Correlation-length–exponent relation for the two-dimensional random Ising model

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We consider the two-dimensional (2D) random Ising model on a diagonal strip of the square lattice, where the bonds take two values, $J_1 > J_2$, with equal probability. Using an iterative method, based on a successive application of the star-triangle transformation, we have determined at the bulk critical temperature the correlation length along the strip ξ_L for different widths of the strip $L \leq 21$. The ratio of the two lengths $\xi_L / L = A$ is found to approach the universal value $A = 2/\pi$ for large L, independent of the dilution parameter J_1/J_2. With our method we have demonstrated with high numerical precision, that the surface correlation function of the 2D dilute Ising model is self-averaging, in the critical point conformally covariant and the corresponding decay exponent is $\eta_\parallel = 1$.

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I. INTRODUCTION

In the presence of quenched, i.e., time independent disorder one generally considers different random samples and the physical observables are characterized by their distribution and (nth) moments. (For $n=1$ and $n=0$ we have the *average* and *typical* value, respectively.) The extensive quantities, which are connected to the free-energy and its derivatives, have normal distribution, thus in a single sample one measures their *average* value with probability one in the thermodynamic limit. These quantities are called *self-averaging*. There are, however, other observables, typically correlation functions, which are broadly distributed and the typical (or most probable) value is different from the average value, even in the thermodynamic limit.

Such type of phenomena takes place in disordered quantum systems [1], where the typical and average behavior of correlations and critical singularities are even qualitatively different. As known by exact results [2,3], renormalization group [4] and numerical calculations [5] in the *infinite randomness fixed point* the average behavior is dominated by the *rare events*, which occur with vanishing probability, whereas the typical behavior is seen in any large sample with probability one.

In a classical system the effect of disorder is comparatively weaker [since in quantum systems the disorder is strictly correlated along the (imaginary) time direction]. Here the critical singularities are controlled by a *random fixed point* and there are usually quantitative differences between the average and typical behavior.

In this respect a well known example is the one-dimensional (1D) random bond Ising model [6], defined by the Hamiltonian $H = -\Sigma_j J_j s_j s_{j+1}$. Here the spin correlation function

$$G(r) = \langle s_{j+r} s_j \rangle = \prod_{k=j}^{j+r-1} t_k, \quad t_k = \tanh(J_k / k_B T), \quad (1.1)$$

is given as a product of random numbers and has a log-normal distribution. Consequently its most probable, or typical value $G(r)_{\rm typ} = \exp\{\ln[G(r)]_{\rm av}\} = [t]_{\rm av}^r$ and average value $[G(r)]_{\rm av}$ are different, even in the thermodynamic limit. In the following we use $\langle \cdots \rangle$ and $[\cdots]_{\rm av}$ to denote thermal and disorder averaging, respectively.

In higher-dimensional classical spin systems with random ferromagnetic couplings, such as the random Ising and Q-state Potts models, the effect of disorder is expected to be even weaker than in 1D. In calculating the correlation function the thermal average in higher dimensions involves several random couplings, not only those connecting directly the two points, therefore the disorder fluctuations are smoothed down. There is a class of random systems which, in the vicinity of their critical point, are homogeneous in macroscopic scales, thus the effect of quenched disorder is *irrelevant*. The corresponding criterion for weak randomness due to Harris [7] requires $\alpha^{\rm pure} < 0$, where $\alpha^{\rm pure}$ is the specific heat exponent of the pure system.

For systems with $\alpha^{\rm pure} > 0$ the disorder is a *relevant* perturbation so that the critical properties are controlled by a (new) *random fixed point* in which unconventional scaling behavior is expected. A detailed study, both (field) theoretical [8,9] and numerical [10–12], about the two-dimensional random $Q>2$ state Potts model has revealed that the critical bulk spin correlation function has *multifractal* behavior: the different moments of the correlation function at the critical point decay as a power

$$[G^n(r)]_{\rm av}^{1/n} \sim r^{-2x^{(n)}}, \quad (1.2)$$

with n dependent decay exponents $x^{(n)}$. We note that for conventional scaling the $x^{(n)}$'s have no n dependence.

