Nonequilibrium phase transitions induced by multiplicative noise: Effects of self-correlation

Sergio E. Mangioni* and Roberto R. Deza†
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Deán Funes 3350, 7600 Mar del Plata, Argentina

Raúl Toral‡
Departament de Física, Universitat de les Illes Balears and Instituto Mediterráneo de Estudios Avanzados, IMEDEA (CSIC-UIB), E-07071 Palma de Mallorca, Spain

Horacio S. Wio§
Centro Atómico Bariloche, Comisión Nacional de Energía Atómica and Instituto Balseiro, 8400 San Carlos de Bariloche, Argentina
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A recently introduced lattice model, describing an extended system which exhibits a *reentrant* (symmetry-breaking, second-order) noise-induced nonequilibrium phase transition, is studied under the assumption that the multiplicative noise leading to the transition is *colored*. Within an effective Markovian approximation and a mean-field scheme it is found that when the self-correlation time τ of the noise is different from zero, the transition is *also reentrant* with respect to the spatial coupling D. In other words, at variance with what one expects for equilibrium phase transitions, a large enough value of D favors *disorder*. Moreover, except for a small region in the parameter subspace determined by the noise intensity σ and D, an increase in τ usually *prevents* the formation of an ordered state. These effects are supported by numerical simulations.

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I. INTRODUCTION

In the last few decades we have witnessed a paradigmatic shift regarding the role of fluctuations, from the equilibrium picture of merely being a $N^{-1/2}$ perturbation on thermodynamic averages—or triggering at most phase transitions between well defined minima of the free energy [1]—to lead a host of new and amazing phenomena in far from equilibrium situations. As examples we may cite noise-induced unimodal-bimodal transitions in some zero-dimensional models [2], stochastic resonance in zero-dimensional and extended systems [3,4], noise-induced spatial patterns [5], noise-delayed decay of unstable states [6], ratchets [7], shifts in critical points [8], etc. An attempt to find a noise-induced *phase* transition that preceded in almost a decade recent studies is the work in Ref. [9], resulting in a ''no-go'' theorem for the existence of such a transition in a system of Ito-like differential equations with *linear* drift and *multiplicative* noise.

Recently it has been shown that a white and Gaussian *multiplicative* noise can lead an *extended* dynamical system (fulfilling appropriate conditions) to undergo a *phase* transition towards an *ordered* state, characterized by a nonzero order parameter and by the breakdown of ergodicity [10]. This result—first obtained within a Curie-Weiss-like mean-field approximation, and further extended to consider the simplest correlation function approach—has been confirmed through extensive numerical simulations [11]. In addition to its *critical* nature as a function of the noise intensity σ, the newly found noise-induced phase transition has the noteworthy feature of being *reentrant*: for each value of D above a threshold one, the ordered state exists only inside a window $[\sigma_1,\sigma_2]$. At variance with the known case of *equilibrium* order-disorder transitions that are induced (in the simplest lattice models) by the nearest-neighbor coupling constant D and rely on the bistability of the local potential, the transition in the case at hand is led by the *combined effects* of D and σ through the nonlinearities of the system. Neither the zero-dimensional system (corresponding to the $D=0$ limit) nor the deterministic one ($\sigma=0$) show any transition.

This counterintuitive ordering role of noise has also been found afterwards in different models in the literature [12–18]. In Refs. [12,13], the authors study a noise-induced reentrant transition in a time-dependent Ginzburg-Landau model with both additive and multiplicative noises. Reference [14] introduces another simple model with a purely multiplicative noise, which also presents a noise-induced reentrant transition. This reference also gives evidence that the universality class of its critical behavior is that of the *multiplicative noise* [19,20] (see also Ref. [21] for a discussion of the universality class of these models). In Refs. [15,16], a first-order phase transition induced by noise is obtained in a system of globally coupled oscillators. A similar first-order phase transition is also found in Ref. [17]. Finally, Ref. [18] addresses the role of multiplicative noise in the context of phase-separation dynamics.

Although for the sake of mathematical simplicity all these references (in particular, Refs. [10,11]) studied only the white-noise case (the only exception is Ref. [18] in which colored noise in space, white in time, is considered) it is expected that, because of their nature, fluctuations coupled

*Electronic address: smangio@mdp.edu.ar
†Electronic address: deza@mdp.edu.ar
‡URL: http://formentor.uib.es/~raul. Electronic address: dfsrtg0@ps.uib.es
§Electronic address: wio@cab.cnea.gov.ar

multiplicatively to the system will show some degree of time-correlation or ''color'' [2,22–25], and hence new effects may arise from this fact. For example, a reentrant behavior has been found recently as a consequence of color in a noise-induced transition [26] and an ordering nonequilibrium phase transition can be induced in a Ginzburg–Landau model by varying the correlation time of the additive noise [27,28]. Thus motivated, we have studied the consequences of a *finite* (but still very short as compared with the ''deterministic'' or coarse-grained time scales) self-correlation time τ of the multiplicative noise in systems of this kind. We now recall some of the new effects that emerge in this colored-noise case, which have been briefly reported in a recent work [29].

Our main finding is that, as a consequence of the multiplicative character of the noise, a strong enough spatial coupling D leads invariably (for $\tau>0$) to a *disordered* state, contrary to what would be expected to happen in equilibrium statistical-mechanical models.

Another important result is that, except for large values of σ and *very small* values of τ, color has an *inhibiting* role for ordered states. Moreover, there exists a *finite* and not very large value of τ beyond which order is impossible.

These results represent a major departure from what one can expect on the basis of equilibrium thermodynamics, according to which one should tend to think that as $D\to\infty$ an *ordered* situation is favored. Whereas that ''intuitive'' notion remains valid if the multiplicative noise that induces the nonequilibrium ordering phase transition is white [10,11], it ceases to be so for $\tau>0$. In the former case, the results could be interpreted in terms of a ''freezing'' of the short-time behavior by a strong enough spatial coupling: for $D/\sigma^2\to\infty$, the stationary probability distribution could be considered to be δ-like, just as the initial one. In our case, an analysis of the short-time behavior of the order parameter up to first order in τ reveals that the disordering effect of D can only be felt for *nonzero* self-correlation time. As τ increases, the minimum value of D required to stabilize the disordered phase decreases and the region in parameter space available to the ordered phase shrinks until it vanishes. Thus, the foregoing results can only be interpreted once we recall that the ordered phase arises from the cooperation of *two* factors: the presence of spatial coupling *and* the multiplicative character of the noise (which may eventually lead to ''counterintuitive'' results).

