hat{y}^a$ may be evaluated, and the limit $n\to 0$ may be taken as outlined in Appendix A 1. G^x and G^y evaluated at the RS saddle point are symmetric with respect to an interchange of the players X and Y. Thus the maximum of $S(\nu^x,\nu^y)$ occurs at equal payoffs and in the thermodynamic limit NE with any other combination of payoffs will be exponentially rare by comparison. Hence we may restrict our discussion to the case $\nu^x=\nu^y=\nu$, where all order parameters are symmetric under interchange of the players.

We thus obtain the average entropy of the number of NE within the RS ansatz,

$$S_\kappa(\nu)=\frac{1}{N}\langle\langle\ln\mathcal{N}(\nu,\nu)\rangle\rangle$$
$$=2\ \mathrm{extr}_{q_1,\hat{q}_1,q_0,\hat{q}_0,R_1,R_0,E,p}\left[\frac{q_1\hat{q}_1}{2}+\frac{q_0\hat{q}_0}{2}-\frac{\kappa R_1^2}{2}+\frac{\kappa R_0^2}{2}+E-\frac{p}{2}+\int da\ db\ p_{\tilde{\kappa}}(a,b)\ln L(a,b)\right], \qquad (17)$$

where $p_{\tilde{\kappa}}(a,b)$ with $\tilde{\kappa}=\kappa R_0/\sqrt{q_0\hat{q}_0}$ denotes

$$p_{\tilde{\kappa}}(a,b)=\frac{1}{2\pi\sqrt{1-\tilde{\kappa}^2}}\exp\left(-\frac{a^2-2\tilde{\kappa}ab+b^2}{2(1-\tilde{\kappa}^2)}\right), \qquad (18)$$

and thus echoes the original distribution of the payoffs, and

$$L(a,b)=H\left(-\frac{\nu-\sqrt{q_0}b}{\sqrt{q_1-q_0}}\right)+\sqrt{\frac{p}{(q_1-q_0)(\hat{q}_1+\hat{q}_0)+\kappa^2(R_1-R_0)^2}}\times G\left(-\frac{\nu-\sqrt{q_0}b}{\sqrt{q_1-q_0}}\right)\times K\left(\frac{\dfrac{\kappa(R_1-R_0)(\sqrt{q_0}b-\nu)}{q_1-q_0}-\sqrt{q_0}a+E}{\sqrt{\hat{q}_1+\hat{q}_0+\dfrac{\kappa^2(R_1-R_0)^2}{q_1-q_0}}}\right), \qquad (19)$$

where $K(x)$ is a shorthand for $H(x)/G(x)$ with $G(x)=(1/\sqrt{2\pi})\exp(-x^2/2)$, and $H(x)=\int_x^\infty dy\ G(y)$. The extremum is to be taken over all order parameters, The saddle-point equations corresponding to Eq. (17) may be solved numerically, their solutions will be discussed in detail in Sec. IV.

A. Distribution of the strategy strengths and the potential payoffs

In this section we calculate the distribution of strategy strengths $\rho_x(x)=\langle\langle(1/N)\Sigma_i\delta(x_i-x)\rangle\rangle$ and the potential payoffs $\rho_{\lambda_x}(\lambda)=\langle\langle(1/N)\Sigma_i\delta(\Sigma_j a_{ij}y_j-\lambda)\rangle\rangle$ at NE. Due to the symmetry of Eq. (14) under an interchange of players for $\nu^x=\nu^y$ it is sufficient to calculate these distributions for a single player only. We make use of the set of variables $\tilde{x}_i$ introduced in Sec. II, since the distribution $\rho_{\tilde{x}}(\tilde{x})$ is equal to $\rho_x(x)$ for $\tilde{x}>0$ and equal to $\rho_{\lambda_x}(\lambda-\nu^x)$ for $\tilde{x}<0$. By the same token, the fraction of strategies with $x_i=0$ is equal to $\int_{-\infty}^0 d\tilde{x}\ \rho_{\tilde{x}}(\tilde{x})=1-p$ and the fraction of potential payoffs with $\lambda_i^x=\nu^x$ is $\int_0^\infty d\tilde{x}\ \rho_{\tilde{x}}(\tilde{x})=p$.

Since all pure strategies are equivalent after averaging over the payoffs (translation invariance), we calculate

$$\rho_{\tilde{x}}(\tilde{x})=\left\langle\left\langle\frac{\int d\mu(\tilde{\mathbf{x}})d\mu(\tilde{\mathbf{y}})\,\delta(\tilde{x}_1-\tilde{x})\prod_i\ \delta(I_i^x)\prod_j\ \delta(I_j^y)\left\|\frac{\partial(\mathbf{I}^x,\mathbf{I}^y)}{\partial(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}\right\|}{\int d\mu(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\prod_i\ \delta(I_i^x)\prod_j\ \delta(I_j^y)\left\|\frac{\partial(\mathbf{I}^x,\mathbf{I}^y)}{\partial(\tilde{\mathbf{x}},\tilde{\mathbf{y}})}\right\|}\right\rangle\right\rangle$$
$$=\lim_{n\to 0}\left\langle\left\langle\prod_{a=1}^n\ \int d\mu(\tilde{\mathbf{x}},\tilde{\mathbf{y}})\,\delta(\tilde{x}_1^1-\tilde{x})\prod_{i,a}\ \delta(I_i^{xa})\prod_{j,a}\ \delta(I_j^{ya})\prod_a\ \left\|\frac{\partial(\mathbf{I}^{xa},\mathbf{I}^{ya})}{\partial(\tilde{\mathbf{x}}^a,\tilde{\mathbf{y}}^a)}\right\|\right\rangle\right\rangle. \tag{20}$$

In order to be able to perform the average over payoffs occurring in both the numerator and the denominator, we have represented the denominator by $n-1$ replicas. The average over payoffs now proceeds exactly as in Sec. II. Introducing the matrices of order parameters q_{ab} again, the $i=1$ term may be split off from the saddle point integral without distorting the saddle point in the thermodynamic limit. Taking the replica symmetric ansatz, one obtains

$$\rho_{\tilde{x}}(\tilde{x})=\lim_{n\to 0}\prod_a\ \int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}\exp\{\mathcal{L}^x(\{\tilde{x}^a,\hat{x}^a\})\}\delta(\tilde{x}^1-\tilde{x})$$
$$=\begin{cases}\displaystyle\int da\ db\ p_{\tilde{\kappa}}(a,b)\frac{1}{\sqrt{2\pi(q_1-q_0)}L(a,b)}\exp\left\{-\frac{(\tilde{x}+\nu-\sqrt{q_0}b)^2}{2(q_1-q_0)}\right\}, & \tilde{x}<0\\ \displaystyle\int da\ db\ p_{\tilde{\kappa}}(a,b)\frac{\sqrt{p}}{\sqrt{2\pi(q_1-q_0)}L(a,b)} & \\ \displaystyle\qquad\times\exp\left\{-\frac{(-\kappa(R_1-R_0)\tilde{x}+\nu-\sqrt{q_0}b)^2}{2(q_1-q_0)}-\frac{1}{2}(\hat{q}_1+\hat{q}_0)\tilde{x}^2+a\sqrt{\hat{q}_0}\tilde{x}-E\tilde{x}\right\}, & \tilde{x}>0,\end{cases} \tag{21}$$

where $\mathcal{L}^x(\{\tilde{x}^a,\hat{x}^a\})$ was defined in Eq. (15), the order parameters take on their saddle-point values, and we have dropped the player indices of the order parameters. These functions have to be evaluated numerically, and will be discussed in Sec. IV C.