An important question concerning random magnetic systems is whether the critical point correlations in Eq. (1.2) transform covariantly under conformal transformations [13]. Although correlations in one sample are not translationally invariant the *average* correlations are translationally and rotationally invariant and—it is generally believed—they are also conformally covariant. Indeed numerical studies in the strip and rectangle geometry for the two-dimensional $Q>2$ state random Potts model show that average critical correlations transform covariantly under conformal transformations [14]. Recently conformal properties of correlation functions and density profiles have been used to determine the scaling dimensions of different operators [12].

The two-dimensional random Ising model with $\alpha^{\mathrm{pure}}=0$ represents the marginal situation of the Harris criterion and detailed studies have been performed to clarify its critical properties [15]. Disorder is predicted as a marginally irrelevant perturbation by field theory, so that the critical singularities of the random model are characterized by the exponents of the pure Ising model supplemented by logarithmic corrections [16,17,8]. Numerical studies are in favor of this scenario [15,18–22], although conflicting interpretation of the numerical results has also been suggested [23,24].

Considering the spin correlation function of the random Ising model according to field theory the decay of the different moments of the bulk correlations at the critical point are given by the power law of the pure model with $x^{\mathrm{pure}}=1/8$, but the logarithmic corrections to the different moments are n dependent [8,25]. Numerically the bulk critical correlations are studied in the infinite plane geometry by MC simulations in Ref. [11], however, the possible logarithmic corrections of the moments have not been analyzed. On the other hand in Refs. [18,26] the transfer matrix method is used in the strip geometry and the decay exponent of the typical and average bulk correlations are deduced from the assumption of conformal invariance.

In the semi-infinite geometry the critical surface correlation function has been studied in Ref. [21] by the star-triangle (ST) method . It was found that for any dilution the numerically calculated average correlations are compatible with the form

$$[G_s(r)]_{\mathrm{av}} \sim r^{-1}(\ln r)^{1/2}. \tag{1.3}$$

Thus the decay exponent of the critical surface magnetization is $\eta_\parallel=1$, as for the pure system, however, in the random model there are also logarithmic corrections.

In this paper we continue to study the surface correlation function of the 2D random Ising model. New features of our investigations are the following.

(i) We considered the strip geometry, rather than the semi-infinite geometry.

(ii) As a numerical method we used an iterative procedure based on the star-triangle transformation. By this ST method a finite strip of random Ising model is formally transformed to a chain of Ising spins with smoothly inhomogeneous bonds. Then, using the exact expression in Eq. (1.1), we have calculated very accurately the correlation length parallel to the strip, ξ_L, and studied its distribution and different moments.

(iii) The advantage of the ST method to the transfer matrix (TM) technique, applied previous for bulk correlations is twofold. First, we could investigate larger widths of the strip, going up to $L=21$, which is approximately twice of the widths available by the TM technique [26]. The second advantage of the ST method that one can consider correlations between two largely separated spins, $r=O(10^3)$, which is at least one order of magnitude larger, than for the TM method. In this way we obtained more accurate averaging and could go deeper into the asymptotic region of the correlations.

(iv) Finally, we calculated different moments of the correlation function and studied the validity of the correlation length-exponent relation, as follows from the assumption of conformal invariance.

The structure of the paper is the following. The model and the ST method to calculate surface correlations are presented in Sec. II. Our results about surface correlations and the corresponding correlation lengths are given in Sec. III. We conclude our paper with a discussion in the final section.

II. STAR-TRIANGLE APPROACH TO SURFACE CORRELATIONS

We consider the Ising model on a diagonal strip of the square lattice, with $i=1,2,\ldots,L$ columns and $j=1,2,\ldots,K$ rows. At $i=1$ and $i=L$ there are two (1,1) surfaces, whereas in the vertical direction with $K \gg L$ we impose periodic boundary conditions. The nearest neighbor spins are connected with ferromagnetic couplings, $J_{i,j}>0$, which could take two values $J_1>J_2$ with equal probability. In the thermodynamic limit $L,K\to\infty$ the model is self-dual [27] and the self-duality point

$$\tanh(J_1/k_BT)=\exp(-2J_2/k_BT), \tag{2.1}$$

corresponds to the critical point, since according to numerical studies there is one phase-transition in the system. The degree of dilution can be varied by changing the ratio of the strong and weak couplings $\rho=J_1/J_2$. At $\rho=1$ one recovers the perfect Ising model, whereas for $\rho\to\infty$ we are in the percolation limit, where $T_c=0$.