It is our purpose in this work to render an explicit account of our calculation and, at the same time, to expose and to discuss the results more thoroughly. After presenting the model in Sec. II, we begin Sec. III by introducing the approximations needed to render the problem accessible to mathematical analysis. We resort to a mean-field approximation such as that in Refs. [10,11] and to a ''unified colored noise approximation'' (UCNA) [30,31], devised to deal with self-correlated noises. In Sec. III A a simplified treatment using the aforementioned approximations is given and in Sec. III B we expose the more sophisticated approach that was actually used to obtain the phase diagrams. In this approach the UCNA is complemented with an appropriate interpolation scheme [31]. In Sec. IV we expose and discuss the results obtained within the last approach, comparing the phase diagram with the ones arising from the simplified approach and (for $\tau\simeq 0$) from a perturbative expansion, and the D dependence of the order parameter m for nonzero τ with a numerical simulation [29]. A final discussion of the approach and its results is made in Sec. V.

II. THE MODEL

The model under consideration has been introduced in Refs. [10,11,29]: a d-dimensional extended system of typical linear size L is restricted to a hypercubic lattice of $N=L^d$ points, whereas time is still regarded as a continuous variable. The state of the system at time t is given by the set of stochastic variables $\{x_i(t)\}$ $(i=1,\ldots,N)$ defined at the sites $\mathbf{r}_i$ of this lattice, which obey a system of coupled ordinary stochastic differential equations (SDE's)

$$\dot{x}_i = f(x_i) + g(x_i)\,\eta_i + \frac{D}{2d}\sum_{j\in n(i)}(x_j - x_i) \tag{1}$$

(throughout the paper, the Stratonovich interpretation for the SDE's will be meant). Equation (1) is the discrete version of the *partial* SDE which in the continuum would determine the state of the extended system: we recognize in the first two terms the generalization of Langevin's equation for site i to the case of multiplicative noise (η_i is the *colored* multiplicative noise acting on site $\mathbf{r}_i$). For the specific example analyzed in Ref. [10], perhaps the simplest one exhibiting the transition under analysis (see, however, Ref. [14]),

$$f(x) = -x(1+x^2)^2, \quad g(x) = 1+x^2. \tag{2}$$

The last term in Eq. (1) is nothing but the lattice version of the Laplacian $\nabla^2 x$ of the extended stochastic variable $x(\mathbf{r},t)$ in a reaction-diffusion scheme. $n(i)$ stands for the set of nearest neighbors in a d-dimensional square lattice $(2d)$ sites which form the immediate neighborhood of the site $\mathbf{r}_i$, and the coupling constant D between neighboring lattice sites is the diffusion coefficient.

As previously stated, it is our aim in this work to investigate the effects of the self-correlation time τ of the multiplicative noise on the model system just described. To that end we must assume a specific form for the noises $\{\eta_i\}$: we choose Ornstein-Uhlenbeck noises, i.e., Gaussian distributed stochastic variables with zero mean and exponentially decaying correlations

$$\langle \eta_i(t)\,\eta_j(t')\rangle = \delta_{ij}(\sigma^2/2\tau)\exp(-|t-t'|/\tau). \tag{3}$$

They arise as solutions of an *uncoupled* set of Langevin SDE:

$$\tau\dot{\eta}_i = -\eta_i + \sigma\xi_i, \tag{4}$$

where the $\{\xi_i(t)\}$ are white noises—namely, Gaussian stochastic variables with zero mean and δ correlated: $\langle \xi_i(t)\xi_j(t')\rangle = \delta_{ij}\delta(t-t')$. For $\tau\to 0$, the Ornstein-Uhlenbeck noise $\eta_i(t)$ approaches the white-noise limit $\xi_i^W(t)$ with correlations $\langle \xi_i^W(t)\xi_i^W(t')\rangle = \sigma^2\delta_{ij}\delta(t-t')$, which was the case studied in Refs. [10,11].

III. THE APPROXIMATIONS

The non-Markovian character of the process $\{x_i\}$ (due to the presence of the colored noise $\{\eta_i\}$) makes it difficult to study. However, some approximations have been devised which render a *Markovian* process that—whereas nicely simplifying the treatment—can still capture some of the essential features of the complete non-Markovian one. The aforementioned UCNA is one of them: in fact a very reliable one, because of its ability to reproduce the small- and large-τ limits [30]. By resorting to interpolation schemes, one can retrieve meaningful results in wider vicinities of these limits [31].

As already declared, our approach is a mean-field-like one. The earlier we make this kind of assumptions in the calculation, the cruder the approximation will turn out to be. In order to find the phase diagram in the presence of colored noise we have made the mean-field approximation at some late stage, so enhancing the precision of the calculation. However, since this calculation is a tedious one, we shall first expose a simpler approximation which brings out most qualitative results. We aim in this way to underline the physical origin of the results presented in Sec. IV. The differences arising from both calculations are pointed out there.

A. A simpler approach

The simpler mean-field approximation follows closely Curie-Weiss' mean-field approach to magnetism, and consists in replacing the last term in Eq. (1)

$$\Delta_i \equiv \frac{D}{2d} \sum_{j \in n(i)} (x_j - x_i) \tag{5}$$

with

$$\overline{\Delta}_i \equiv D(\overline{x} - x_i), \tag{6}$$

where $\overline{x}$ is a *parameter* that will be determined self-consistently. In other words, the (short-ranged) interactions are substituted by a time- and space-independent ''external'' field whose value *depends on the state* of the system. Since in this approximation Eq. (1) gets immediately decoupled, there is no use in keeping the subindex *i* and we may refer to the systems in Eqs. (1) and (4) as if they were single equations. Hereafter, the primes will indicate derivatives with respect to x (clearly $\overline{\Delta}' = -D$).