The same procedure may be used to calculate another quantity of interest, namely, the fraction w of pure strategies which are *both* played with nonzero probability in two randomly chosen Nash equilibria. Like q_0, this quantity is a measure of the degree of similarity of two randomly chosen NE. However, w does not directly depend on the self-overlap of the mixed strategies, and may serve to test if there are strategies which are more likely to be played at a NE than others. From Eqs. (14) and (16) one obtains

$$w=\lim_{n\to 0}\prod_a\ \int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}\exp\{\mathcal{L}^x(\{\tilde{x}^a,\hat{x}^a\})\}\Theta(\tilde{x}^1)\Theta(\tilde{x}^2)$$
$$=\int da\ db\ p_{\tilde{\kappa}}(a,b)\frac{p}{[(q_1-q_0)(\hat{q}_1+\hat{q}_0)+\kappa^2(R_1-R_0)^2]L^2(a,b)}G^2\left(-\frac{\nu-\sqrt{q_0}b}{\sqrt{q_1-q_0}}\right)$$
$$\times K^2\left(\frac{\dfrac{\kappa(R_1-R_0)(\sqrt{q_0}b-\nu)}{q_1-q_0}-\sqrt{\hat{q}_0}a+E}{\sqrt{\hat{q}_1+\hat{q}_0+\dfrac{\kappa^2(R_1-R_0)^2}{q_1-q_0}}}\right). \tag{22}$$

B. Stability of the replica-symmetric saddle point

The results for the quenched average were derived on the basis of the replica-symmetric ansatz [Eq. (16)]. In this section we investigate the stability of this ansatz with respect to small fluctuations around Eq. (16) in order to check if this ansatz is at least locally stable. We restrict ourselves to the special case $\kappa=0$, where the payoff matrices are uncorrelated and the order parameters R^x_{ab} and R^y_{ab} do not arise [15]. We consider small transversal fluctuations around the RS saddle point, and expand Eq. (14) to second order in these fluctuations to obtain

$$S=S_{RS}+\frac{1}{2}\Delta^T M\Delta+O(\Delta^3), \tag{23}$$

where Δ denotes a vector of small fluctuations in the off-diagonal elements of the order parameters q^x_{ab}, $\hat{q}^x_{ab}$, q^y_{ab}, and $\hat{q}^y_{ab}$, and M is given by

$$M=\begin{pmatrix} \frac{\partial^2 G^y}{\partial q^x_{ab}\partial q^x_{cd}} & -I & 0 & \frac{\partial^2 G^y}{\partial q^x_{ab}\partial \hat{q}^y_{cd}} \\ -I & \frac{\partial^2 G^x}{\partial \hat{q}^x_{ab}\partial \hat{q}^x_{cd}} & \frac{\partial^2 G^x}{\partial \hat{q}^x_{ab}\partial q^y_{cd}} & 0 \\ 0 & \frac{\partial^2 G^x}{\partial q^y_{ab}\partial \hat{q}^x_{cd}} & \frac{\partial^2 G^x}{\partial q^y_{ab}\partial q^y_{cd}} & -I \\ \frac{\partial^2 G^y}{\partial \hat{q}^y_{ab}\partial q^x_{cd}} & 0 & -I & \frac{\partial^2 G^y}{\partial \hat{q}^y_{ab}\partial \hat{q}^y_{cd}} \end{pmatrix}. \tag{24}$$

Due to the symmetry of Eq. (14) under an interchange of the players at the RS saddle point, only three different nontrivial submatrices need to be evaluated. The criterion for the RS ansatz to be locally stable needs to be determined by working out the paths of integration in the complex plane of $\hat{q}^x$ and $\hat{q}^y$. Denoting the replicon eigenvalues of $\partial^2 G^x/\partial \hat{q}^x_{ab}\partial \hat{q}^x_{cd}$, $\partial^2 G^y/\partial q^x_{ab}\partial q^x_{cd}$, and $\partial^2 G^y/\partial q^x_{ab}\partial \hat{q}^y_{cd}$ by λ_1, λ_2, and λ_3, respectively, one obtains the criterion for the local stability of the RS ansatz;

$$\frac{1}{\lambda_2}[\lambda_1\lambda_2-(\lambda_3-1)^2]<0,$$
$$\frac{1}{\lambda_2}[\lambda_1\lambda_2-(\lambda_3+1)^2]<0. \tag{25}$$

For details of the calculation, see Appendix B.

IV. DISCUSSION OF THE RESULTS

The quantity $S_\kappa(\nu)$ contains a wealth of information. We begin by discussing the general shape of $S_\kappa(\nu)$ and the number of NE as a function of κ, then turn to the statistical properties of NE and give a geometric interpretation of the results, and finally discuss the distribution of potential payoffs and the strategy strengths.

A. $S_\kappa(\nu)$ and the number of NE

Expression (17) for $S^{RS}_\kappa(\nu)$ defines a family of curves with a pronounced maximum shown exemplarily for $\kappa=0$ in Fig. 1. As argued in Sec. II in the thermodynamic limit the maximum of $S_\kappa(\nu)$ dominates the spectrum of NE.

Another point of interest is the value of ν where $S_\kappa(\nu)$ crosses the $S=0$ axis. Due to the symmetry of $S(\nu^x,\nu^y)$ under an interchange of the players this point indicates the NE with the maximum sum of the payoffs. For $\kappa=-1$ it takes on the value 0 and increases monotonously with κ. At $\kappa=\kappa_c$ it diverges to infinity; $S_\kappa(\nu)$ no longer crosses the $S=0$ axis. In this case there is an exponentially large number of NE offering an arbitrarily large payoff to either player, where an arbitrarily small fraction of strategies are played. From the annealed approximation one obtains $\kappa_c\approx-0.59$, the corresponding result from the RS expression for the quenched average is $\kappa_c\approx-0.58$. This effect may be explained as follows: At large values of κ players may pick a few of the pairs of strategies (i,j), which offer a large payoff to both of them and play them with a large probability. An exponential number of NE may be constructed in this way, even though their number is exponentially small compared to the total number of NE. The entropy of NE given by the maximum S_κ of $S_\kappa(\nu)$ is shown in Fig. 2.