For a given distribution of the couplings correlations between two surface spins $G_s(r)=\langle s_{1,j+r}s_{1,j}\rangle$, can be conveniently calculated by the star-triangle method. The ST method was introduced by Hilhorst and van Leeuwen [28] and later by others [29] to study the surface critical properties of triangular lattice Ising models with a layered structure. Recently, the method has been generalized for arbitrary distribution of the couplings and applied for the random semi-infinite Ising model in Ref. [21], hereafter referred to as paper I. In the following we recapitulate the method for the *strip geometry* with free boundary conditions at the two edges of the strip.

Central to the method is the ST transformation by which one replaces all right-pointing triangles of the strip by a star, which yields a hexagonal lattice of spins, denoted by dashed lines in Fig. 1. In the second step of the mapping the left pointing stars of the hexagonal lattice are replaced by triangles resulting in a new triangular lattice, which is denoted by dotted lines in Fig. 1. Iterating the procedure a sequence of triangular Ising models is generated ($m=0,1,2,\ldots$) from the original model with $m=0$. Neither the width of the strip nor the number of spins is changed under the transformation.

As seen in Fig. 1 the surface spins of the mth and the $(m+1)$th models are connected by the surface couplings of the intermediate hexagonal lattice and, as shown in paper I, there is an explicit relation between the thermal average of the surface spins in the two models. Then, from the fact that the surface magnetization of a finite strip with free boundary conditions is vanishing, follows that the surface couplings of the hexagonal lattice goes to zero as $m\to\infty$. As a consequence the surface spins of the triangular lattice decouple asymptotically from the rest of the system. The surface cor-

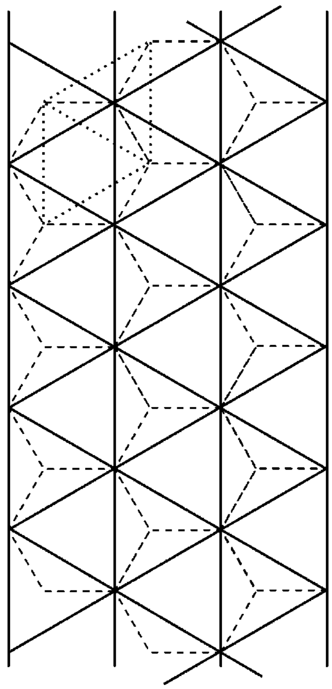

FIG. 1. Strip of triangular lattice with $L=4$ layers. The hexagonal lattice, obtained by the ST transformation is denoted by dashed lines, the triangles of the new triangular lattice are shown by dotted lines. In a diagonal square lattice the vertical couplings, which are zero in the original model, are generated during the iteration process.

relation functions of the mth and $(m+1)$-th triangular models are similarly related and one can use the results of paper I to calculate the surface correlation length from this relation. One can, however, proceed in a simpler way noticing that the surface spin correlation function stays asymptotically invariant under the mapping. Then $G_s(r)$ in the original model can be expressed in the form of the one-dimensional Ising model in Eq. (1.1) replacing J_j by the asymptotic value of the surface coupling $J_j^{(s)}(m)\equiv J_{1,j}(m)$.

For a given strip of width L the surface correlations show an asymptotic exponential decay $G_s(r)\sim\exp(-r/\xi_L)$, $r\gg L$, where the correlation length ξ_L is approximated by

$$\frac{1}{\xi_L(m)}=-\frac{1}{r}\ln\left(\prod_{k=j}^{j+r-1}\tanh[J_k^{(s)}(m)/k_BT]\right), \qquad (2.2)$$

and $\lim_{m\to\infty}\xi_L(m)=\xi_L$. Averaging over different disorder realizations one obtains the *typical* correlation length

$$\xi_L^{\mathrm{typ}}=[\xi_L]_{\mathrm{av}}. \qquad (2.3)$$

In physical applications one should average the nth power of the correlation function the asymptotic behavior of which, $[G_s(r)^n]_{\mathrm{av}}^{1/n}\sim\exp(-r/\xi_L^{(n)})$, defines the corresponding correlation length $\xi_L^{(n)}$. As already mentioned for $n=1$ and $n=0$ we obtain the average and typical correlation lengths, respectively. From the different moments of the distribution of the inverse correlation length $p(1/\xi_L)$ one obtains $\xi_L^{(n)}$ in a cumulant expansion