If we take the time derivative of Eq. (1), replace first $\dot{\eta}$ in terms of η and ξ from Eq. (4) and then η in terms of $\dot{x}$ and x from Eq. (1), we obtain the following *non-Markovian* SDE:

$$\tau\left(\ddot{x} - \frac{g'}{g}\dot{x}^2\right) = -\left(1 - \tau\left[(f+\overline{\Delta})' - \frac{g'}{g}(f+\overline{\Delta})\right]\right)\dot{x} + (f+\overline{\Delta}) + \sigma g \xi. \tag{7}$$

The aforementioned UCNA allows us to recover a Markovian SDE: for our particular problem it amounts, on one hand, to a usual adiabatic elimination (namely, neglecting $\ddot{x}$) and, on the other, to neglect $\dot{x}^2$ so that the system's dynamics be governed by a Fokker-Planck equation [32]. The resulting equation, being *linear* in $\dot{x}$ (but of course not in x), can be immediately solved for $\dot{x}$ giving

$$\dot{x} = Q(x;\overline{x}) + S(x;\overline{x})\xi, \tag{8}$$

with

$$Q(x;\overline{x}) \equiv (f+\overline{\Delta})\theta, \tag{9}$$

$$S(x;\overline{x}) \equiv \sigma g \theta, \tag{10}$$

$$\theta(x;\overline{x}) \equiv \{1 - \tau g[(f+\overline{\Delta})/g]'\}^{-1}. \tag{11}$$

The parametric dependence of $Q(x)$ and $S(x)$ on $\overline{x}$ has been written explicitly.

The Fokker-Planck equation associated to the SDE Eq. (8) is

$$\partial_t P(x,t;\overline{x}) = -\partial_x[R_1(x;\overline{x})P(x,t;\overline{x})] + \frac{1}{2}\partial_x^2[R_2(x;\overline{x})P(x,t;\overline{x})], \tag{12}$$

with drift and diffusion coefficients given by [33]

$$R_1(x;\overline{x}) = Q + \frac{1}{4}(S^2)', \tag{13}$$

$$R_2(x;\overline{x}) = S^2. \tag{14}$$

The solution of the time-independent Fokker-Planck equation leads to the stationary probability density

$$P^{st}(x;\overline{x}) = \mathcal{N}^{-1}\exp\left[\int_0^x dx' \frac{2R_1(x';\overline{x}) - \partial_{x'}R_2(x';\overline{x})}{R_2(x';\overline{x})}\right], \tag{15}$$

$\mathcal{N}$ being its norm. The partial-derivative notation $\partial_{x'}$ in Eq. (15)—as in Eqs. (18) and (19) below—is only a reminder of the parametric dependence of R_1, R_2, on $\overline{x}$.

The value of $\overline{x}$ arises from a self-consistency relation, once we equate it to the average value of the random variable x_i in the stationary state

$$\overline{x} = \langle x \rangle \equiv \int_{-\infty}^{\infty} dx\, x P^{st}(x;\overline{x}) \equiv F(\overline{x}). \tag{16}$$

As in the known Curie-Weiss mean-field approach, the condition

$$\left.\frac{dF}{d\overline{x}}\right|_{\overline{x}=0} = 1 \tag{17}$$

allows us to find the transition line between the ordered and the disordered phases. Here also, $F(\overline{x})$ is a smooth odd function such that Eq. (16) has always a root at $\overline{x}=0$ and for $dF/d\overline{x}|_{\overline{x}=0} > 1$ it has two nontrivial roots which differ only in sign. The condition Eq. (17) thus reads

$$\mathcal{N}^{-1}\int_{-\infty}^{\infty}dx\,x\int_0^x dx'\exp\left[\int_0^{x'}dx''\frac{2R_1-\partial_{x''}R_2}{R_2}\right]\partial_{\bar{x}}\left(\frac{2R_1-\partial_{x'}R_2}{R_2}\right)\Bigg|_{\bar{x}=0}=1, \tag{18}$$

where

$$\mathcal{N}=\int_{-\infty}^{\infty}dx\exp\left[\int_0^x dx'\frac{2R_1-\partial_{x'}R_2}{R_2}\right]\Bigg|_{\bar{x}=0}. \tag{19}$$

Equations (18) and (19) must be solved numerically in order to find the lines in parameter space (σ,τ,D) that separate ordered $(\bar{x}\neq 0)$ from disordered $(\bar{x}=0)$ phases. The results of this calculation will be shown in Sec. V. Next, we introduce a more refined approach in which the mean-field approximation is made at a later stage in the calculation.

B. A more refined approach

As we shall see, the relations obtained with this more sophisticated approach are similar to Eqs. (15)–(19), but with different expressions for the functions $R_1(x;\bar{x})$ and $R_2(x;\bar{x})$. The idea here is to *introduce first* the UCNA, without yet resorting to the mean-field approximation. In the following, Δ_i has the same meaning as in Eq. (5) and, as it occurred previously with $\bar{\Delta}'$, it satisfies $\Delta_i'=-D$. Note however that whereas for $f_i\equiv f(x_i)$ and $g_i\equiv g(x_i)$ the prime has the meaning of a total derivative with respect to x_i, for Δ_i' and all the functions involving them, its meaning is really that of a *partial* derivative with respect to x_i. Proceeding as before, i.e. taking the time derivative of Eqs. (1) and using Eqs. (1) and (4) to eliminate the η's in favor of the x's and ξ's, we obtain the following system of (non Markovian) SDE's:

$$\tau\left(\ddot{x}_i-\frac{g_i'}{g_i}\dot{x}_i^2\right)=-\left[1-\tau g_i\left(\frac{f_i+\Delta_i}{g_i}\right)'\right]\dot{x}_i+(f_i+\Delta_i)+\sigma g_i\xi_i+\frac{D\tau}{2d}\sum_{j\in n(i)}\dot{x}_j. \tag{20}$$

The UCNA proceeds here through the neglect of $\ddot{x}_i$ and of $(\dot{x}_i)^2$, so retrieving a *linear* equation in the $\dot{x}$'s (but of course not in the x's), which can be rewritten as

$$\dot{x}_i=\left[1-\tau g_i\left(\frac{f_i+\Delta_i}{g_i}\right)'\right]^{-1}\left[(f_i+\Delta_i)+\sigma g_i\xi_i+\frac{D\tau}{2d}\sum_{j\in n(i)}\dot{x}_j\right]$$

$$=[Q_i+S_i\xi_i]+\frac{\beta_i}{2d}\sum_{j\in n(i)}\dot{x}_j. \tag{21}$$

Here the quantities

$$\theta_i\equiv[1-\tau g_i(f_i+\Delta_i/g_i)']^{-1}, \tag{22}$$

$$Q_i\equiv(f_i+\Delta_i)\theta_i, \tag{23}$$

$$S_i\equiv\sigma g_i\theta_i, \tag{24}$$

$$\beta_i\equiv D\tau\theta_i \tag{25}$$

have been defined in order to simplify the notation. Note that although only the dependence upon x_i has been made explicit in this notation, these quantities depend also (through Δ_i) on the values x_j at the neighboring sites.