We find $S_{\kappa=-1}=0$, since there is only a single equilibrium point for zero-sum games. S_κ increases with κ, so for all $\kappa>-1$ the typical number of NE scales exponentially with N. The maximum of the typical number of NE is reached for the case of symmetric games, where $S_{\kappa=1}\sim 0.358$. This result may be compared with a rigorous upper bound for the maximal number of NE in a bimatrix game derived using geometric methods [3,4]: For any nondegenerate $N\times N$ bimatrix game with large N there are at most $e^{0.955N}$ equilibrium points. As expected, the typical-case scenario does not saturate this bound, at least not for the distribution of payoffs considered here. Nevertheless for $\kappa>-1$

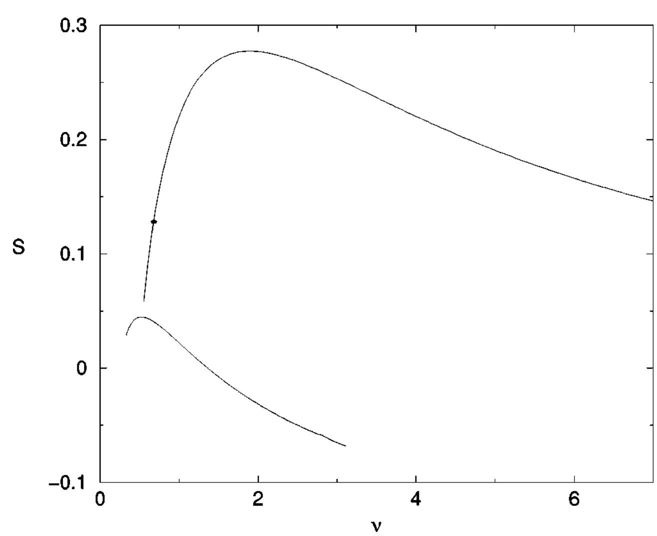

FIG. 1. The results of the quenched averages of $S_\kappa(\nu)$ for $\kappa=-0.8$ and 0, respectively (bottom and top, respectively). For $\kappa=0$ replica symmetry is locally stable for $\nu>0.67$ as indicated by the black dot.

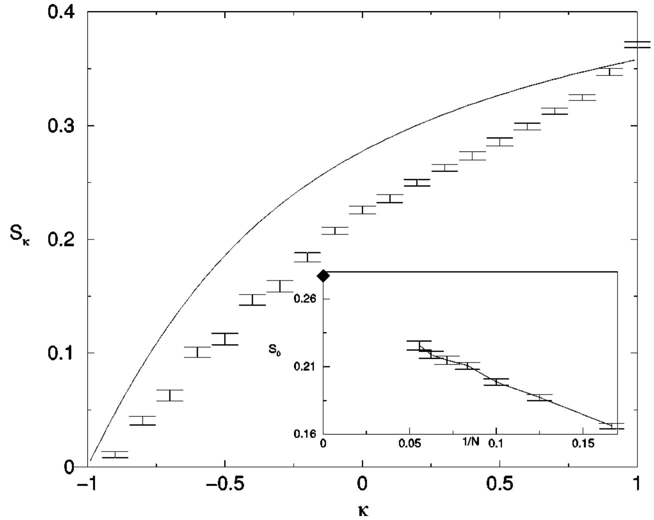

FIG. 2. The RS entropy of NE S_κ as a function of κ (solid line). The numerical results stem from enumerations with system size $N=18$ averaged over 100 samples, and the error bars denote the statistical error. The inset shows the finite-size effects for the case $\kappa=0$. S_0 is plotted against $1/N$, and the analytic result for $N\to\infty$ is indicated by the filled diamond.

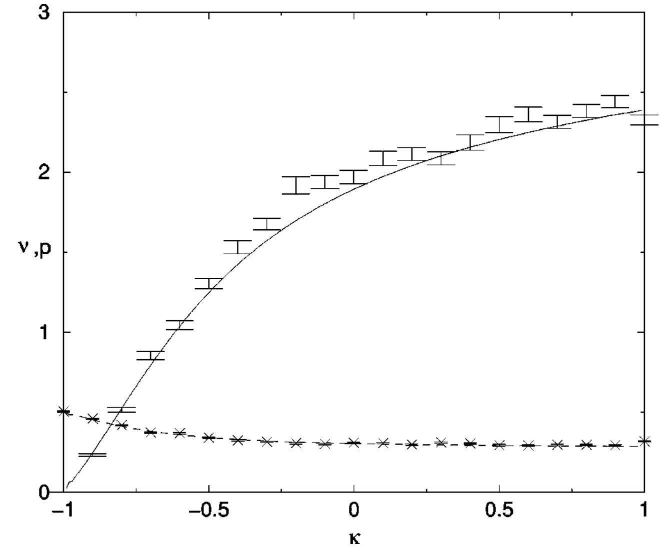

FIG. 3. The payoff ν (solid line) and the fraction p (dashed line) of strategies played with nonzero probability of the typical NE. The analytic results are compared with numerical simulations for $N=50$ averaged over 200 samples.

the typical number of NE investigated here and the maximal number of NE both scale exponentially with N.

The increase of the number of NE with the correlation between the payoff matrices may be explained as follows: As will be discussed in Sec. IV B, the payoff ν to both players increases with κ. For increasing values of $\nu=\nu^x=\nu^y$ the necessary (but not sufficient) conditions for a NE,

$$\sum_j a_{ij}y_j \leqslant \nu^x \; x_i \geqslant 0 \;\; \forall i,$$
$$\sum_i x_i b_{ij} \leqslant \nu^y \; y_j \geqslant 0 \;\; \forall j \tag{26}$$

become increasingly easy to fulfill. In fact for $\nu=0$ only a single point on the simplexes of the two players fulfills Eq. (26), whereas for large ν a correspondingly large section of the simplexes qualify as a candidate for equilibrium points [12]. As a result the number of points which apart from Eq. (26) obey $(\Sigma_j a_{ij}y_j-\nu^x)x_i=0$ and $(\Sigma_i x_i b_{ij}-\nu^y)y_j=0$, and thus constitute NE increases with κ.

B. Statistical properties of Nash equilibria

In the thermodynamic limit not only the number of NE will be dominated by the maximum of $S_\kappa(\nu)$, but a randomly chosen NE will also give the payoff $\nu=\nu^x=\nu^y=\mathrm{argmax}S_\kappa(\nu)$ with probability 1, because the number of NE with this payoff is exponentially larger than the number of all other NE. Similarly, the self-overlap, the mutual overlap, and the fraction of strategies played with nonzero probability will take on their saddle-point values evaluated at the maximum of $S_\kappa(\nu)$ with probability 1. Figure 3 shows the payoff dominating the spectrum of NE and the corresponding fraction p of strategies played with nonzero probability, whereas the self-overlap of mixed strategies q_1 and the overlap q_0 between the mixed strategies of different NE are shown in Fig. 4.

At $\kappa=-1$ we recover the results for zero-sum games with $q_1=q_0=\pi$, $\nu=0$, and $p=1/2$ [12]. As κ rises, the payoff increases. This effect may be understood as follows: At increasing κ the outcome of a pair of strategies (i,j) which is beneficial to player X say, tends to become more beneficial to player Y. As a result players focus on these strategies and the payoff at a NE to both players rises. By the same token, the fraction p of strategies which are played with nonzero probability at a NE decreases with κ and the self-overlap q_1 of the mixed strategies increases.

