Delayed stochastic systems

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(Received 1 September 1999)

Noise and time delay are two elements that are associated with many natural systems, and often they are sources of complex behaviors. Understanding of this complexity is yet to be explored, particularly when both elements are present. As a step to gain insight into such complexity for a system with both noise and delay, we investigate such delayed stochastic systems both in dynamical and probabilistic perspectives. A Langevin equation with delay and a random-walk model whose transition probability depends on a fixed time-interval past (delayed random walk model) are the subjects of in depth focus. As well as considering relations between these two types of models, we derive an approximate Fokker-Planck equation for delayed stochastic systems and compare its solution with numerical results.

PACS number(s): 02.50.−r, 02.30.Ks, 05.40.−a

I. INTRODUCTION

Many natural and artificial systems are associated with noise or fluctuation. Issues relating to noise have consequently been a major topic in a variety of fields (see, e.g., Refs. [1] and [2]). Though not as commonly recognized as noise, complex behaviors due to time delays are also found in many systems. Examples include differential equations with delay [3], delays in biophysiological controls [4], signal transmission delays in biological and artificial neural networks [5–9] and coupled oscillators [10] traffic models [11], electrical circuits [12], physics [13], and so on. This research has revealed that time delay can introduce surprisingly rich behaviors to otherwise simple systems. From the point of view of information processing, these factors have been considered obstacles; however, it was recently found or suggested that noise and delay can actually be an integral part of biological information processing [6,14,15].

Against this background, systems with both noise and delay are beginning to gain attention. They have recently been considered numerically [16,8] and analytically [17–22]. Such systems can be viewed as a special case of stochastic systems with memory, which has been studied in physics [23–28]. However, the understanding of complex behaviors associated with noise and delay together is far from complete. The main theme of this paper is to present some analytical and numerical results on such delayed stochastic systems from both dynamical and probabilistic perspectives.

The dynamical model we investigate is given by a Langevin equation with delay, or, equivalently, it can be viewed as a delay differential equation with noise. Exact and approximate analytical expressions are derived for its correlation functions. We then consider the corresponding model in the probability space by proposing delayed random-walk models [18]. They are random walks whose transition probability depends on the walker's position at a fixed time-interval (delay) past. The behavior of the correlation function is investigated both analytically and numerically. The correspondences between the two descriptions of delayed stochastic systems is also illustrated by deriving an approximate Fokker-Planck equation.

II. MODELS

We present here in general forms the types of models investigated in the following sections.

A. Delayed Langevin equation

The dynamical stochastic equation with delay we study here is generally given by the following forms:

$$\frac{d}{dt}X_t = -\mu(X_{t-\tau}) + \xi_t, \quad \langle \xi_{t_1}\xi_{t_2}\rangle = \delta(t_1 - t_2). \tag{1}$$

This equation can be viewed as an extension of either Langevin equation with delay, or delay differential equation with noise. We refer to this equation hereafter as ''delayed Langevin equation.'' It is an extension of the normal Langevin equation with delay and a function μ. We place the following conditions on μ:

$$\mu(s) > 0 \ \ (s > 0), \quad \mu(-s) = -\mu(s) \ \ (\forall s). \tag{2}$$

These restrictions make the dynamics symmetric with respect to the origin $X = 0$, which is stochastically attractive when $\tau = 0$. Noise ξ_t is the time-uncorrelated Gaussian noise. It should be noted that the equation is normalized by the ''width'' of the noise ξ_t.

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B. Delayed random walk

We consider a random walk that takes a unit step in a unit time. The delayed random walk we start with is an extension of a position dependent random walk whose step toward the origin is more likely when no delay exists. Formally, it has the following definition:

$$\begin{aligned} P(X_{t+1}=n;X_{t+1-\tau}=s) &= g(s-1)P(X_t=n-1;X_{t+1-\tau}=s;X_{t-\tau}=s-1) \\ &\quad + g(s+1)P(X_t=n-1;X_{t+1-\tau}=s;X_{t-\tau}=s+1) \\ &\quad + f(s-1)P(X_t=n+1;X_{t+1-\tau}=s;X_{t-\tau}=s-1) \\ &\quad + f(s+1)P(X_t=n+1;X_{t+1-\tau}=s;X_{t-\tau}=s+1), \end{aligned} \tag{3}$$

$$f(x)+g(x)=1, \tag{4}$$

where the position of the walker at time t is X_t, and $P(X_{t_1}=u_1;X_{t_2}=u_2)$ is the joint probability for the walker to be at u_1 and u_2 at time t_1 and t_2, respectively. $f(x)$ and $g(x)$ are transition probabilities to take a step in the negative and positive directions, respectively, at position x. In this paper, we further place the conditions:

$$f(x)>g(x) \quad (x>0), \qquad f(-x)=g(x) \quad (\forall x). \tag{5}$$

As in Eq. (2), these conditions make the delayed random walks symmetric with respect to the origin, which is attractive without delay ($\tau=0$).

We now proceed to obtain a few properties from this general definition. By the symmetry with respect to the origin, the average position of the walker is 0. This symmetry can be further used to inductively show in the stationary state ($t\to\infty$) that

$$P(X_{t+1}=n;X_t=n+1)=P(X_{t+1}=n+1;X_t=n). \tag{6}$$

We can show the above as follows. By the definition of the stationarity, we have

$$\begin{aligned} P(X_{t+1}=n;X_t=n+1)+P(X_{t+1}=n;X_t=n-1) &= P(X_{t+1}=n+1;X_t=n) \\ &\quad +P(X_{t+1}=n-1;X_t=n). \end{aligned} \tag{7}$$

For $n=0$, we note that due to the symmetry, we have

$$P(X_{t+1}=0;X_t=1)=P(X_{t+1}=1;X_t=0). \tag{8}$$

Using these two equations inductively leads us to the desired relation (6). We will derive the stationary probability distribution using this property.

Also, the multiplication of Eq. (3) for the stationary state by $\cos(\alpha n)$ and summation over n and s yields for the generating function:

$$\langle\cos(\alpha X_t)\rangle=\cos(\alpha)\langle\cos(\alpha X_t)\rangle + \sin(\alpha)\langle\sin(\alpha X_t)\{f(X_{t-\tau})-g(X_{t-\tau})\}\rangle. \tag{9}$$

In particular, we have a following invariant relationship with respect to the delay.

$$\frac{1}{2}=\langle X_t\{f(X_{t-\tau})-g(X_{t-\tau})\}\rangle. \tag{10}$$

This invariant property is used later as well.