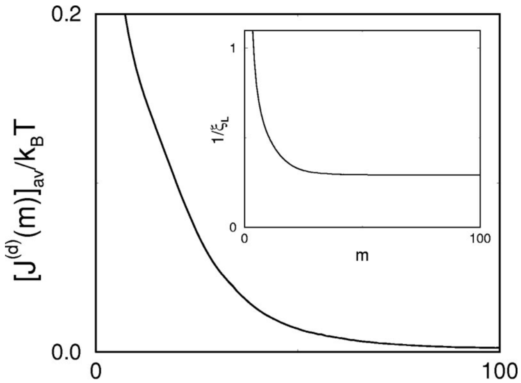

FIG. 2. The average surface diagonal coupling, connecting the surface spins to the rest of the system, as a function of the number of iteration m. The calculation is performed on a given random sample with dilution $\rho=4$ and on a strip of size $L=11$ and $K=1024$. Inset: The approximate inverse correlation length, defined in Eq. (2.2), as a function of the iteration parameter.

$$\frac{1}{\xi_L^{(n)}}=\frac{1}{\xi_L^{\mathrm{typ}}}-\frac{1}{2}nr\left[\left(\frac{1}{\xi_L}-\frac{1}{\xi_L^{\mathrm{typ}}}\right)^2\right]_{\mathrm{av}}+\cdots. \qquad (2.4)$$

For isotropic systems the correlation length parallel to the strip, $\xi_L^{(n)}$ and the width of the system L are asymptotically proportional and for *conformally invariant* systems their ratio takes the universal value [13]:

$$\frac{\xi_L^{(n)}}{L}=\frac{1}{\pi x_s^{(n)}}. \qquad (2.5)$$

Here $x_s^{(n)}$ is the anomalous dimension of the surface magnetization, defined through the asymptotic decay of the critical surface correlation in the semi-infinite geometry, $[G_s(r)^n]_{\mathrm{av}}^{1/n}\sim r^{-2x_s^{(n)}}$. Thus $x_s^{(n)}$ is the surface counterpart of $x^{(n)}$ in Eq. (1.2) and satisfies the scaling relation $\eta_\parallel^{(n)}=2x_s^{(n)}$.

III. RESULTS

We studied the spin correlations of the random Ising model on the (1,1) surface of the square lattice by the ST method. Evidently the original model with $m=0$ can be considered as a special triangular lattice model with vanishing vertical bonds. During iteration, however, nonzero vertical couplings are generated so that also the surface couplings $J_j^{(s)}(m)$ become nonzero. In the actual calculations we considered strips of width $L=2l+1$ up to $L=21$ [30], whereas for the length of the strip K the condition $K\gg L$ is always satisfied. Typically we took $K=1024$ and checked that the numerical results are insensitive on the variation of K in this region.

As indicated in the previous chapter, under iteration the surface spins asymptotically decouple from the rest of the system and the surface correlations have one-dimensional character. For an illustration we have calculated the average value of the first diagonal coupling $[J^{(d)}(m)]_{\mathrm{av}}$ connecting the first and second line of spins, as a function of the iteration m. It is given in Fig. 2 together with the correlation

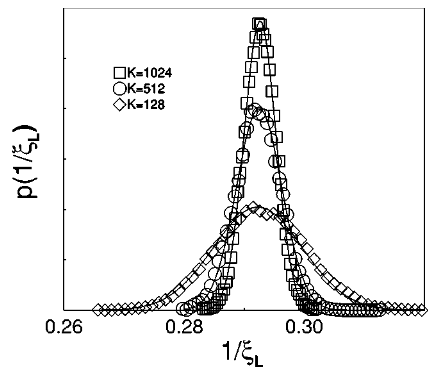

FIG. 3. Probability distribution of the inverse of the correlation length for $L=11$ and for different lengths of the strip K. We studied 20 000 samples for each lengths at a dilution $\rho=4$. The full lines represent the Gaussian approximations to the distributions.