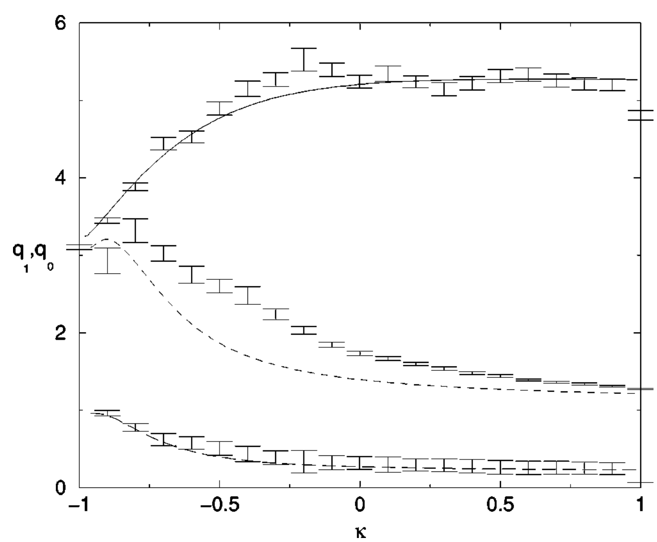

FIG. 4. The self-overlap of mixed strategies q_1 (solid line), the overlap q_0 between the mixed strategies of different NE (dashed line), and the ratio q_0/q_1 (long-dashed line). The analytic results are compared with numerical simulations for $N=50$ averaged over 200 samples for q_1 and $N=18$ averaged over 100 samples for q_0.

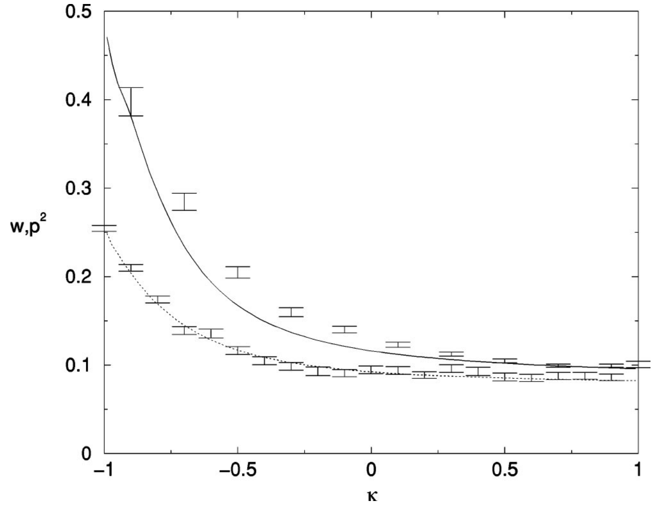

FIG. 5. The fraction of strategies w (top) played in both mixed strategies of two randomly chosen NE and the square p^2 of the fraction of strategies played at a single NE (bottom) against κ. The analytic results are compared with numerical simulations for $N=50$ averaged over 200 samples for p^2, and $N=18$ averaged over 100 samples for w.

The geometric structure of the set of NE may be elucidated by considering the mutual overlap $q_0=(1/N)\Sigma_i x_i^1 x_i^2$ between mixed strategies of different NE. At $\kappa=-1$, where there is only a single NE, q_0 equals the self-overlap q_1. After an initial increase q_0 decreases with increasing κ. The initial increase of q_0 is due to the rapid increase of the lengths of the mixed strategy vectors, and is thus not seen in the ratio between the overlaps. This result may be interpreted geometrically in that the NE become more and more separated with increasing κ, and for $\kappa\to+1$ they end up in nearly uncorrelated positions, $\langle x^1 x^2\rangle-\langle x^1\rangle\langle x^2\rangle=0.21$. At the same time an increasing fraction of components of the mixed strategies have $x_i=0$, i.e., lie on an edge of the simplex.

Even though the NE spread over the simplex with increasing κ, players still tend to focus on specific strategies. This may be seen by comparing the fraction of strategies w played in *both* mixed strategies of two randomly chosen NE with the corresponding result p^2, which would result if players chose p strategies to be played with nonzero probabilities at random. From Fig. 5 one finds that although w decreases with κ consistent with the spread of NE over the simplex, it always remains above p^2. This behavior is consistent with the idea that with increasing κ players focus on pairs of strategies which are beneficial to both, of which there is a large number for large values of κ.

Since NE are isolated points, replica symmetry describes a set of equilibrium points which are distributed uniformly over a part of the simplex with opening angle $\arccos(q_0/q_1)$. A replica-symmetry-breaking scenario would involve clusters of NE, and maybe even clusters of clusters, so an ansatz explicitly including more than two overlap scales would have to be employed along the lines of the Parisi scheme [5]. However, at least for $\kappa=0$, we found that replica symmetry is locally stable for $\nu>0.67$ and most importantly at the maximum of the curve. Replica symmetry remained locally stable across the range of ν investigated; nevertheless it may become locally unstable again at sufficiently large values of ν. Thus we may conclude that for a typical NE at $\kappa=0$ the replica-symmetric ansatz is self-consistent. Since we know from the results of Ref. [12] that replica symmetry is marginally stable at $\kappa=-1$, one may in fact speculate that for the typical NE the RS scenario holds across the entire range of κ. Nevertheless there may well be distributions of the payoffs which lead to nonuniform distributions of NE and to replica-symmetry breaking, presumably distributions with large values of κ, or with correlations between the entries of the payoff matrices at different sites.

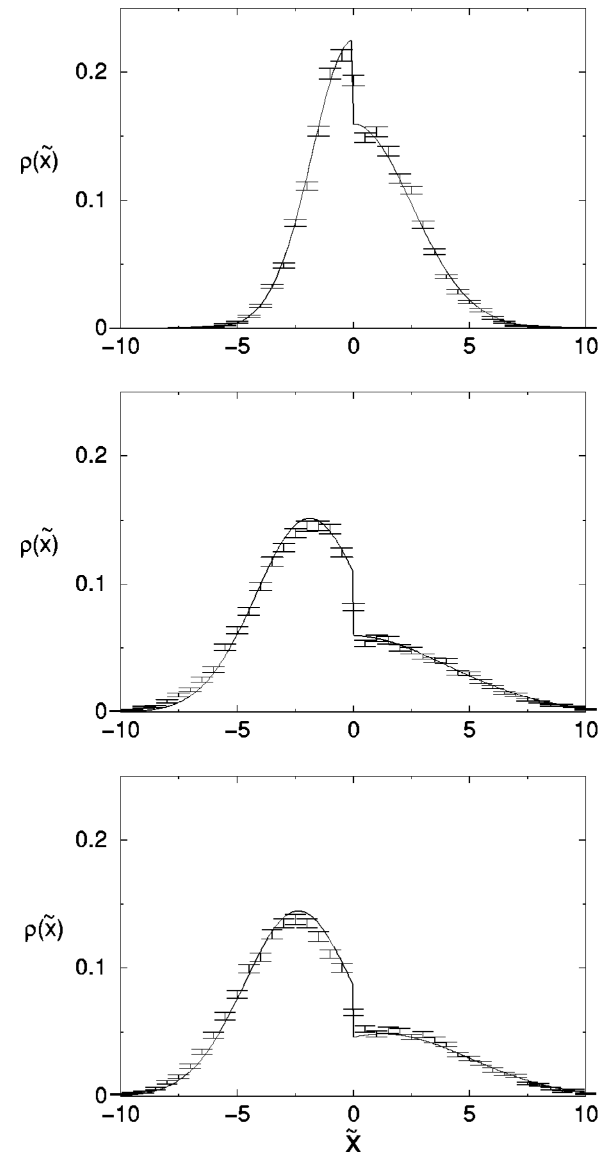

FIG. 6. The distribution of the potential payoffs ($\tilde{x}<0$) and the strategy strengths ($\tilde{x}>0$) for $\kappa=-1$, 0, and 1 from top to bottom.