III. ANALYSIS

In this section, we will investigate the nature of the delayed stochastic systems through two complete examples: a threshold and a linear model.

A. Threshold model

The first model, called a threshold model, is described by the following delayed Langevin equation:

$$\frac{d}{dt}X_t=-\alpha\theta(X_{t-\tau})+\xi_t, \quad \langle\xi_{t_1}\xi_{t_2}\rangle=\delta(t_1-t_2), \tag{11}$$

where θ is a step function,

$$\theta(s)=1 \quad (s>0), \qquad \theta(s)=0 \quad (s=0), \qquad \theta(s)=-1 \quad (s<0). \tag{12}$$

The corresponding delayed random walk is given by defining $f(x)$ and $g(x)$ as

$$\begin{aligned} f(x)&=\frac{1}{2}[1+\eta\theta(x)], \\ g(x)&=\frac{1}{2}[1-\eta\theta(x)], \end{aligned} \tag{13}$$

where η is a constant and θ is the same step function as in Eq. (12). Though the appearance of the definition is different, this is the same delayed random walk studied previously [18]. Here, we will add on some details. Analysis on stationary state with tedious consideration of different cases gives the following exact results.

(1) The stationary probability distribution P_X^τ of X with delay τ is given as follows: For $\tau=0$,

$$\begin{aligned} P_0^0&=\frac{\eta}{1+\eta}, \\ P_X^0&=\frac{\eta}{(1+\eta)(1-\eta)}\left\{\frac{1-\eta}{1+\eta}\right\}^X \quad (1\leqslant X). \end{aligned} \tag{14}$$

For $\tau=1$,

$$\begin{aligned} P_0^1&=\frac{\eta}{1+2\eta}, \\ P_1^1&=\frac{\eta(2+\eta)}{2(1+2\eta)(1+\eta)}, \\ P_X^1&=\frac{\eta}{(1+2\eta)(1-\eta)^2}\left\{\frac{1-\eta}{1+\eta}\right\}^X \quad (2\leqslant X). \end{aligned} \tag{15}$$

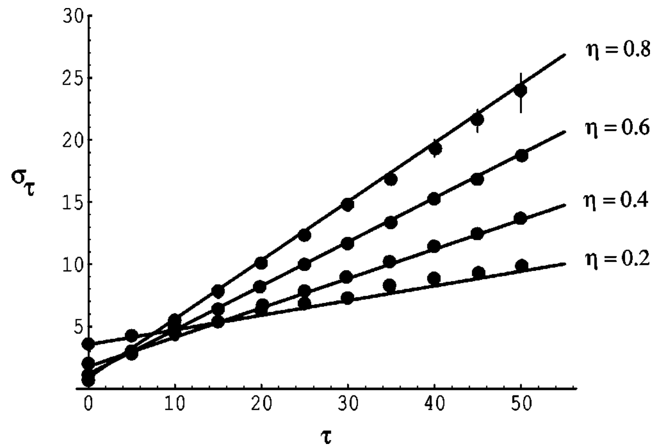

FIG. 1. $\sqrt{\langle X_\tau^2\rangle}$ as a function of τ for various η. (The error bars are the root mean square error of data points.)

For $\tau=2$,

$$P_0^2=\frac{\eta(2-\eta^2)}{2+5\eta-\eta^3},$$

$$P_1^2=\frac{\eta(4+2\eta-\eta^2-\eta^3)}{2(1+\eta)(2+5\eta-\eta^3)},$$

$$P_2^2=\frac{\eta(4+\eta+\eta^2)}{2(1+\eta)^2(2+5\eta-\eta^3)},\tag{16}$$

$$P_X^1=\frac{\eta(2-\eta+\eta^2)}{(1-\eta)^3(2+5\eta-\eta^3)}\left\{\frac{1-\eta}{1+\eta}\right\}^X \quad (3\leqslant X).$$

(2) The stationary state mean square position $\langle X_\tau^2\rangle$ can be calculated using the above probability distributions.

For $\tau=0$,

$$\langle X_0^2\rangle=\frac{1}{2\eta^2}.\tag{17}$$

For $\tau=1$,

$$\langle X_1^2\rangle=\frac{1}{2\eta^2}\,\frac{1+2\eta+2\eta^2+2\eta^3}{1+2\eta}.\tag{18}$$

For $\tau=2$,

$$\langle X_2^2\rangle=\frac{1}{2\eta^2}\,\frac{2+5\eta+8\eta^2+12\eta^3-2\eta^5}{2+5\eta-2\eta^3}.\tag{19}$$

An extension of the above analysis for $3\leqslant\tau$ is quite intricate mainly due to the non-linearity associated with the step function both in the delayed Langevin equation and the delayed random walk, and it is yet to be explored. We have found with the numerical simulation, however, that the stationary root mean square position $\sqrt{\langle X_\tau^2\rangle}$ is approximately a linear function of τ (Fig. 1). The seminumerical investigation yielded the following relationship [18]:

$$\sqrt{\langle X_\tau^2\rangle}=0.59\,\eta\tau+\frac{1}{\sqrt{2}\,\eta}.\tag{20}$$

This linear relationship can be qualitatively explained both from the delayed Langevin equation and the delayed random walk. First, let us start the discussion from the delayed Langevin equation. The formal integration of Eq. (11) leads to

$$X_t=-\alpha\int_0^t\theta(X_{s-\tau})ds+B_t,\tag{21}$$

where B_t is a Brownian motion or a Wiener process. We now rescale the equation by introducing $\rho>0$ and new variables

$$s'=\frac{s}{\rho},\quad t'=\frac{t}{\rho},\quad \epsilon=\frac{\tau}{\rho}.\tag{22}$$

Also, by defining

$$Z_{t'}\equiv\frac{1}{\rho}X_{\rho t'},\tag{23}$$

and using the property

$$\theta\left(\frac{x}{\rho}\right)=\theta(x),\tag{24}$$

we can transform Eq. (21) as follows:

$$Z_{t'}=-\alpha\int_0^{t'}\theta(Z_{s'-\epsilon})ds'+\frac{1}{\rho}B_{\rho t'}.\tag{25}$$