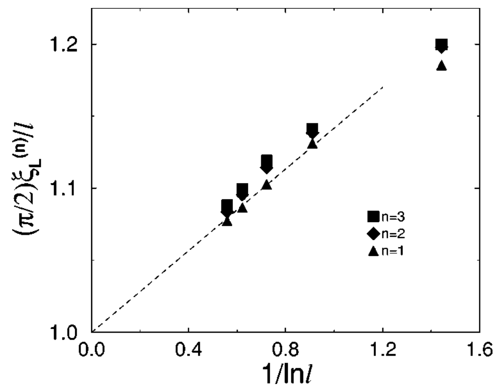

FIG. 5. Ratio of the correlation length corresponding to the nth moment of the correlation function $\xi_L^{(n)}$ and the width of the strip in units of $l=(L-1)/2$ at a dilution $\rho=4$. The dashed line is guide to the eyes, representing logarithmic finite-size corrections as $O(1/\ln l)$.

length $\xi_L(m)$ as defined in Eq. (2.2) between two spins of maximal distance $r=K/2$. As seen in Fig. 2 both $[J^{(d)}(m)]_{\rm av}$ and $\xi_L(m)$ approach their limiting values rapidly, exponentially with m. Analyzing the iteration equations in paper I one can show that $m\sim L^2$ iteration steps are needed to reach the asymptotic region, which is indeed verified numerically.

For a given dilution, ρ, the correlation length, ξ_L, shows variation from sample to sample. The distribution of the inverse correlation length $p(1/\xi_L)$ obtained over 20 000 samples is shown in Fig. 3 for different lengths of the strip K. As seen in Fig. 3 the average value of ξ_L, defining the *typical* correlation length in Eq. (2.3) is independent of K, whereas the width of the distribution is decreasing with the length of the strip as $1/\sqrt{K}$. This observation is in agreement with the cumulant expansion in Eq. (2.4) and with the fact that $\xi_L^{(n)}$ is asymptotically independent of K. The distribution $p(1/\xi_L)$ is found approximately Gaussian, however for finite strips there is always some deviation from the normal distribution.

Next, we are going to study the L dependence of the typical correlation length $\xi_L^{\rm typ}$ for different values of the dilution $\rho=1,2,4$, and 10. First we note that in order to use the same lattice units in the vertical and horizontal directions one should replace $L=2l+1$ by l. In Fig. 4 the ratio $(\pi/2)\xi_L^{\rm typ}/l$ is plotted against $1/l$. As seen in Fig. 4 in the range of dilution we worked, the correlation length is monotonically decreasing with ρ, whereas there is an approximate linear $1/l$ correction to the ratio for all values of ρ. The asymptotic value of the ratio is found dilution independent, we estimated as

$$\lim_{L\to\infty}\frac{\pi}{2}\frac{\xi_L^{\rm typ}}{l}=1\pm 0.003. \tag{3.1}$$

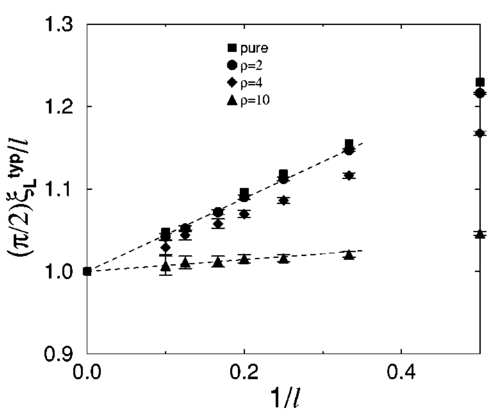

FIG. 4. Ratio of the typical correlation length $\xi_L^{\rm typ}$ and the width of the strip, measured in units $l=(L-1)/2$ for different dilutions. The dashed lines are guide to the eyes, representing the finite-size corrections as $O(1/l)$.

We have also studied the different moments of the correlation function and calculated the corresponding average correlation length $\xi_L^{(n)}$. The ratio $(\pi/2)\xi_L^{(n)}/l$ is found to have a strong size dependence, much stronger than for the typical correlation length. In this case the finite-size corrections are approximately logarithmic, as can be seen in Fig. 5. For any finite L the corrections are increasing with the moment, n, however the asymptotic value of the ratio tends to the same universal value

$$\lim_{L\to\infty}\frac{\pi}{2}\frac{\xi_L^{(n)}}{l}=1\pm 0.03. \tag{3.2}$$

as for the typical ratio in Eq. (3.1). We note that the error of the estimate in Eq. (3.2) is increasing with n.