C. Distribution of potential payoffs and the strategy strengths

Figure 6 shows the distribution of strategy strengths $\rho_x(x)=\langle\langle(1/N)\Sigma_i\delta(x_i-x)\rangle\rangle$ ($\tilde{x}>0$) and the potential payoffs $\rho_{\lambda_x}(\lambda)=\langle\langle(1/N)\Sigma_i\delta(\Sigma_j a_{ij}y_j-\lambda)\rangle\rangle$ ($\tilde{x}<0$) calculated in Sec. III A. The decrease of the fraction of strategies played with nonzero probability, $\int_0^\infty d\tilde{x}\ \rho_{\tilde{x}}(\tilde{x})=p$ with κ is clearly visible. One also finds a marked tendency for both players to use large values of x_i and y_j for decreasing values of p, as is demanded by the normalization condition.

One also observes the formation of a ''shoulder'' in the distribution of $\rho_{\tilde{x}}(\tilde{x})=\rho_{\lambda_x}(\lambda-\nu^x)$ $(\tilde{x}<0)$ centered at $-\nu$. It shows that the distribution of $\Sigma_j a_{ij}y_j$ remains peaked at zero leading to the formation of the shoulder at $\tilde{x}=-\nu$ as ν increases.

D. Comparison with numerical results

The numerical results for Figs. 2 and 6, q_0 in Fig. 4, and w in Fig. 5 were obtained by using so-called vertex enumeration to enumerate all NE. Since the computational effort for vertex enumeration scales with $\sim 2.598^N$, the system size had to be restricted to $N=18$ and averages were taken over 100 samples, resulting in pronounced finite-size effects. Nevertheless the increase of the number of NE with κ is clearly confirmed by the simulations.

The numerical results for Fig. 3, q_1 in Fig. 4, p in Fig. 5, and Fig. 6 were obtained by using an iterated variant of the Lemke-Howson algorithm [16,4] to locate *a single* NE and by averaging the results for $N=50$ over 200 different realizations of the payoffs. Although some finite-size effects remain, there is good agreement between the analytical and the numerical results.

V. SUMMARY AND OUTLOOK

We analyzed the properties of Nash equilibria in large random bimatrix games. To this end we constructed an indicator function which was used to count the number of NE with given payoffs to both players. We found that the number of NE is exponentially dominated by NE with a certain payoff to both players, and a certain set of order parameters. This implies that for a randomly chosen Nash equilibrium quantities such as the fraction of strategies played with a given probability, the self-overlap, and most importantly the payoff to either player take on a specific value with probability 1.

We considered square payoff matrices and argued that for large games and identically and independently distributed elements of the payoff matrices at different sites (i,j), the only relevant parameter of the probability ensemble is the correlation between elements of the same site of the payoff matrices a and b. We then calculated the quenched average of the number of Nash equilibria, from which one may also deduce quantities such as the payoff, the self-overlap, and the mutual overlap of mixed strategies at NE, and the distribution of the strategy strengths and the potential payoffs.

We found that both the number of equilibrium points and the payoff to both players increase with the correlation between the payoff matrices: With increasing correlation the number of pairs of strategies which are beneficial to both players grows. Players may focus on these pairs and achieve a larger payoff; the fraction of strategies played with nonzero probability decreases accordingly. From the values of the saddle-point parameters one may also deduce information on the geometric properties of NE: With increasing correlation between the payoff matrices the NE spread out over wider regions of the simplex. These analytic results were quantitatively compared with extensive numerical simulations, and good agreement was found.

Another point of relevance is that for a sufficiently large correlation between the payoff matrices, an exponentially large number of NE appears which offer arbitrarily large payoffs (on the statistical mechanics scale) to both players. The number of such NE is of course exponentially small compared to the total number of NE; nevertheless these equilibrium points may be relevant if players are free to choose equilibrium points.

A number of generalizations and extensions of these scenarios may be envisaged at this stage, including the investigation of bimatrix games with rectangular payoff matrices or payoff matrices with correlations between the elements at different sites. Furthermore, a scenario of games of several players might be extended to describe cooperative games, where coalitions of players pool their payoffs and seek to maximize the gain of their respective coalition. In this context it may also be interesting to consider the case of $O(N)$ players choosing between $O(1)$ strategies.

ACKNOWLEDGMENTS

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APPENDIX A: QUENCHED AVERAGE

In the following we give a derivation of the quenched average of the entropy of NE. In order to represent the logarithm of $\mathcal{N}$ we replicate n times the expression for the number of NE $\mathcal{N}$ [Eq. (7)] to obtain Eq. (9).

Treating the normalizing determinant as a self-averaging quantity, we may split off $\ln\|\det(D)\|$ with

$$D=\begin{pmatrix} \delta_{ii'}\Theta(-\tilde{x}_i) & -a_{ij}\Theta(\tilde{y}_j) \\ -b_{ij}\Theta(\tilde{x}_i) & \delta_{jj'}\Theta(-\tilde{y}_j) \end{pmatrix} \tag{A1}$$

from Eq. (9), and *separately* average the normalizing determinant over the disorder. Leaving out all the rows and columns which have only the entry 1 along the diagonal and do not contribute to the determinant, we are left with the determinant of a matrix

$$D'=\begin{pmatrix} 0 & -a' \\ -b' & 0 \end{pmatrix}, \tag{A2}$$

where the matrices a' and b' are the pN by pN submatrices of the payoff matrices containing the elements with $\tilde{x}_i>0$ and $\tilde{y}_j>0$. We thus calculate $\langle\langle\|\ln\det(D)\|\rangle\rangle$ as a function of $p_x=p_y$ exploiting the block-structure of the matrix D and using results from the theory of random matrices [17]. Since we have $\ln\|\det(D)\|=\ln\|\det(a')\|+\ln\|\det(b')\|$, the correlation between the elements of these matrices has no effect. We may thus use the circular theorem [18], which gives the average density $\rho(\omega)$ of eigenvalues ω of a pN by pN matrix with identically and independently Gaussian distributed entries with zero mean and variance N^{-1}

$$\rho(\omega)=\begin{cases}(\pi p)^{-1} & \|\omega\|<\sqrt{p}\\ 0 & \text{otherwise,}\end{cases} \tag{A3}$$

giving

$$\langle\langle \ln\|\det(D)\|\rangle\rangle = 2Np\int_S d\omega\,\rho(\omega)\|\omega\| = Np(\ln p-1), \tag{A4}$$

where the integral is over the region in the complex plane with $\|\omega\|<\sqrt{p}$. After this step, the only terms in Eq. (9) where the disorder is present are