We note the property that if B_t is a Wiener process, so is $(1/\sqrt{\rho})B_{\rho t}$. Hence, we can deduce from Eq. (25) that

$$\frac{d}{dt}Z_t=-\alpha\theta(Z_{t-\epsilon})+\frac{1}{\sqrt{\rho}}\xi_t.\tag{26}$$

By comparing Eq. (11) and Eq. (26), we see that the system can be scaled. Now let us consider the case that delay is τ and that B_t has a variance ν^2. Then, if X has a variance $\sigma^2(\tau,\nu)=\langle X_\tau^2\rangle$, we obtain by noting Eq. (23) that

$$\sigma^2\left(\frac{\tau}{\rho},\frac{1}{\sqrt{\rho}}\right)=\frac{1}{\rho^2}\sigma^2(\tau,1),\tag{27}$$

or, by taking the square root of both sides,

$$\sigma\left(\frac{\tau}{\rho},\frac{1}{\sqrt{\rho}}\right)=\left\{\frac{1}{\sqrt{\rho}}\right\}^2\sigma(\tau,1).\tag{28}$$

This scaling relation, (with the consideration of $\tau=0$ case), can be satisfied if σ has the following form:

$$\sigma(\tau,\nu)=\frac{1}{\sqrt{2\alpha}}(\kappa\tau+\nu^2),\tag{29}$$

where κ is a constant. This equation shows that the stationary root-mean-square position has a linear dependence on τ.

Let us now approach the same issue from the delayed random-walk side. In the stationary state, we may assume that the number of steps between the two zero crossings by the walker (zero crossing interval) is a constant value h. By

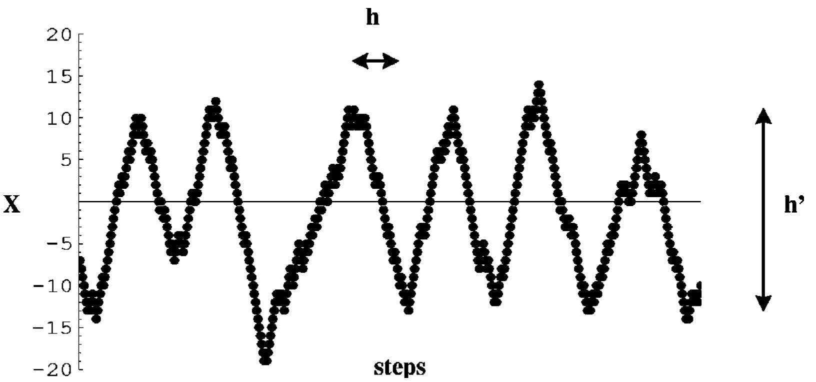

FIG. 2. A sample path by the walker of the threshold model. The data is taken with $\eta = 0.5$ and $\tau = 20$ between 4000 to 4500 steps. h and h' are the estimates explained in the text.

the symmetry of the walk about the origin, we further assume that the motion of the walker can be approximated as shown in Fig. 2. With these assumptions, the duration of time step the walker moves in the negative direction is given by h and the length of such motion can be approximated as $h' = [\frac{1}{2}(1+\eta) - \frac{1}{2}(1-\eta)]h = \eta h$. The root-mean-square position then can be approximated as follows in terms of h:

$$\sqrt{\langle X_\tau^2 \rangle} = \sqrt{\frac{2}{h} \sum_{k=1}^{h/2} \eta^2 k^2}$$

$$= \frac{\eta}{\sqrt{6}} h \sqrt{\left(\frac{1}{2} + \frac{1}{h}\right)\left(1 + \frac{1}{h}\right)}$$

$$\approx \frac{\eta}{\sqrt{6}} h \left(\frac{1}{2}\right) \approx \left(\frac{0.58}{2}\right) \eta h. \tag{30}$$

On the other hand, if we consider the relation between τ and h, we can infer that the fact that the walker takes positive moves for time step duration of h means that the walker was earlier in the negative site during τ steps for the duration of h. This leads to the following relationship:

$$\tau = \frac{h}{2} - \frac{h_0}{2}, \tag{31}$$

where h_0 is the value of h with $\tau = 0$. By putting together Eqs. (30) and (31), we obtain

$$\sqrt{\langle X_\tau^2 \rangle} \approx 0.58 \eta \tau + \sqrt{\langle X_{\tau=0}^2 \rangle}. \tag{32}$$

This result is in good agreement with our earlier seminumerical estimate given in Eq. (20). Thus, we have a slightly more quantitative argument from the delayed random-walk model for the linear relationship between the root-mean-square position and the delay τ.

Numerical simulations of the threshold model have shown that with sufficiently large τ given α or η, its correlation function oscillates both in nonstationary and stationary states (Fig. 3). Unlike the linear model in the next section, we have not found a way to capture this oscillatory behavior again, due primarily to the nonlinearity associated with the step function in the model.

In summary, on the threshold model we have some understanding of the following:

(1) Linear relationship between the delay and the root mean square position of the model.

(2) Stationary probability distribution of the delayed random walk with small τ.

However, the following questions are yet to be resolved:

(1) Stationarity of the model with respect to increasing delay is not clear. Does the model reach the stationary state for all τ?

(2) Oscillatory behavior of the correlation function has not been clarified analytically either in stationary or nonstationary regime.

(3) Probability distribution with $\tau > 3$ in the delayed random walk has not been obtained analytically.

B. Linear model

We now turn our attention to the second model that we call a linear model. It turns out that the linear models are more tractable analytically.

The linear delayed Langevin equation is given as follows by setting $\mu(s) = \gamma s$ with γ a constant.

$$\frac{d}{dt} X_t = -\gamma X_{t-\tau} + \xi_t, \quad \langle \xi_{t_1} \xi_{t_2} \rangle = \delta(t_1 - t_2). \tag{33}$$

This Langevin equation is a special case of the equation considered in Ref. 17. It has been shown that:

(1) The equation is stationary if and only if $\tau < \pi/2\gamma$.

(2) The stationary correlation function $K(r) \equiv \langle X_{t+r} X_t \rangle$ has the following form when $r < \tau$:

$$K(r) = K(0)\cos(\gamma r) - \frac{1}{2\gamma}\sin(\gamma r), \quad K(0) = \frac{1+\sin(\gamma\tau)}{2\gamma\cos(\gamma\tau)}. \tag{34}$$

In the Appendix, we present a connection of the analytical result of Eq. (34) and integral representation of the correlation function [29].