Now, assuming the lattice to be isotropic, we apply to this set a *mean-field-like approximation* (but not yet the main one) consisting in *replacing* in all the functions appearing in Eq. (21) the $2d$ neighbors x_j of the variable x_i *by a common value* y_i. Through this procedure one reduces the number of different coupled SDE's to 2: one for $x\equiv x_i$ and another for its nearest neighbor variable $y\equiv y_i$. These are

$$\dot{a}=h_a+g_{ab}\xi_b, \tag{26}$$

where a sum over the values x,y is implied for the indices a,b, and the noise variables satisfy $\langle\xi_a(t)\xi_b(t')\rangle=\delta_{ab}\delta(t-t')$. If, similarly as before, we define

$$\theta(x,y)=\left[1-\tau g(x)\frac{\partial}{\partial x}\left(\frac{f(x)+D(y-x)}{g(x)}\right)\right]^{-1}, \tag{27}$$

$$Q(x,y)=[f(x)+D(y-x)]\theta(x,y), \tag{28}$$

$$S(x,y)=\sigma g(x)\theta(x,y), \tag{29}$$

$$\beta(x,y)=\tau D\theta(x,y), \tag{30}$$

$$A(x,y)=[1-\beta(x,y)\beta(y,x)]^{-1}=A(y,x), \tag{31}$$

and write $\bar{a}=y$ if $a=x$ and vice versa, then the explicit forms of the functions in Eq. (26) are

$$h_a=A(x,y)[Q(a,\bar{a})+\beta(a,\bar{a})Q(\bar{a},a)] \tag{32}$$

and

$$g_{ab}=A(x,y)S(a,\bar{a}) \quad \text{if } b=a, \tag{33}$$

$$=A(x,y)\beta(a,\bar{a})S(\bar{a},a) \quad \text{if } b=\bar{a}. \tag{34}$$

The bivariate Fokker–Planck equation associated with Eq. (26) is

$$\partial_t P=-\partial_a(R_aP)+\frac{1}{2}\partial_a\partial_b(R_{ab}P), \tag{35}$$

where $P=P(x,y;t)$. According to standard techniques [33], the drift and diffusion coefficients are given respectively by

$$R_a(x,y)=h_a+\frac{1}{2}g_{bc}\partial_b(g_{ac}), \tag{36}$$

$$R_{ab}(x,y)=g_{ac}g_{bc}. \tag{37}$$

Since the denominators occurring in these equations may become zero for some values of x or y, we resort to an *interpolation procedure* (analogous to the one used in Refs. [26,31]) consisting in replacing the expression (31) for $A(x,y)$ by

$$A(x,y)=\frac{1-\beta(x,y)\beta(y,x)}{1+\beta(x,y)^2\beta(y,x)^2}. \tag{38}$$

Since $\beta(x,y)$ is proportional to τ, it follows that the expression in Eq. (31) coincides with the interpolated one, Eq. (38), as $\tau\to 0$ and $\tau\to\infty$ (the latter limit meaning indeed ''τ comparable with the 'deterministic' time scales'').

By integrating the bivariate Fokker-Planck Eq. (35) with respect to y we obtain a single-variable equation, which in the stationary case reads

$$0=-\partial_x(R_1P^{st})+\frac{1}{2}\partial_x^2(R_2P^{st}), \tag{39}$$

where R_1P^{st} and R_2P^{st} are functions of x only:

$$R_1P^{st}\equiv\int_{-\infty}^{\infty}dy\,P^{st}(x,y)R_x(x,y), \tag{40}$$

$$R_2P^{st}\equiv\int_{-\infty}^{\infty}dy\,P^{st}(x,y)R_{xx}(x,y). \tag{41}$$

Here it is when we resort to the *main mean-field-type approximation*, resembling the Curie-Weiss' type of approach used in Ref. [10]: *assuming* $P^{st}(x,y)\approx P^{st}(x)\,\delta(y-\overline{x})$, R_1 and R_2 in Eqs. (40) and (41) are approximated by

$$R_1=R_x(x;\overline{x}), \tag{42}$$

$$R_2=R_{xx}(x;\overline{x}). \tag{43}$$

In this way, from the stationary *joint* probability density function $P^{st}(x,y)$ we retrieve an *effective* single-variable one $P^{st}(x;\overline{x})$ whose expression in terms of $R_1(x;\overline{x})$ and $R_2(x;\overline{x})$ arising from Eq. (39) is the same as in Eq. (15). The value of $\overline{x}$ follows again from a self-consistency relation such as Eq. (16). The procedure to find the phase diagram is the same as in the foregoing subsection and the explicit expression of the condition $dF/d\overline{x}|_{\overline{x}=0}=1$ is given by Eqs. (18) and (19), this time in terms of the corresponding R_1, R_2 given by Eqs. (42) and (43).

As discussed in Ref. [29], although the kind of approximation leading to Eq. (38) is not of a perturbative nature, it has provided sound results in the cases analyzed heretofore [26,31]. Nonetheless, for the sake of comparison we have also adapted to the case of a multiplicative noise a known perturbative procedure [34]. Within this context, the expressions for $R_1(x;\overline{x})$ and $R_2(x;\overline{x})$ come from a consistent small-τ expansion of a Fokker-Planck equation [Eqs. (12) or (35)]:

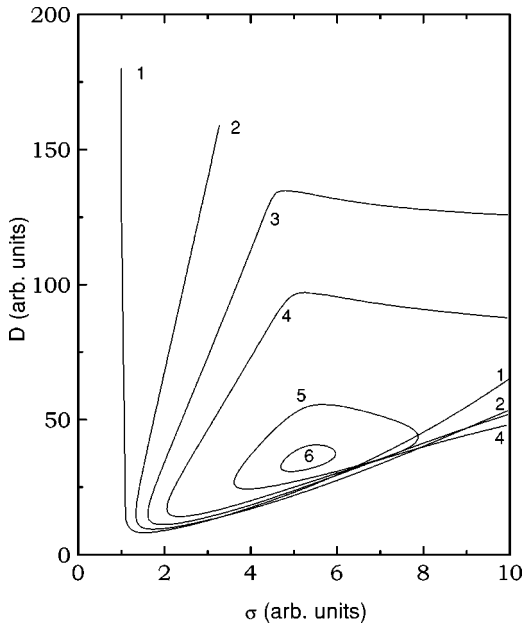

FIG. 1. Phase diagram in the σ-D plane, for different values of τ: (1) $\tau=0$; (2) $\tau=0.015$; (3) $\tau=0.03$; (4) $\tau=0.05$; (5) $\tau=0.1$; (6) $\tau=0.123$. The ordered phase exists only *inside* the corresponding curves.

$$R_1(x;\overline{x})=(f+\overline{\Delta})+\sigma^2x\{g[1+\tau(f+\overline{\Delta})']+\tau(f+\overline{\Delta})\}, \tag{44}$$

$$R_2(x;\overline{x})=\{\sigma g[1+\tau(f+\overline{\Delta})']\}^2. \tag{45}$$

IV. THE RESULTS

A. Phase diagram

In the following we shall describe the results obtained through the numerical solution of Eqs. (18) and (19) in the more refined approach, i.e., with R_1 and R_2 as prescribed by Eqs. (42) and (43). We shall also compare these results with the ones arising from Eqs. (13) and (14), and with a perturbative expansion for small τ. Figures 1–3 are, respectively, the projections onto the σ-D, τ-σ, and τ-D planes, of cuts of the boundary separating the ordered and disordered phases performed at fixed values of the remaining parameters.

Let us first concentrate on Fig. 1: it corresponds to Fig. 1 in Ref. [29], but is the result of an improved calculation based on the more refined mean-field approach described in Sec. III B. The novelty is that, at least for τ not too small, it is now evident that the region available to the ordered phase is *bounded*. The noteworthy aspects are the following.

(A). As in the white-noise case $\tau=0$ (Refs. [10,11]), the ordering phase transition is *reentrant with respect to* σ: for a range of values of D that depends on τ, ordered states can only exist within a window $[\sigma_1,\sigma_2]$. The fact that this window shifts to the right *for small* τ means that, for fixed D, color *destroys* order just above σ_1 but *creates* it just above σ_2.

(B). For fixed $\sigma>1$ and $\tau\neq 0$, ordered states exist *only within a window* of values for D. Thus the ordering phase transition is *also reentrant with respect to* D. For τ small enough the maximum value of D compatible with the or-

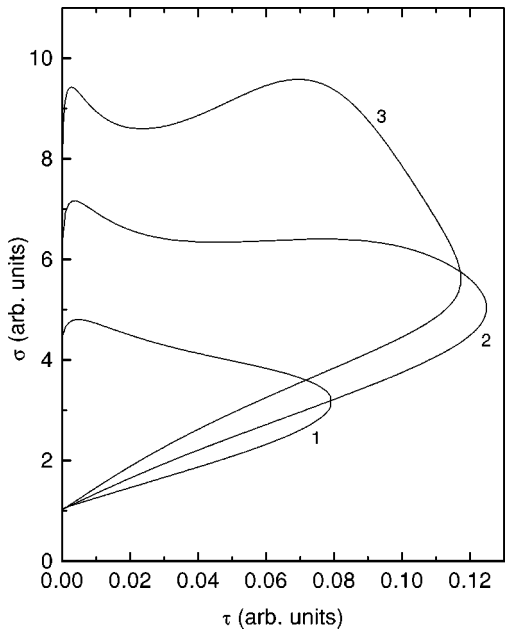

FIG. 2. Phase diagram in the τ-σ plane, for different values of D: (1) $D=20$; (2) $D=32$; (3) $D=45$. The ordered phase exists only *inside* the corresponding curves.

dered phase increases rather steeply with σ, reaching a maximum around $\sigma\sim 5$ and then decreases gently. For $\tau \geqslant 0.1$ it becomes evident (in the ranges of D and σ analyzed) that the region sustaining the ordered phase is *closed*, and shrinks to a point for a value slightly larger than $\tau=0.123$.

(C). For fixed values of $\sigma>1$ and D larger than its minimum for $\tau=0$, the system *always* becomes disordered for τ large enough. The maximum value of τ consistent with order altogether corresponds to $\sigma\sim 5$ and $D\sim 32$. In other words, ordering is possible *only* if the multiplicative noise inducing it has short memory.

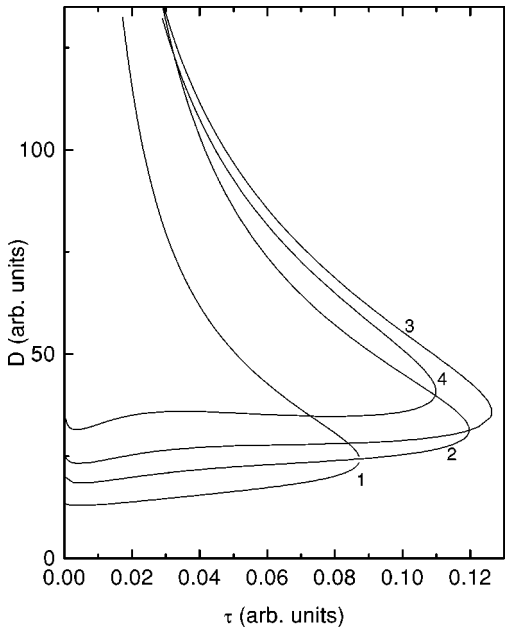

FIG. 3. Phase diagram in the τ-D plane, for different values of σ: (1) $\sigma^2=10$; (2) $\sigma^2=20$; (3) $\sigma^2=30$; (4) $\sigma^2=50$. The ordered phase exists only *inside* the corresponding curves.