$$\begin{aligned}\left\langle\left\langle \prod_{i,j}\exp\left\{ i\sum_a \hat{x}_i^a a_{ij}\tilde{y}_j^a\Theta(\tilde{y}_j^a)+i\sum_a \tilde{x}_i^a\Theta(\tilde{x}_i^a)b_{ij}\hat{y}_j^a\right\}\right\rangle\right\rangle\\ =\exp\Bigg\{-1/(2N)\sum_{a,b}\Bigg(\sum_i \hat{x}_i^a\hat{x}_i^b\sum_j \tilde{y}_j^a\Theta(\tilde{y}_j^a)\tilde{y}_j^b\Theta(\tilde{y}_j^b)\\ -2\kappa\sum_i i\hat{x}_i^a\tilde{x}_i^b\Theta(\tilde{x}_i^b)\sum_j i\hat{y}_j^b\tilde{y}_j^a\Theta(\tilde{y}_j^a)\\ +\sum_i \tilde{x}_i^a\Theta(\tilde{x}_i^a)\tilde{x}_i^b\Theta(\tilde{x}_i^b)\sum_j \hat{y}_j^a\hat{y}_j^b\Bigg)\Bigg\},\end{aligned} \tag{A5}$$

where the indices a and b denote the replicas, $a,b=1,\ldots,n$, and the average has been taken over the distribution of payoffs [Eq. (3)]. In order to obtain expressions which factorize in i and j, we introduce the matrices of order parameters

$$\begin{aligned} q_{ab}^x&=\frac{1}{N}\sum_i \tilde{x}_i^a\Theta(\tilde{x}_i^a)\tilde{x}_i^b\Theta(\tilde{x}_i^b),\\ q_{ab}^y&=\frac{1}{N}\sum_j \tilde{y}_j^a\Theta(\tilde{y}_j^a)\tilde{y}_j^b\Theta(\tilde{y}_j^b),\\ R_{ab}^x&=\frac{1}{N}\sum_i i\hat{x}_i^a\tilde{x}_i^b\Theta(\tilde{x}_i^b),\quad R_{ab}^y=\frac{1}{N}\sum_j i\hat{y}_j^b\tilde{y}_j^a\Theta(\tilde{y}_j^a),\\ p_a^x&=\frac{1}{N}\sum_i \Theta(\tilde{x}_i^a),\quad p_a^y=\frac{1}{N}\sum_j \Theta(\tilde{y}_j^a)\end{aligned} \tag{A6}$$

using integrals over δ functions. The last pair of order parameters is introduced so the normalizing determinant may be included as a function of p_a^x and p_a^y. This procedure turns Eq. (A5) into

$$\begin{aligned}\prod_{a\geqslant b}\int\frac{dq_{ab}^{x,y}d\hat{q}_{ab}^{x,y}}{2\pi/N}\prod_{a,b}\int\frac{dR_{ab}^x dR_{ab}^y}{2\pi/(\kappa N)}\prod_a\int\frac{dp_a^{x,y}d\hat{p}_a^{x,y}}{2\pi/N}\delta(p_a^x-p_a^y)\exp\Bigg\{-iN\sum_{a\geqslant b} q_{ab}^{x,y}\hat{q}_{ab}^{x,y}-i\kappa N\sum_{a,b} R_{ab}^x R_{ab}^y\\ +iN\sum_a p_a^{x,y}\hat{p}_a^{x,y}\Bigg\}\exp\Bigg\{\sum_{a\geqslant b}\hat{q}_{ab}^x\tilde{x}^a\Theta(\tilde{x}^a)\tilde{x}^b\Theta(\tilde{x}^b)+i\kappa\sum_{a,b} R_{ab}^y i\hat{x}^a\tilde{x}^b\Theta(\tilde{x}^b)-\frac{1}{2}\sum_{a,b} q_{ab}^y\hat{x}^a\hat{x}^b-i\sum_{a,i}\tilde{x}_i^a\Theta(-\tilde{x}_i^a)\hat{x}_i^a\\ -i\nu^x\sum_{a,i}\hat{x}_i^a-i\sum_a\hat{p}_a^x\Theta(\tilde{x}^a)\Bigg\}\exp\Bigg\{\sum_{a\geqslant b}\hat{q}_{ab}^y\tilde{y}^a\Theta(\tilde{y}^a)\tilde{y}^b\Theta(\tilde{y}^b)+\kappa\sum_{a,b} R_{ab}^x\tilde{y}^a\Theta(\tilde{y}^a)i\hat{y}^b-\frac{1}{2}\sum_{a,b} q_{ab}^x\hat{y}^a\hat{y}^b\\ -i\sum_{a,i}\tilde{y}_i^a\Theta(-\tilde{y}_i^a)\hat{y}_i^a-i\nu^y\sum_{a,i}\hat{y}_i^a-i\sum_a\hat{p}_a^y\Theta(\tilde{y}^a)\Bigg\}.\end{aligned} \tag{A7}$$

All order parameters have been introduced via conjugate variables, except R_{ab}^x and R_{ab}^y, which are conjugate to each other. Care must be taken to scale all order parameters so they are of $O(1)$ in the thermodynamic limit. Expression (A7) may now be substituted back into Eq. (9). The simplex constraint is incorporated by including yet another set of integrals

$$\begin{aligned}\prod_a\int\frac{dE_a^{x,y}}{2\pi/N}\exp\Bigg\{ iN\sum_a E_a^{x,y}-i\sum_a E_a^x\sum_i\tilde{x}_i^a\Theta(\tilde{x}_i^a)\\ -i\sum_a E_a^y\sum_i\tilde{y}_i^a\Theta(\tilde{y}_i^a)\Bigg\}.\end{aligned} \tag{A8}$$

The integrals over $\tilde{x}_i^a$ and $\hat{x}_i^a$ now factorize and form a product of N identical terms and may thus be written as the Nth power of a single such term. The same point applies to the integrals over $\tilde{y}_j^a$ and $\hat{y}_j^a$. Anticipating saddle points of the integrals over conjugate order parameters along the imaginary axis, we also perform a change of variables $i\hat{q}_{ab}^{x,y}\to\hat{q}_{ab}^{x,y}$, and analogously for R_{ab}^y, $E_a^{x,y}$, and $\hat{p}_a^{x,y}$. Including the normalizing determinant (A4), we finally obtain Eqs. (14) and (15).

Replica-symmetric ansatz

In the thermodynamic limit $N\to\infty$ the integrals over order parameters are dominated by their saddle point. Yet in order to carry out the replica limes $n\to 0$ we have to make an ansatz for the values of the order parameter matrices. The simplest ansatz is the replica-symmetric one given by Eq. (16). Since G^x and G^y are symmetric under an interchange of the players we may drop the superscripts x and y. We obtain

$$G=\ln\prod_a\int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}\exp\Bigg\{-\frac{1}{2}\sum_a(\hat{q}_1+\hat{q}_0)\tilde{x}^a\tilde{x}^a\Theta(\tilde{x}^a)$$
$$+\frac{1}{2}\sum_{a,b}\hat{q}_0\tilde{x}^a\Theta(\tilde{x}^a)\tilde{x}^b\Theta(\tilde{x}^b)$$
$$+\kappa(R_1-R_0)\sum_a i\hat{x}^a\tilde{x}^a\Theta(\tilde{x}^a)+\kappa R_0\sum_{a,b}i\hat{x}^a\tilde{x}^b\Theta(\tilde{x}^b)$$
$$-\frac{1}{2}(q_1-q_0)\sum_a\hat{x}^a\hat{x}^a-\frac{1}{2}q_0\sum_{a,b}\hat{x}^a\hat{x}^b$$
$$-i\sum_a\tilde{x}^a\Theta(-\tilde{x}^a)\hat{x}^a-i\nu\sum_a\hat{x}^a$$
$$-E\sum_a\tilde{x}^a\Theta(\tilde{x}^a)-\hat{p}\sum_a\Theta(\tilde{x}^a)\Bigg\}. \tag{A9}$$