For the corresponding delayed random walk, we define $f(x)$ and $g(x)$ as

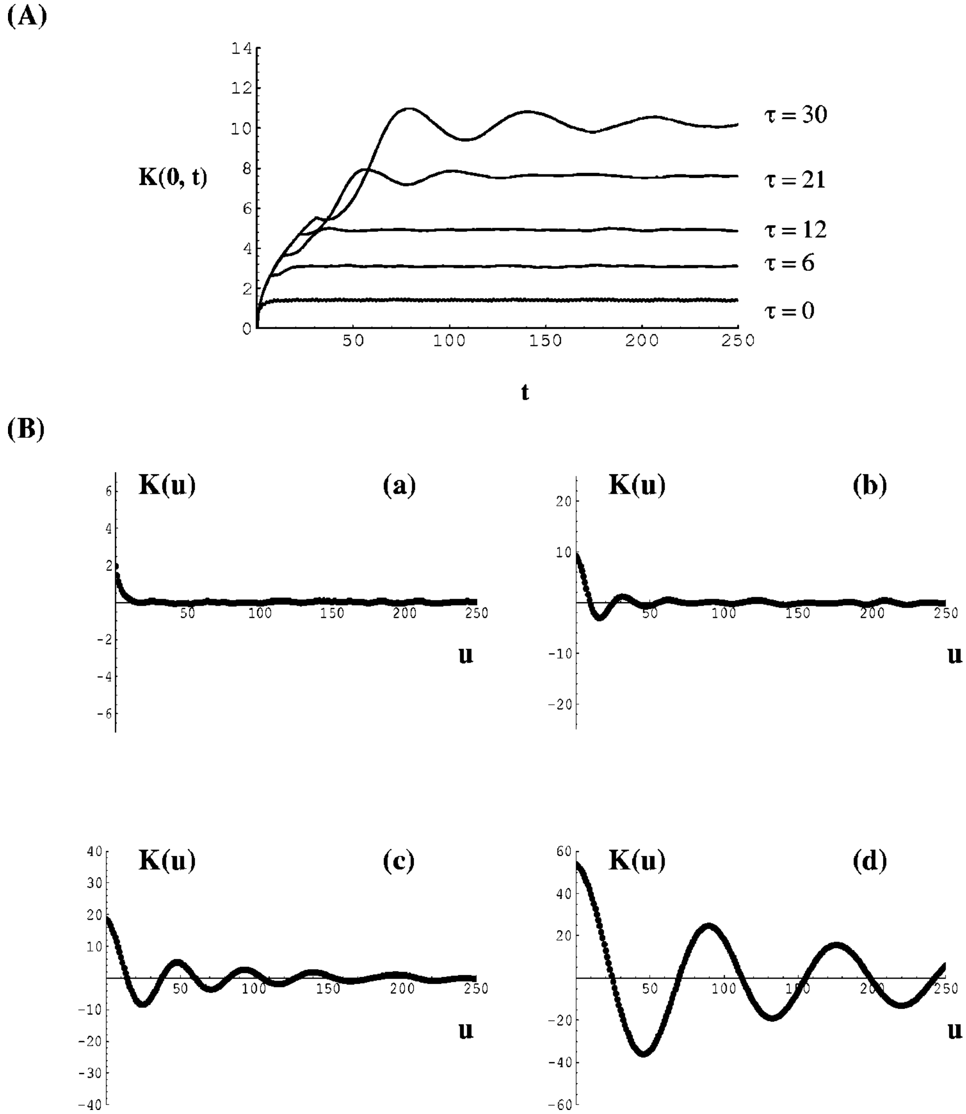

FIG. 3. (a) Dynamics of the root-mean-square position for various τ with $\eta=0.5$. (b) Samples of stationary correlation functions $K(u)=\langle X_{t+u}X_t\rangle$ for the threshold model with $\eta=0.5$ and $\tau=0$ (a), 6 (b), 10 (c), and 20 (d).

$$f(x)=\frac{1}{2}\quad(1+2d)\quad(x>a),\quad\frac{1}{2}(1+\beta x)\quad(-a\leqslant x\leqslant a),\quad\frac{1}{2}(1-2d)\quad(x<-a),$$

$$g(x)=\frac{1}{2}\quad(1-2d)\quad(x>a),\quad\frac{1}{2}(1-\beta x)\quad(-a\leqslant x\leqslant),\quad\frac{1}{2}(1+2d)\quad(x<-a).\tag{35}$$

Physically, this model implies that when $\tau=0$ the transition probability for the walker to move toward the origin increases linearly at a rate of $\beta\equiv 2d/a$ as the distance increases from the origin up to the position a after which the transition probability holds constant. We assume that with sufficiently large a, we can ignore the probability that the walker is outside of the range $(-a,a)$.

Then, the previous invariant relation in Eq. (10) becomes the following with this model:

$$\langle X_{t+\tau}X_t\rangle=K(\tau)=\frac{1}{2\beta}.\tag{36}$$

This invariance with respect to τ of the correlation function with τ steps apart is a simple characteristic of this delayed random walk model. This property is a key to obtaining the analytical expression for the correlation function, to which we now turn our attention.

For the stationary state and $0\leqslant u\leqslant\tau$, the following is obtained the definition (3):