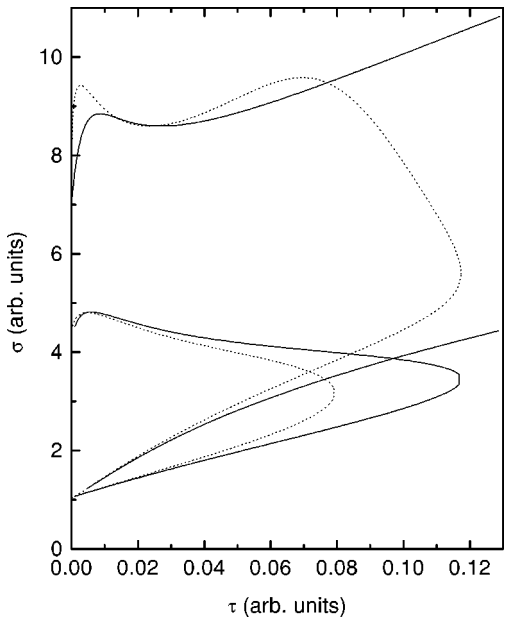

FIG. 4. Comparison between the simpler mean-field approach (solid line) and the refined one (dots) in the τ-σ plane, for $D=20$ (lower two curves) and $D=45$ (upper two ones). Not only does the simpler approach tend to overestimate the size of the ordered region, but it even predicts unbounded ordered regions for some values of D.

(D). The fact that the region sustaining the ordered phase finally shrinks to a point means that even for that small region in the σ–D plane for which order is induced by color, a further increase in τ destroys it. In other words, the phase transition is *also reentrant with respect to* τ. For D large enough there may exist even *two* such windows.

Some of the features just described become more evident by looking at Fig. 2: the existence of a maximum correlation time consistent with ordering for each value of D (occurring for an optimal value of σ) (C); the ordering ability (A) of a very small amount of color for $\sigma>\sigma_2(D)$ [$\sigma_2(\tau,D)$ increases very rapidly at first]; and the reentrance with respect to τ and even the occurrence of a *double* reentrance for D large enough (D). Figure 3 represents another way of seeing the reentrance with respect to D for constant σ (large enough) and the fact that there exists a maximum τ consistent with order for each value of σ (it being largest for $\sigma\sim 5$). The scarce dependence of D on τ in the lower branch—as well as its almost linear dependence on σ—is easily understood by looking at the rightmost branch of Fig. 1.

In Figs. 4–6 we compare the results just shown—obtained as we said using the more refined approach of Sec. III B—with the ones arising from the simpler one (Sec. III A). Figure 4 corresponds to Fig. 2, whereas Figs. 5 and 6 focus respectively on the $\tau=0.03$ and $\tau=0.015$ curves in Fig. 1. Not only does the simpler approach (grossly) overestimate the size of the ordered region but also—as one may infer from Figs. 4 and 5—it seems to predict unbounded ordered regions.

Figure 6 corresponds to a rather small value of τ, so that a comparison with the results obtained by using expressions (44) and (45) makes sense. For σ and D small enough the curves almost coincide. As it is well known, the simulta-

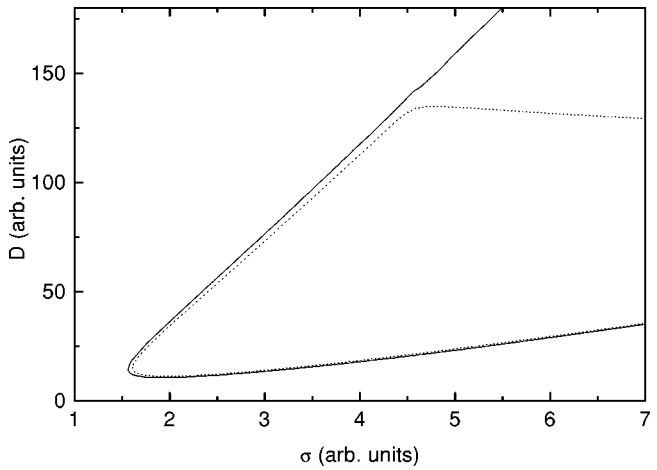

FIG. 5. Comparison between the two mean-field approaches in the σ-D plane, for $\tau=0.03$. Solid line: simpler one; dots: refined one.

neous consideration of small τ and large σ cannot be done independently [23–25]. In the present case, a similar effect arises when we consider large values of D, as we discuss below. Hence, in order to consider larger values of σ and D, one should take *extremely low* values of τ. As Fig. 6 shows, already for $\tau=0.015$ there is an apparent discrepancy between the perturbative results and the mean-field ones even for not so large values of σ and D. The noteworthy fact is that the perturbative expansion also tends to indicate the existence of a reentrance with respect to D.

B. Order parameter

The order parameter in this system is $m\equiv|\overline{x}|$, namely, the positive solution of the consistency equation [see Eq. (16) in Sec. III A]. In Fig. 7 we plot m vs σ for $D=20$ and different values of τ. Consistently with what has been discussed in (A) and (C) and shown in Fig. 1, we see that as τ increases the window of σ values where ordering occurs shrinks until it disappears. One also notices that at least for this D, the value of σ corrresponding to the maximum order parameter varies very little with τ. Figure 8 is a plot of m vs τ for $D=45$ and two values of σ (≈ 7.07 and ≈ 8.94) that illustrates the case of double reentrance in τ.

Since the previous results have been obtained in the mean-field approximation, we have also performed numerical simulations in order to have an independent check of the predictions. As a representative example—corresponding to the phenomenon (B) above (the destruction of the ordered

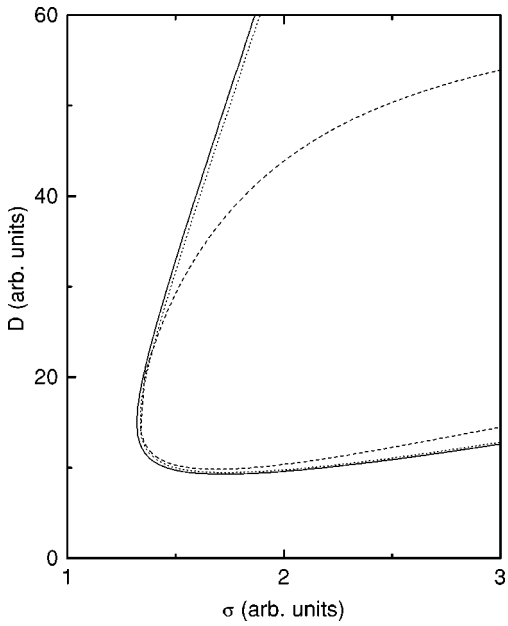

FIG. 6. Comparison between the two mean-field approaches and with a perturbative expansion in the σ-D plane, for $\tau=0.015$. Solid line: simpler MF; dotted line: refined MF; dashed line: perturbative.