A particularly efficient way to disentangle the three sums over the replica-replica couplings is to use two coupled Gaussian integrals over variables termed a and b echoing the original average over the payoff matrices, which yield

$$G=\ln\int da\,db\,p_{\tilde{\kappa}}(a,b)\prod_a\int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}$$
$$\times\exp\Bigg\{-\frac{1}{2}\sum_a(\hat{q}_1+\hat{q}_0)\tilde{x}^a\tilde{x}^a\Theta(\tilde{x}^a)$$
$$+\kappa(R_1-R_0)\sum_a i\hat{x}^a\tilde{x}^a\Theta(\tilde{x}^a)$$
$$-\frac{1}{2}(q_1-q_0)\sum_a\hat{x}^a\hat{x}^a+a\sqrt{\hat{q}_0}\sum_a\tilde{x}^a\Theta(\tilde{x}^a)$$
$$+ib\sqrt{q_0}\sum_a\hat{x}^a-i\sum_a\tilde{x}^a\Theta(-\tilde{x}^a)\hat{x}^a-i\nu^x\sum_a\hat{x}^a$$
$$-E\sum_a\tilde{x}^a\Theta(\tilde{x}^a)-\hat{p}\sum_a\Theta(\tilde{x}^a)\Bigg\}$$
$$:=\ln\int da\,db\,p_{\tilde{\kappa}}(a,b)\prod_a\int\mathcal{D}(\tilde{x}^a,\hat{x}^a), \tag{A10}$$

where $p_{\tilde{\kappa}}(a,b)$ with $\tilde{\kappa}=\kappa R_0/\sqrt{q_0\hat{q}_0}$ is defined by Eq. (18). The resulting expression factorizes, giving n identical integrals over $\tilde{x}$ and $\hat{x}$, which may be easily performed by considering the cases $\tilde{x}<0$ and $\tilde{x}>0$ separately. The limit $n\to 0$ of Eq. (14) may now be taken, yielding Eqs. (17)–(19).

APPENDIX B: STABILITY OF THE REPLICA-SYMMETRIC SADDLE POINT

In this section we outline the calculation of the eigenvalues of the Hessian matrix of Eq. (14) in order to check if ansatz (14) is locally stable against small fluctuations of the order parameters. We focus on the so-called replicon modes [19], and restrict ourselves to the case $\kappa=0$. In this case the Hessian matrix of fluctuations of Eq. (14) around Eq. (16) is given by Eq. (24).

The derivatives of G^x and G^y are evaluated at the RS saddle point. Due to the symmetry of G^x and G^y under an interchange of the players we have to find the replicon eigenvalues of three different submatrices of M, beginning with $\partial^2G/\partial\hat{q}_{ab}\partial\hat{q}_{cd}$: At the replica-symmetric saddle point there are three different entries in the $n(n-1)/2\times n(n-1)/2$ matrix of derivatives with respect to the off-diagonal elements of $\hat{q}_{ab}$ with $a>b$. These are

$$\frac{\partial^2G}{\partial\hat{q}_{ab}\partial\hat{q}_{cd}}=\begin{cases}P_1 & \text{for } a=c,b=d\\ Q_1 & \text{for exactly one pair of indices equal}\\ R_1 & \text{for } a\neq c,b\neq d,\end{cases} \tag{B1}$$

where

$$P_1=\langle x_a^2x_b^2\rangle-\langle x_ax_b\rangle\langle x_ax_b\rangle,$$
$$Q_1=\langle x_a^2x_bx_c\rangle-\langle x_ax_b\rangle\langle x_ax_c\rangle, \tag{B2}$$
$$R_1=\langle x_ax_bx_cx_d\rangle-\langle x_ax_b\rangle\langle x_cx_d\rangle,$$

the angular brackets denote the normalized averages over

$$\langle(\cdots)\rangle=\frac{\prod_a\int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}\exp\{\mathcal{L}^x(\{\tilde{x}^a,\hat{x}^a\})\}(\cdots)}{\prod_a\int\frac{d\tilde{x}^a d\hat{x}^a}{2\pi}\exp\{\mathcal{L}^x(\{\tilde{x}^a,\hat{x}^a\})\}}, \tag{B3}$$

and $\mathcal{L}^x$ is defined in Eq. (15) and the order parameters take on their saddle-point values. In the limit $n\to 0$ the replicon eigenvalue of this matrix equals

$$\lambda_1=P_1-2Q_1+R_1$$
$$=\int da\,db\,p_{\tilde{\kappa}}(a,b)\Bigg[\frac{\int\mathcal{D}(\tilde{x},\hat{x})\tilde{x}^2\Theta(\tilde{x})}{\int\mathcal{D}(\tilde{x},\hat{x})}-\left(\frac{\int\mathcal{D}(\tilde{x},\hat{x})\tilde{x}\Theta(\tilde{x})}{\int\mathcal{D}(\tilde{x},\hat{x})}\right)^2\Bigg]^2$$
$$=\int da\,db\,p_{\tilde{\kappa}}(a,b)\left[\frac{\partial^2}{\partial E^2}\ln L(a,b)\right]^2, \tag{B4}$$

where $\mathcal{D}(\tilde{x},\hat{x})$ is defined in Eq. (A10) and $L(a,b)$ is defined in Eq. (15).

The replicon eigenvalue of $\partial^2G/\partial q_{ab}\partial q_{cd}$ is evaluated in the same fashion. We obtain

$$\frac{\partial^2G}{\partial q_{ab}\partial q_{cd}}=\begin{cases}P_2 & \text{for } a=c,b=d\\ Q_2 & \text{for exactly one pair of indices equal}\\ R_2 & \text{for } a\neq c,b\neq d,\end{cases} \tag{B5}$$

where

$$P_2 = \langle \hat{x}_a^2 \hat{x}_b^2 \rangle - \langle \hat{x}_a \hat{x}_b \rangle \langle \hat{x}_a \hat{x}_b \rangle,$$

$$Q_2 = \langle \hat{x}_a^2 \hat{x}_b \hat{x}_c \rangle - \langle \hat{x}_a \hat{x}_b \rangle \langle \hat{x}_a \hat{x}_c \rangle, \tag{B6}$$

$$R_2 = \langle \hat{x}_a \hat{x}_b \hat{x}_c \hat{x}_d \rangle - \langle \hat{x}_a \hat{x}_b \rangle \langle \hat{x}_c \hat{x}_d \rangle.$$