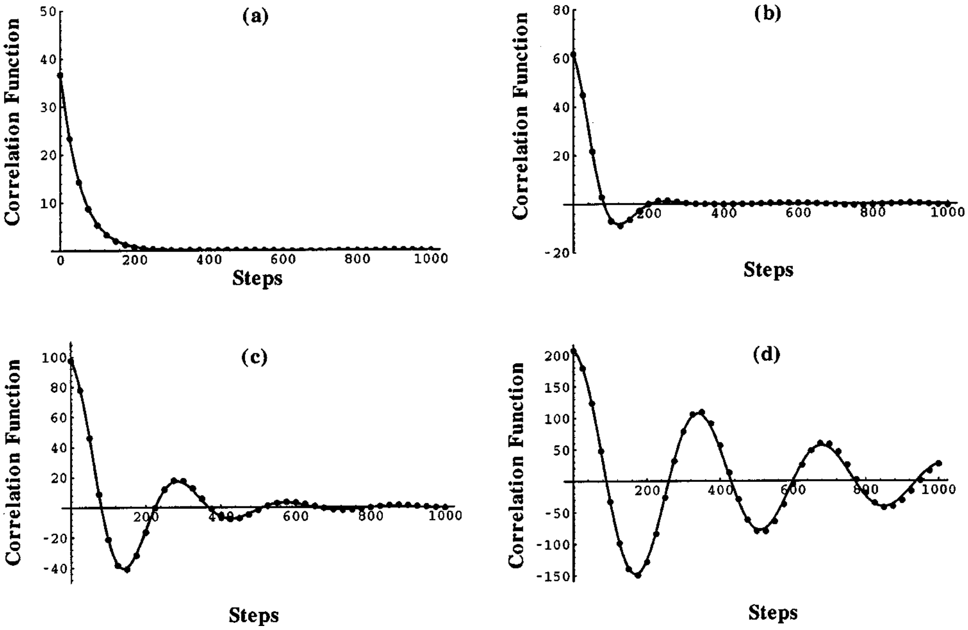

FIG. 4. Stationary correlation function $K(u)$ from simulations (dots) as a function of steps u with varying τ compared with the analytical solution obtained in the text (line). The parameters are set as $a=50$, $d=0.4$, and $\tau=10$ (a), 40 (b), 60 (c), and 80 (d). The simulation performed random walks of 6000 steps starting from the origin. The position data after 4500 steps are used to compute the correlation and averaged over 10 000 trials.

$$P(X_{t+u}=n;X_t=l)=\sum_s g(s)P(X_{t+u}=n-1;X_{t+1}=l;X_{t+u-\tau}=s)$$
$$+\sum_s f(s)P(X_{t+u}=n+1;X_{t+1}=l;X_{t+u-\tau}=s). \tag{37}$$

We can derive the following equation for the correlation function by multiplication of this equation by nl and summing over.

$$K(u)=K(u-1)-\beta K(\tau+1-u),\quad (1\leqslant u\leqslant \tau). \tag{38}$$

A similar argument can be given for $\tau<u$,

$$K(u)=K(u-1)-\beta K(u-1-\tau),\quad (\tau<u). \tag{39}$$

Equations (38) and (39) can be solved explicitly using Eq. (36). In particular, for $0\leqslant u\leqslant\tau$ we obtain

$$K(u)=K(0)\frac{(m_+^u-m_+^{u-1})-(m_-^u-m_-^{u-1})}{m_+-m_-}-\frac{1}{2}\frac{(m_+^u-m_-^u)}{m_+-m_-}$$
$$K(0)=\frac{1}{2\beta}\frac{(m_+-m_-)+\beta(m_+^\tau-m_-^\tau)}{(m_+^\tau-m_+^{\tau-1})-(m_-^\tau-m_-^{\tau-1})} \tag{40}$$
$$m_\pm=\left(1-\frac{\beta^2}{2}\right)\pm\frac{\beta}{2}\sqrt{\beta^2-4}.$$

For $\tau<u$, it is possible to write $K(u)$ in a multiple summation form, though the expression becomes rather complex. For example, with $\tau<u\leqslant 2\tau$,

$$K(u)=\frac{1}{2\beta}-\beta\sum_{i=1}^{u-\tau}K(i), \tag{41}$$

where $K(i)$s summed are given by Eq. (40).

When $\gamma\ll 1$ (or $a\gg d$), the delayed random-walk model approximately corresponds to this Langevin equation with delay by associating $\gamma\approx\beta$. In particular, we can obtain Eq. (34) from the result (40) obtained for the delayed random walk, by expanding in small β. We see that

$$\frac{(m_+^u-m_+^{u-1})-(m_-^u-m_-^{u-1})}{m_+-m_-}\approx\cos(\beta u) \tag{42}$$

$$\frac{\beta(m_+^u-m_-^u)}{m_+-m_-}\approx\sin(\beta u). \tag{43}$$

The behavior of the correlation function is shown in Fig. 4. As we increase τ, oscillatory behavior of the correlation function appears. The decay of the peak envelope is found numerically to be exponential. The decay rate of the envelope for the small u is approximately $1/[2K(0)]$. Also, the mean-square position $[K(0)]$ increases with increasing delay τ as shown in Fig. 5.

Analysis of the correlation function for the transient state can be done in a similar argument as in the stationary state. Let us first approach this issue from the delayed random-walk model. We can derive the set of coupled dynamical equations as follows:

$$K(0,t+1)=K(0,t)+1-2\beta K(\tau,t-\tau)$$

$$K(u,t+1)=K(u-1,t+1)-\beta K[\tau-(u-1),t+u-\tau],$$

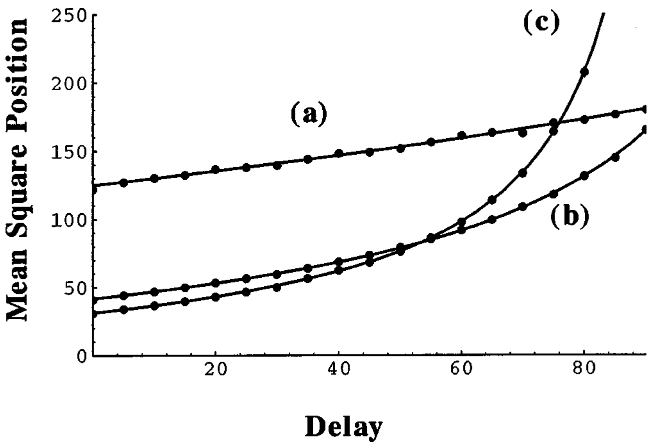

FIG. 5. The mean-square position $\langle X^2 \rangle = K(0)$ with varying delay τ. The data is from simulations (dots) averaged over 10 000 trials with 60 000 steps as in Fig. 4, and from the analytical solution obtained in the text (line). The parameters are set as $a=50$, and $d=0.1$ (a), 0.3 (b), and 0.4 (c).

$$(1 \leqslant u \leqslant \tau) \tag{44}$$

$$K(u,t+1)=K(u-1,t+1)-\beta K[(u-1)-\tau,t+1],$$

$$(u>\tau).$$

For the initial condition, we need to specify the correlation function for the interval of initial τ steps. Let us consider a random walk, which is held at the origin before the walker began to take a step, thus performing a homogeneous random walk for the steps $(1,\tau)$. This translates to the initial condition for the correlation function as

$$K(u,t)=t \quad (0 \leqslant u \leqslant \tau). \tag{45}$$

The solution can be iteratively generated for Eq. (44) given this initial condition. We have plotted some examples for the dynamics of the mean-square displacement $K(0)$ in Fig. 6. Again, the oscillatory behavior arises with increasing τ. Hence, the model discussed here shows the oscillatory behavior with increasing delay that appears in both its stationary and transient states.