At this point we compare our results in Eqs. (3.1) and (3.2) to that obtained in the semi-infinite geometry in Eq. (1.3) and comment about the validity of the correlation length-exponent relation in Eq. (2.5). The universal ratio found in Eqs. (3.1) and (3.2) according to the correlation length-exponent relation corresponds to a typical and average surface magnetization decay exponent of $\eta_\parallel=1$ in complete agreement with the result in the semi-infinite geometry in Eq. (1.3). Furthermore, for the average correlations we observed logarithmic corrections in both geometries. Thus we can conclude that the correlation length-exponent relation is valid for the typical and average surface correlations of the random 2D Ising model, thus they transform covariantly under conformal transformations.

IV. DISCUSSION

In the previous section the surface correlation function of the 2D random Ising model is studied in the strip geometry. In particular we have calculated the ratio of the correlation length $\xi_L^{(n)}$ obtained from the average of the nth moment of the surface correlation function and the width L of the system. We found that asymptotically this ratio goes to a universal value, irrespective of the degree of dilution ρ and the value of the moment, n. For typical correlations the correction terms are of $O(1/L)$, whereas for the average, $n=1$, and for the higher moments $n>1$. The finite-size corrections are logarithmic, they are in the form of $O(1/\ln L)$.

Several qualitative features of the above results can be understood by analyzing the ST iteration procedure. As mentioned before the surface spins are asymptotically decoupled after $m \sim L^2$ iteration steps, when the expression of the new surface couplings $J_k^{(s)}(m)$ contains a set of the original couplings $J_{i,j}$, taken from a region of $1 \leqslant i \leqslant L$ and $k-L<j<k+L$. Consequently the $J_k^{(s)}(m)$ are correlated for short distances, but they are practically independent between two sites which are separated by at least a distance of $O(L)$. Now we can use an approximate coarse-grained description: introduce block-spin variables to replace each $O(L)$ number of surface spins which are connected by correlated couplings. The new couplings between the block spins $J_{k'}^{(B)}$ are approximately independent random variables and they satisfy $\tanh(J_{k'}^{(B)}/k_BT)=O(1)$, since with the original couplings $\tanh(J_k^{(s)}/k_BT)=O(1/L)$. Then from Eq. (2.2) follows that the correlation length of the system is self-averaging, in any sample the expression in Eq. (2.2) in the thermodynamic limit goes to the average correlation length $[\xi_L]_{\rm av}$, with probability 1. Evaluating the variance of $1/\xi_L$ in the coarse-grained picture one gets

$$\left[\left(\frac{1}{\xi_L}-\frac{1}{\xi_L^{\rm typ}}\right)^2\right]_{\rm av} \simeq \frac{\beta}{rL}, \tag{4.1}$$

thus the first correction term in the cumulant expansion in Eq. (2.4) is of $O(1/L)$, as it should be. The eventual correlations between the block-spin couplings will tend to reduce the value of β in Eq. (4.1). According to our numerical studies $\beta \sim 1/\ln L$ and this effect is the source of the logarithmic corrections in the random Ising model.

The results in the semi-infinite geometry and in the strip geometry are in complete correspondence. Comparing the conformal result in Eq. (2.5) with the numerical estimates in Eqs. (3.1) and (3.2) one obtains the following conclusions.

(i) The correlation length-exponent relation is valid for the random Ising model, thus the (surface) correlations of the system are conformally covariant.

(ii) For the typical and average correlations at the critical point the decay is given by the same exponent, which does not depend on the degree of dilution. Consequently there is no multifractal behavior for the critical correlations of the model.

(iii) The typical surface correlations are free of logarithmic corrections, whereas the average correlations and the higher moments are subject of logarithmic corrections, the strength of those is increased with n.

(iv) Finally, our numerical results give strong and accurate numerical support to the field-theoretical conjecture that the random and pure Ising models in 2D belong to the same universality class.

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[30] In principle one could study much larger strips, for example in the perfect system one can easily go up to $L=1000$. For the random system, however, there occur regions in the sample with many weak (or strong) couplings and therefore the iteration procedure could become numerically instable with increasing width of the system.