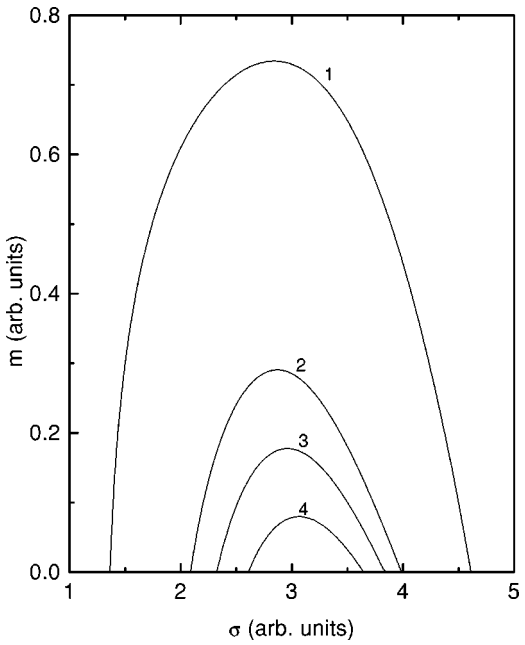

FIG. 7. Order parameter m vs σ, for $D=20$ and four values of τ: (1) $\tau=0.015$; (2) $\tau=0.05$; (3) $\tau=0.06$; (4) $\tau=0.07$.

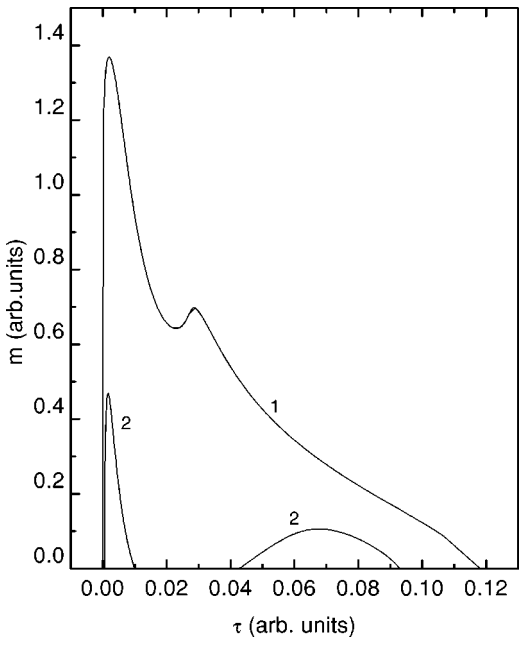

FIG. 8. Plot of m vs τ for $D=45$, showing cases of double reentrance: (1) $\sigma^2=50$; (2) $\sigma^2=80$.

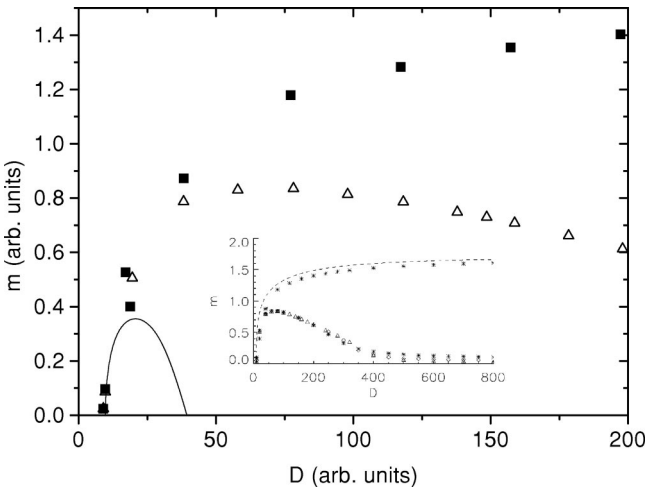

FIG. 9. Plot of m vs D for $\sigma=2$ and $\tau=0.01$, showing the predictions of the more refined mean-field approach together with results coming from a numerical integration of the original SDE's, Eq. (1).

phase by an increasing coupling constant D)—we plot jointly in Fig. 9, for rather small values of $\sigma(=2)$ and τ $(=0.01)$, the D dependence of the order parameter as predicted by our mean-field approach and as resulting from a numerical integration of the original SDE's, Eq. (1). We have taken three different lattice sizes in order to assess finite-size effects. As we see, the numerical simulations *do* predict the disordering for large enough D, and even the maximum ordering occurs in a region which is consistent with the mean-field prediction. This comparison warns us, however, that the mean-field approximation can severely underestimate the size of the ordered region.

The *short-time evolution* of $\langle x\rangle$ can be obtained multiplying Eq. (12) by x and integrating:

$$\frac{d\langle x\rangle}{dt}=\int_{-\infty}^{\infty}dx\,R_1(x;\overline{x})P(x,t;\overline{x}). \tag{46}$$

Let us assume an initial condition such that at early times $P(x,t\sim 0;\overline{x})=\delta(x-\overline{x})$. Equating $\overline{x}=\langle x\rangle$ as before, we obtain

$$\frac{d\langle x\rangle}{dt}=R_1(\overline{x},\overline{x}) \tag{47}$$

[again, we can use for R_1 the result Eq. (13) of the simple approximation or the more elaborate one given by Eq. (42)]. The numerical solution of Eq. (47) has an initial *rising* period (it is initially *unstable*) reaching very soon a maximum and tending to zero afterwards.

For $D/\sigma^2\to\infty$, Eq. (47) results to be valid also in the *asymptotic regime* since $P^{st}(x)=\delta(x-\overline{x})$ [11]. In Ref. [11] an equivalent equation is obtained in the $\tau=0$ case for *both* limits ($D=0$ *and* $D/\sigma^2\to\infty$) being there interpreted in terms of a ''freezing'' of the short-time behavior. According to this criterion, in the $D/\sigma^2\to\infty$ limit the system undergoes a second-order phase transition *if* the corresponding zero-dimensional model presents *a linear instability in its short-time dynamics*, i.e., if after linearizing Eq. (47):

$$\langle\dot{x}\rangle=-\alpha\langle x\rangle \tag{48}$$

one finds that $\alpha<0$. We then see that the trivial (disordered) solution $\langle x\rangle=0$ is stable only for $\alpha>0$. For $\alpha<0$ other stable solutions with $\langle x\rangle\neq 0$ appear, and the system develops order through a genuine *phase* transition. In this case, $\langle x\rangle$ can be regarded as the *order parameter*. In the white noise limit $\tau=0$ this is known to be the case for sufficiently large values of the coupling D and for a window of values for the noise amplitude $\sigma\in[\sigma_1,\sigma_2]$.