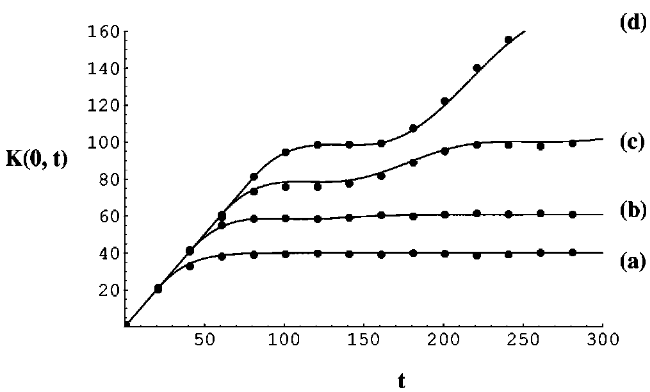

FIG. 6. Examples of dynamics of the mean square position $\langle X^2 \rangle = K(0)$ with varying delay τ. The data is from simulations (dots) averaged over 10 000 trials, and from the analytical solutions (line). The parameters are set as $a=50$, $d=0.45$, and $\tau=20$ (a), 40 (b), 60 (c), and 80 (d).

From the point of view of random walks, this delayed random-walk model provides an example whose correlation function behaves differently from commonly known random walks with memory, such as self-avoiding, or persistent walks [30]. In addition, we note that oscillatory or chaotic behavior associated with delays is generally difficult to analyze [16]. Hence, this model also serves as one of the rare analytically tractable examples among models with delay.

We will now show the corresponding dynamics of correlation functions for the delayed Langevin equation. (The detail of derivation and associated proofs will be discussed elsewhere [31].) The dynamics of the correlation function is given by the following set of coupled equations:

$$\frac{\partial}{\partial t}K(0,t)=-2\beta K(\tau,t-\tau)+1 \tag{46}$$

$$\frac{\partial}{\partial u}K(u,t)=-\beta K(\tau-u,t+u-\tau) \quad (0<u \leqslant \tau) \tag{47}$$

$$\frac{\partial}{\partial u}K(u,t)=-\beta K(u-\tau,t) \quad (\tau<u). \tag{48}$$

These sets of equations clearly correspond to the sets derived from the delayed random-walk model. We can further derive for non-negative integer m that

$$\frac{\partial}{\partial t}K(u,t)=-\beta K(u-\tau,t)-\beta K(\tau+u,t-\tau) + \sum_{i=0}^{m} \frac{(-\beta)^i}{i!}(u-i\tau)^i \quad [m\tau<u \leqslant (m+1)\tau]. \tag{49}$$

Thorough investigation of these coupled partial differential equations for the correlation function is currently underway.

In summary, on the linear model we have some understanding of the following:

(1) Conditions for the stationarity of the model.

(2) Analytical form of stationary correlation function $K(u)$ for $u<\tau$.

(3) A set of equations for the correlation function to obey both in nonstationary and stationary regime, from which one can trace its oscillatory behavior.

However, the following questions are yet to be resolved.

(1) Expressions for nonstationary and stationary probability distribution have not been obtained.

(2) Thorough investigation of a set of equation for the correlation function needs to be made.

IV. FOKKER-PLANCK EQUATION

The main theme of this section is derivation of a Fokker–Planck equation for the linear model using the correspondence between the delayed Langevin equation and the delayed random walk. (A short account of this section is given in Ref. [32].) Our strategy of derivation uses indirect derivation via expansion of the delayed random walk, rather than a direct derivation from the Langevin equation. The derived

Fokker-Planck equation contains a derivative with respect to the delay parameter. The stationary solution of the equation is also determined and is shown to have good agreement with numerical simulation.

We proceed toward a Fokker-Planck equation by expanding the linear model in the delayed random walk description using the ‘‘step operators’’ and its expansion as discussed in Ref. 2. We first rewrite the definition of the delayed random walk with delay τ as follows:

$$P_\tau(n,t+1;s,t-\tau)=g(s)P_\tau(n-1,t;s,t-\tau)+f(s)P_\tau \times(n+1,t;s,t-\tau), \quad (50)$$

where

$$P_\tau(n,t;s,t-\tau)\equiv P(X_t=n;X_{t-\tau}=s) \quad (51)$$

in the earlier definition (3). By subtracting $P(n,t;s,t-\tau)$ from both sides and using $g(s)+f(s)=1$, we obtain

$$\begin{aligned}P_\tau(n,t+1;s,t-\tau)-P_\tau(n,t;s,t-\tau)\\ =g(s)P_\tau(n-1,t;s,t-\tau)+f(s)P_\tau(n+1,t;s,t-\tau)\\ -[g(s)+f(s)]P_\tau(n,t;s,t-\tau). \quad (52)\end{aligned}$$

In order to go from this discrete space and time model to a continuous time and space expansion, we introduce the step operators, defined by the following action on an arbitrary function h:

$$\mathcal{E}_u^+h(u)=h(u+1), \quad \mathcal{E}_u^-h(u)=h(u-1). \quad (53)$$

In effect, $\mathcal{E}_u^+$ and $\mathcal{E}_u^-$ shift u by one. They can be expanded as

$$\mathcal{E}_u^\pm=1\pm\frac{\partial}{\partial u}+\frac{1}{2}\frac{\partial^2}{\partial u^2}\pm\cdots. \quad (54)$$

Using these step operators, we can rewrite the above equation as follows:

$$\begin{aligned}(\mathcal{E}_t^+\mathcal{E}_\tau^+-1)P_\tau(n,t;s,t-\tau)\\ =[(\mathcal{E}_n^--1)g(s)+(\mathcal{E}_n^+-1)f(s)]P_\tau(n,t;s,t-\tau)\end{aligned} \quad (55)$$

We note that we are using these step operators not only for space but with delay which is a parameter in the model. The implicit assumption is given that the following is justified for small parameter ϵ:

$$\begin{aligned}P_{\tau+\epsilon}(n,t;s,t-\tau)\approx P_\tau[n,t;s,t-(\tau+\epsilon)]\approx P_\tau(n,t;s,t-\tau)\\ +\epsilon\frac{\partial}{\partial\tau}P_\tau(n,t;s,t-\tau). \quad (56)\end{aligned}$$

We note that this is a rather strong assumption knowing that delay acts as a bifurcation parameter changing the behavior of the system. Thus, the above assumption can be valid, at most, in the range where the behavior of the system is ‘‘smooth’’ with respect to the delay parameter.