In the limit $n \to 0$ the replicon eigenvalue of this matrix equals

$$\begin{aligned} \lambda_2 &= P_2 - 2Q_2 + R_2 \\ &= \int da\ db\ p_{\tilde{\kappa}}(a,b) \left[\frac{\int \mathcal{D}(\tilde{x},\hat{x}) \hat{x}^2}{\int \mathcal{D}(\tilde{x},\hat{x})} - \left(\frac{\int \mathcal{D}(\tilde{x},\hat{x}) \hat{x}}{\int \mathcal{D}(\tilde{x},\hat{x})} \right)^2 \right]^2 \\ &= \int da\ db\ p_{\tilde{\kappa}}(a,b) \left[\frac{\partial^2}{\partial \nu^2} \ln L(a,b) \right]^2. \end{aligned} \tag{B7}$$

The matrix $\partial^2 G / \partial q_{ab} \partial \hat{q}_{cd}$ also consists of three different entries. These are

$$\frac{\partial^2 G}{\partial \hat{q}_{ab} \partial q_{cd}} = \begin{cases} P_3 & \text{for } a=c, b=d \\ Q_3 & \text{for exactly one pair of indices equal} \\ R_3 & \text{for } a \neq c, b \neq d, \end{cases} \tag{B8}$$

where

$$P_3 = -\langle x_a x_b \hat{x}_a \hat{x}_b \rangle + \langle x_a x_b \rangle \langle \hat{x}_a \hat{x}_b \rangle,$$

$$Q_3 = -\langle x_a x_b \hat{x}_a \hat{x}_c \rangle + \langle x_a x_b \rangle \langle \hat{x}_a \hat{x}_c \rangle, \tag{B9}$$

$$R_3 = -\langle x_a x_b \hat{x}_c \hat{x}_d \rangle + \langle x_a x_b \rangle \langle \hat{x}_c \hat{x}_d \rangle.$$

In the limit $n \to 0$ the replicon eigenvalue of this matrix equals

$$\begin{aligned} \lambda_3 &= P_3 - 2Q_3 + R_3 \\ &= -\int da\ db\ p_{\tilde{\kappa}}(a,b) \left[\frac{\int \mathcal{D}(\tilde{x},\hat{x}) \tilde{x} \hat{x} \Theta(\tilde{x})}{\int \mathcal{D}(\tilde{x},\hat{x})} - \frac{\int \mathcal{D}(\tilde{x},\hat{x}) \tilde{x} \Theta(\tilde{x})}{\int \mathcal{D}(\tilde{x},\hat{x})} \frac{\int \mathcal{D}(\tilde{x},\hat{x}) \hat{x}}{\int \mathcal{D}(\tilde{x},\hat{x})} \right]^2 \\ &= \int da\ db\ p_{\tilde{\kappa}}(a,b) \left[\frac{\partial^2}{\partial \nu \partial E} \ln L(a,b) \right]^2. \end{aligned} \tag{B10}$$

Since the replicon eigenvectors of these three matrices are parallel, the eigenvalues of Eq. (24) are those of the matrix

$$\begin{pmatrix} \lambda_2 & -1 & 0 & \lambda_3 \\ -1 & \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_3 & \lambda_2 & -1 \\ \lambda_3 & 0 & -1 & \lambda_1 \end{pmatrix}, \tag{B11}$$

and we denote the coefficients of replicon-fluctuations as δq^x, $\delta \hat{q}^x$, δq^y, and $\delta \hat{q}^y$. In order to determine the criterion for local stability of the RS saddle point, we first eliminate the fluctuations in the conjugate order parameters $\delta \hat{q}^x$ and $\delta \hat{q}^y$ near the saddle point. From $\partial S / \partial \delta \hat{q}^y = 0$ and $\partial S / \partial \delta \hat{q}^y = 0$ one obtains $\delta \hat{q}^x = (1/\lambda_2)(\delta q^x - \lambda_3 \delta q^y)$ and $\delta \hat{q}^y = (1/\lambda_2)(\delta q^y - \lambda_3 \delta q^x)$, respectively.

This allows us to write the matrix of replicon fluctuations in terms of δq^x and δq^y only yielding after some algebra

$$S = S_{RS} + \frac{1}{2} (\delta q^x \delta q^y) M' \begin{pmatrix} \delta q^x \\ \delta q^y \end{pmatrix} + O(\delta^3), \tag{B12}$$

with

$$M' = \frac{1}{\lambda_2} \begin{pmatrix} \lambda_1 \lambda_2 - \lambda_3^2 - 1 & 2\lambda_3 \\ 2\lambda_3 & \lambda_1 \lambda_2 - \lambda_3^2 - 1 \end{pmatrix}. \tag{B13}$$

Since the integrals over the variables q^x and q^y are now over a real function, the criterion that the RS ansatz (14) is locally stable is that both eigenvalues of M' are negative, giving

$$\begin{aligned} &\frac{1}{\lambda_2} [\lambda_1 \lambda_2 - (\lambda_3 - 1)^2] < 0, \\ &\frac{1}{\lambda_2} [\lambda_1 \lambda_2 - (\lambda_3 + 1)^2] < 0. \end{aligned} \tag{B14}$$

[1] J.F. Nash, Ann. Math. **54**, 286 (1951).
[2] Wang Jianhua, *The Theory of Games* (Oxford University Press, Oxford, 1988).
[3] H. Keiding, Games Economic Behavior **21**, 148 (1997).
[4] B. von Stengel, in *Handbook of Game Theory*, edited by R.J. Aumann and S. Hart (North-Holland, Amsterdam, in press), Vol. 3.
[5] M. Mézard, G. Parisi, and M.A. Virasoro, *Spin Glass Theory and Beyond* (World Scientific, Singapore, 1987).
[6] *Spin Glasses and Random Fields*, edited by A.P. Young (World Scientific, Singapore, 1997).
[7] J.A. Hertz, A. Krogh, and R.G. Palmer, *Introduction to the Theory of Neural Computation* (Addison-Wesley, Redwood City, CA, 1991).
[8] S. Diederich and M. Opper, Phys. Rev. A **39**, 4333 (1989).
[9] M. Opper and S. Diederich, Phys. Rev. Lett. **69**, 1616 (1992).

[10] M. Mézard and G. Parisi, J. Phys. (France) Lett. **46**, L771 (1985).
[11] R. Monasson and R. Zecchina, Phys. Rev. Lett. **76**, 3881 (1996).
[12] J. Berg and A. Engel, Phys. Rev. Lett. **81**, 4999 (1998).
[13] J. Berg and M. Weigt, Europhys. Lett. **48**, 129 (1999).
[14] M. Opper, P. Kuhlmann, and A. Mietzner, Europhys. Lett. **37**, 31 (1997).
[15] At nonzero κ, cross-terms, e.g., of the form $\partial^2 S/\partial q_{ab}^y \partial R_{cd}^y = \langle x_a x_b i\hat{x}_c x_d \rangle$, arise. At the RS saddle point these matrices consist of four different elements, and the standard analysis of Ref. [19] is no longer applicable.
[16] C.E. Lemke and J.T. Howson, J. Soc. Ind. Appl. Math. **12**, 413 (1964).
[17] A. Crisanti, G. Paladin, and A. Vulpiani, *Products of Random Matrices in Statistical Physics* (Springer-Verlag, Berlin, 1993).
[18] E.P. Wigner, Ann. Math. **67**, 325 (1958).
[19] J.R.L. de Almeida and D. Thouless, J. Phys. A **11**, 983 (1978).