We discuss now how the stability of the ordered phase is altered by nonzero values of τ. If we linearize Eq. (47) using the expression of $R_1(\overline{x},\overline{x})$ from Eq. (13), we obtain

$$\alpha=\frac{(1+\tau+\tau D)^2-\sigma^2(1-3\tau+2\tau D)}{(1+\tau+\tau D)^3}. \tag{49}$$

If we use instead Eq. (42), the result can be written exclusively in terms of τD:

$$\alpha=1-\sigma^2\frac{B(\tau D)}{A(\tau D)} \tag{50}$$

with

$$A(x)=1+5x+8x^2+3x^3-3x^4-x^5+2x^6+x^7, \tag{51}$$

$$B(x)=1+3x+x^2-5x^3-3x^4+3x^5+x^6, \tag{52}$$

so the instability occurs at $\sigma^2=A/B$. Now, the ratio B/A keeps always below 1 in the positive range, has a minimum of ~ 0.05 at $\tau D\sim 1.09$ and a maximum of ~ 0.36 at τD ~ 2.33. If we consider, e.g., the conditions in Fig. 5 (namely, $\tau=0.03$) and take D large enough so that $\tau D>2.33$, one can see that Eq. (50) approximates the left boundary better than Eq. (49) does. In the limit $\tau\ll 1$ (but still finite) Eq. (50) can be approximated to the expression reported in Ref. [29], namely,

$$\alpha=\frac{1+\tau D-\sigma^2}{1+\tau D}. \tag{53}$$

It is worthwile to stress the fact [evident from Eqs. (49), (50), and (53)] that the value of D has no effect on the *location* of the instability by itself, but only through the combination τD: according to Eq. (53) the stable phase is the disordered one ($\langle x\rangle=0$) for $1+\tau D>\sigma^2$ (since $\alpha>0$) and the ordered one ($\langle x\rangle\neq 0$) for $1+\tau D<\sigma^2$. In summary, whereas the noise intensity σ has a destabilizing effect on the disordered phase, as soon as $\tau\neq 0$ the spatial coupling D tends to *stabilize* it. For $\tau=0$ the last effect is not present, being then $\sigma>1$ the condition for ordering [10,11]. Considering that the effect of even a tiny correlation is enhanced by D, we can understand the abrupt change in slope (from negative to positive) shown in Fig. 1 as soon as $\tau\neq 0$. Note the approximately inverse relation between τ and D for fixed σ on the upper branches of Fig. 3, even when Eq. (48) is strictly valid for $D/\sigma^2\to\infty$.

V. CONCLUSIONS

This work has focused on the effects of a self-correlation in the multiplicative noise, on the reentrant noise-induced phase transition introduced in Ref. [10]. Whereas in a recent paper we have reported the most relevant results [29], it has been our purpose in the present work to expose in more detail the techniques and the approximations employed. We also discuss more thoroughly the results, adding new figures and enriching the contents of others.

Through the use of the UCNA we recovered a Markovian behavior for the system, and through an interpolation scheme similar to the one introduced in Ref. [31] we resolved indeterminacies in the equations describing it. We stress that the equations resulting from this interpolation scheme are *exact* in the limits $\tau=0$ and $\tau\to\infty$. In addition to the fact that the interpolation scheme has been already applied with success in other works [26,31], the goodness of our approximations for small but nonetheless finite values of τ has been checked against a standard perturbative expansion [34] (adapted for multiplicative noise). It is worth emphasizing the fact that these approximations are, so far, the *only* tool available for an analytical treatment of this essentially non-Markovian problem.

The main result is that for $\tau\neq 0$, the order established as a consequence of the multiplicative character of the noise can be *destroyed* by a strong enough spatial coupling. Figure 1 shows that for given τ (0.03) and σ, the ordered phase can only exist between definite values of D. In particular, the upper bound on D decreases roughly as τ^{-1} for given σ.

The foregoing result can be understood by recalling that the ordered phase arises as a consequence of the *collaboration* between the multiplicative character of the noise and the presence of spatial coupling. The disordering effect of D arises *only* when $\tau\neq 0$ (the results in Ref. [11]—rightly interpreted in terms of a ''freezing'' of the short-time behavior by a strong enough spatial coupling—are thus consistent with ours). As τ increases, the minimum value of D required to stabilize the disordered phase decreases rapidly, and the region in parameter space available to the ordered phase shrinks until it disappears.

The example worked throughout this paper shows vividly the fact that the conceptual inheritance from equilibrium thermodynamics (though often useful) is not always applicable. The equilibrium-thermodynamic lore would induce us to think that as $D\to\infty$ an ordered situation is favored [1]. Although this is certainly true for the Curie-Weiss model (since in that case the deterministic potential is itself bistable and an increase of spatial coupling has the effect of raising the potential barrier between the stable states), it is *not* in the case we are dealing with, since the deterministic potential is *monostable*. Hence, it is the combined effects of the multiplicative noise *and* the spatial coupling that induce the transition.

As a summary, whereas one might say that the value of Refs. [10,13,11] is that they tell experimentalists where *not* to look for a noise-induced phase transition—namely, in those systems which are prone to exhibit a usual (zero-dimensional) noise-induced transition, and for too large noise intensity—the present work tells moreover that, due to the consideration of the more realistic colored noise source, *an ordered phase is not to be found for large values of the spatial coupling either*. Though the specific choice of the forms for the functions $f(x)$ and $g(x)$ may appear to some as physically unmotivated, is up to our knowledge the *simplest* one exhibiting this phenomenon. We believe nonetheless that the phenomenon is robust and transcends the specific choice made in this work.

The next obvious step is to consider a finite correlation length in the lattice model, which requires to go beyond the mean field approach. This problem is presently being studied.

ACKNOWLEDGMENTS

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