With the expansion of step operator to second order in n and the first order in t and τ, we obtain the Fokker-Planck equation:

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial\tau}\right)P_\tau(n,t;s,t-\tau)=\frac{\partial}{\partial n}[(\beta s)P_\tau(n,t;s,t-\tau)]\\ +\frac{1}{2}\frac{\partial^2}{\partial n^2}P_\tau(n,t;s,t-\tau).\end{aligned} \quad (57)$$

We note that this equation is for the joint probability between two points with time specifically τ apart. Also, it should be noted that the delay parameter τ appears in a derivative term in a dimensionally correct way, which, however, is not apparent from the form of the Langevin equation given in Eq. (33).

Let us investigate the stationary solution of the Fokker-Planck equation,

$$\frac{\partial}{\partial\tau}P_\tau^e(n,s,\tau)=\frac{\partial}{\partial n}[(\beta s)P_\tau^e(n,s,\tau)]+\frac{1}{2}\frac{\partial^2}{\partial n^2}P_\tau^e(n,s,\tau), \quad (58)$$

with a boundary condition,

$$P_\tau^e(n,s,\tau=0)=\sqrt{\frac{\beta}{\pi}}e^{-\beta n^2}\delta(n-s). \quad (59)$$

This boundary condition is chosen to satisfy the requirement that with $\tau=0$, the stationary solution should be identical to that for the Ornstein-Uhlenbeck process.

The solution can be found using Fourier transforms, which involves only Gaussian integrations for this equation. We first transform Eq. (58) as follows [33]:

$$\frac{\partial}{\partial\tau}\mathcal{P}^e(k,s,\tau)=\left(ik\beta s-\frac{1}{2}k^2\right)\mathcal{P}^e(k,s,\tau), \quad (60)$$

where

$$\mathcal{P}^e(k,s,\tau)=\int_{-\infty}^{+\infty}P_\tau^e(n,s,\tau)e^{-ikn}dn. \quad (61)$$

Solving this transformed equation using the boundary condition leads to

$$\begin{aligned}\mathcal{P}^e(k,s,\tau)=\mathcal{P}^e(k,s,\tau=0)e^{(ik\beta s-(1/2)k^2)\tau},\\ \mathcal{P}^e(k,s,\tau=0)=e^{-(1/4\beta)k^2}.\end{aligned} \quad (62)$$

The desired result is given by the inverse transform as follows:

$$\begin{aligned}P^e(n,s,\tau)=\left(\sqrt{\frac{\beta}{\pi}}e^{-(1/2)\beta^2s^2\tau-\beta ns}\right)\\ \times\left(\sqrt{\frac{1}{2\pi\tau}}e^{-[(n-s)^2/2\tau]}\right). \quad (63)\end{aligned}$$

We note that the second factor approaches the delta function with τ approaching 0 consistent with the boundary condition. A sample plot of this solution is shown in Fig. 7. To verify

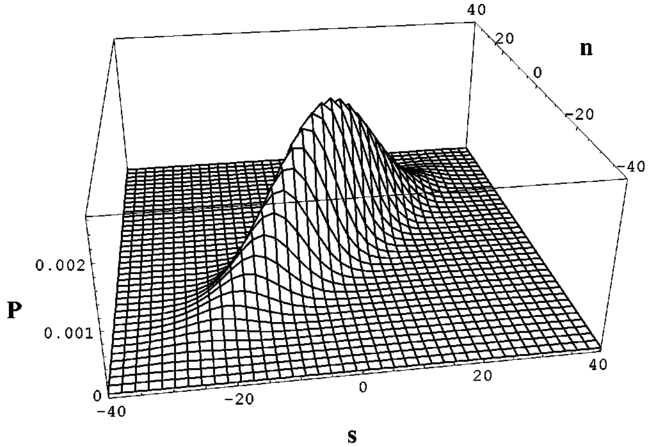

FIG. 7. A plot of the stationary joint probability distribution function. The parameters are set as $\beta = d/a = 0.004$ and $\tau = 30$.

our analytical derivation, we have compared the solution (63) with numerical simulation with various τ. Samples are presented in Figs. 8 and 9, which show good agreement.

We stress again that the derived Fokker-Planck equation is just for the joint probability between two points with time specifically τ apart. Derivation of a general equation as well as investigation of the behavior of its nonstationary solution are left for the future. Nevertheless, our systematic derivation here has illustrated a correspondence between the delayed Langevin equations and delayed random walks, and provided another tool in the form of the Fokker-Planck equation to study delayed stochastic systems. It is expected to be useful, particularly with respect to gaining an understanding of their probability density function. It is also of interest to compare our results from this indirect path with a more direct derivation from the Langevin Eq. (33). Some research based on expansion with delay parameters has been done [22]. How these two lines of derivation relate to each other requires further investigation.

V. DISCUSSION

We have presented a study of delayed stochastic systems from both dynamical equation and random walk perspectives. In particular, we have tried to illustrate the connections between the two descriptions.

Some points of discussion are now in order. The first point is how this model is placed in relation to other models with noise and delay (or memory, to be more general). We particularly note that the Langevin equation discussed here is not a special case of the generalized Langevin equation which is consistent with the fluctuation-dissipation theorem. As argued in Ref. 23 for the generalized Langevin equation, the noise term needs to be ''colored'' in Eq. (33) for consistency. Investigation of the colored noise case in its relation to delayed random walks as well as further studies of the correspondence of dynamical aspects of Eqs. (3) and (33) are currently underway.

We also note that we can find an application of the delayed stochastic models discussed here to the study of such a system as human posture control [34,18]. A resonance behavior between noise and delay by simpler stochastic elements has been studied as well [20]. However, the task of

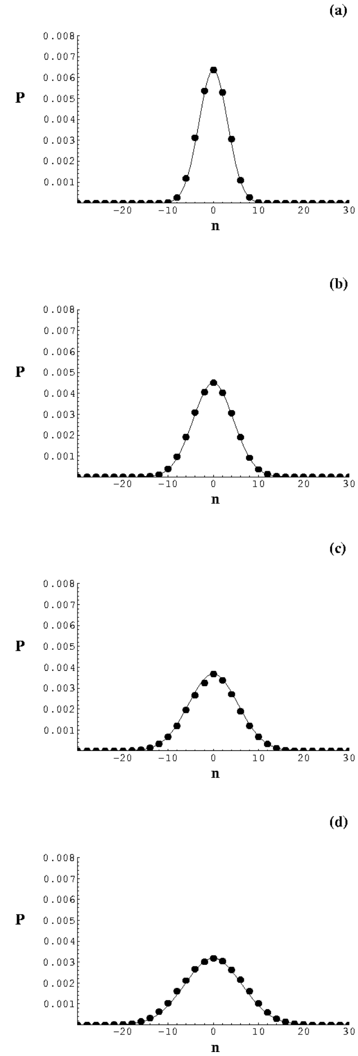

FIG. 8. Stationary probability distribution from simulations (dots) compared with the analytical solution obtained in the text (line). The parameters are set as $s = 0$, $\beta = d/a = 0.008$ and $\tau = 10$ (a), 20 (b), 30 (c), and 40 (d). The simulation performed delayed random walks of Eq. (2), for 5000 steps starting from the origin for 1000 trials. The position data after 4000 steps were used to compute the stationary joint probability.

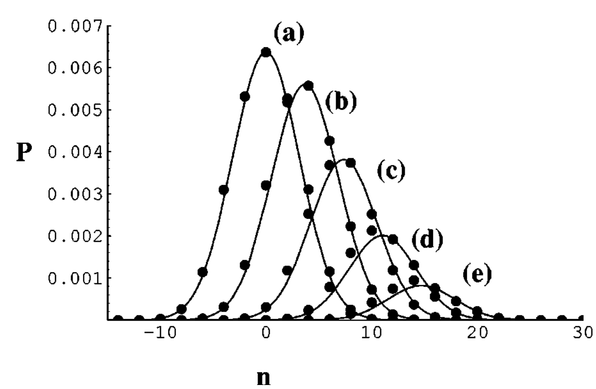

FIG. 9. Comparison of the stationary probability distribution as in Fig. 2 with different parameter settings. $\beta = 0.008$, $\tau = 10$, $s = 0$ (a), 4 (b), 8 (c), 12 (d), and 16 (e).

understanding delayed stochastic systems with many elements involved remains. Such efforts are at the same time looking at a much wider range of important applications such as neural networks, immune systems, economics, the Internet, and so on. Though limited to simple concrete models, it is hoped that our investigation here from probabilistic and dynamical perspectives will be of help toward deeper understanding of delayed stochastic systems.

APPENDIX: CORRELATION FUNCTION OF LANGEVIN EQUATION WITH DELAY

In this section, we connect the two approaches toward the correlation function for the Langevin equation with delay. The first approach is due to Ref. [17], in which the expression given in Eq. (34) is derived. The second approach is along the line of Ref. 23 using Fourier transform. With this approach, $K(u)$ is given as follows in the integral forms:

$$\begin{aligned} K(u) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\omega u}}{\omega^2+\beta^2-2\beta\omega\sin\omega\tau}d\omega \\ &= \frac{1}{\pi}\int_{0}^{+\infty} \frac{\cos\omega u}{\omega^2+\beta^2-2\beta\omega\sin\omega\tau}d\omega \\ &= \frac{1}{2\pi\beta}\int_{-\infty}^{+\infty} \frac{e^{i\beta\gamma u}}{(i\gamma+e^{-i\beta\gamma\tau})(-i\gamma+e^{i\beta\gamma\tau})}d\gamma. \end{aligned} \tag{A1}$$

Thus, for $u \leqslant \tau$, these two approaches indirectly established the identity between the Eqs. (34) and (A1). Direct verification of the identity is hindered by the fact that the poles of the integrand in Eq. (A1) are solutions of a transcendental equation. In Fig. 10, we have shown an example of numerical integration of the integral and Eq. (34) for $K(0)$ [35].

Finally, we note that when we have a colored noise of the following type instead of the white noise in Eq. (33),

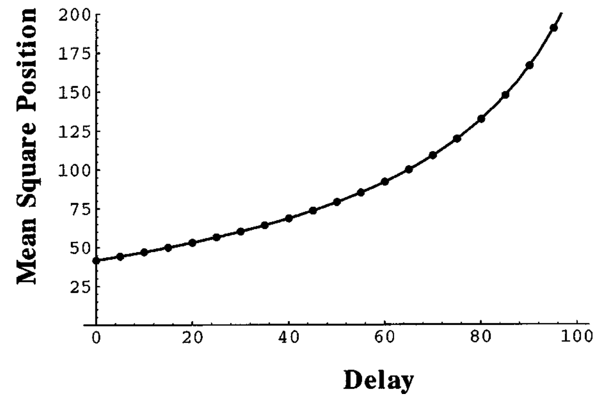

FIG. 10. Example of comparison of the mean-square position $\langle X^2\rangle = K(0)$ from the analytical solution (34) (line) and from the numerical integration of Eq. (A1) (dots). The parameters are set as $a=50$, $d=0.3$.

$$\langle \xi_{t_1}\xi_{t_2}\rangle = \delta[(t_1-\tau)-t_2]. \tag{A2}$$

we have the following integral form for the correlation function $C(u)$:

$$C(u) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\omega(u+\tau)}}{\omega^2+\beta^2-2\beta\omega\sin\omega\tau}d\omega. \tag{A3}$$

We note that from the comparison of integral forms, we have a relationship between the correlation function for the white and the colored noise case. In particular, when $u=0$,

$$C(0) = K(\tau) = \frac{1}{2\beta}. \tag{A4}$$

In the colored noise case considered here, the stationary mean square position is constant even with changing delay. This is a natural consequence of the fact that our choice of the colored noise is based on the consistency condition with the fluctuation-dissipation theorem discussed in Ref. 23.

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