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High-frequency dynamics of wave localization

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We study the effect of localization on the propagation of a pulse through a multimode disordered waveguide. The correlator $\langle u(\omega_1)u^*(\omega_2)\rangle$ of the transmitted wave amplitude u at two frequencies differing by $\delta\omega$ has for large $\delta\omega$ the stretched exponential tail $\propto\exp(-\sqrt{\tau_D\delta\omega/2})$. The time constant $\tau_D=L^2/D$ is given by the diffusion coefficient D, even if the length L of the waveguide is much greater than the localization length ξ. Localization has the effect of multiplying the correlator by a frequency-independent factor $\exp(-L/2\xi)$, which disappears upon breaking time-reversal symmetry. [S1063-651X(99)50412-1]

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The frequency spectrum of waves propagating through a random medium contains dynamical information of interest in optics [1], acoustics [2], and seismology [3]. A fundamental issue is how the phenomenon of wave localization [4] affects the dynamics. The basic quantity is the correlation of the wave amplitude at two frequencies differing by $\delta\omega$. A recent microwave experiment by Genack *et al.* [5] measured this correlation for a pulse transmitted through a waveguide with randomly positioned scatterers. The waves were not localized in that experiment, because the length L of the waveguide was less than the localization length ξ, so the correlator could be computed from the perturbation theory for diffusive dynamics [6]. The characteristic time scale in that regime is the time $\tau_D=L^2/D$ it takes to diffuse (with diffusion coefficient D) from one end of the waveguide to the other. According to diffusion theory, for large $\delta\omega$ the correlator decays $\propto\exp(-\sqrt{\tau_D\delta\omega/2})$ with time constant τ_D.

What happens to the high-frequency decay of the correlator if the waveguide becomes longer than the localization length? That is the question addressed in this Rapid Communication. Our prediction is that, although the correlator is suppressed by a factor $\exp(-L/2\xi)$, the time scale for the decay remains the diffusion time τ_D, even if diffusion is only possible on length scales $\ll L$. The exponential suppression factor disappears if time-reversal symmetry is broken (by some magneto-optical effect). Our analytical results are based on the formal equivalence between a frequency shift and an imaginary absorption rate, and are supported by a numerical solution of the wave equation.

We consider the propagation of a pulse through a disordered waveguide of length L. In the frequency domain the transmission coefficient $t_{nm}(\omega)$ gives the ratio of the transmitted amplitude in mode n to the incident amplitude in mode m. (The modes are normalized to carry the same flux.) We seek the correlator $C(\delta\omega)=\langle t_{nm}(\omega+\delta\omega)t^*_{nm}(\omega)\rangle$. (The brackets $\langle\cdots\rangle$ denote an average over the disorder.) We assume that the (positive) frequency increment $\delta\omega$ is sufficiently small compared to ω that the mean free path l and the number of modes N in the waveguide do not vary appreciably, and may be evaluated at the mean frequency ω [7]. We also assume that $l\gg c/\omega$ (with c the wave velocity). The localization length is then given by [8] $\xi=(\beta N+2-\beta)l$, with $\beta=1$ (2) in the presence (absence) of time-reversal symmetry. For $N\gg1$ the localization length is much greater than the mean free path, so that the motion on length scales below ξ is diffusive (with diffusion coefficient D).

Our approach is to map the dynamic problem without absorption onto a static problem with absorption [9]. The mapping is based on the analyticity of the transmission amplitude $t_{nm}(\omega+iy)$, at complex frequency $\omega+iy$ with $y>0$, and on the symmetry relation $t_{nm}(\omega+iy)=t^*_{nm}(-\omega+iy)$. The product of transmission amplitudes $t_{nm}(\omega+z)t_{nm}(-\omega+z)$ is therefore an analytic function of z in the upper half of the complex plane. If we take z real, equal to $\frac{1}{2}\delta\omega$, we obtain the product of transmission amplitudes $t_{nm}(\omega+\frac{1}{2}\delta\omega)t^*_{nm}(\omega-\frac{1}{2}\delta\omega)$ considered above [the difference with $t_{nm}(\omega+\delta\omega)t^*_{nm}(\omega)$ being statistically irrelevant for $\delta\omega\ll\omega$]. If we take z imaginary, equal to $i/2\tau_a$, we obtain the transmission probability $T=|t_{nm}(\omega+i/2\tau_a)|^2$ at frequency ω and absorption rate $1/\tau_a$. We conclude that the correlator C can be obtained from the ensemble average of T by analytic continuation to imaginary absorption rate,

$$C(\delta\omega)=\langle T\rangle \quad \text{for} \quad 1/\tau_a\rightarrow -i\,\delta\omega. \tag{1}$$

Two remarks on this mapping: (i). The effect of absorption (with rate $1/\tau^*$) on $C(\delta\omega)$ can be included by the substitution $1/\tau_a\rightarrow -i\,\delta\omega+1/\tau^*$. This is of importance for comparison with experiments, but here we will for simplicity ignore this effect. (ii). Higher moments of the product $\mathcal{C}=t_{nm}(\omega+\frac{1}{2}\delta\omega)t^*_{nm}(\omega-\frac{1}{2}\delta\omega)$ are related to higher moments of T by $\langle\mathcal{C}^p\rangle=\langle T^p\rangle$ for $1/\tau_a\rightarrow -i\,\delta\omega$. This is not sufficient to determine the entire probability distribution $P(\mathcal{C})$, because moments of the form $\langle\mathcal{C}^p\mathcal{C}^{*q}\rangle$ cannot be obtained by analytic continuation [10].

To check the validity of this approach and to demonstrate how effective it is we consider briefly the case $N=1$. A disordered single-mode waveguide is equivalent to a geometry of parallel layers with random variations in composition and thickness. Such a randomly stratified medium is studied in seismology as a model for the subsurface of the Earth [3]. The correlator of the reflection amplitudes $K(\delta\omega) = \langle r(\omega+\delta\omega)r^*(\omega)\rangle$ has been computed in that context by White *et al.* [11] (in the limit $L\to\infty$). Their result was

$$K(\delta\omega)=(2l/c)\delta\omega\int_0^\infty dx\,\exp[-x(2l/c)\delta\omega]\frac{x}{x-i}. \quad (2)$$

The distribution of the reflection probability $R=|r|^2$ through an absorbing single-mode waveguide had been studied many years earlier as a problem in radio-engineering [12], with the result

$$\langle R\rangle=(l/c\tau_a)\int_1^\infty dz\,\exp[-(z-1)(l/c\tau_a)]\frac{z-1}{z+1}. \quad (3)$$

One readily verifies that Eqs. (2) and (3) are identical under the substitution of $1/\tau_a$ by $-i\delta\omega$.

In a similar way one can obtain the correlator of the transmission amplitudes by analytic continuation to imaginary absorption rate of the mean transmission probability through an absorbing waveguide. The absorbing problem for $N=1$ was solved by Freilikher, Pustilnik, and Yurkevich [13]. That solution will not be considered further here, since our interest is in the multi-mode regime, relevant for the microwave experiments [5]. The transmission probability in an absorbing waveguide with $N\gg 1$ is given by [14]

$$\langle T\rangle=\frac{l}{N\xi_a\sinh(L/\xi_a)}\exp\left(-\delta_{\beta,1}\frac{L}{2Nl}\right), \quad (4)$$

for absorption lengths $\xi_a=\sqrt{D\tau_a}$ in the range $l\ll\xi_a\ll\xi$. The length L of the waveguide should be $\gg l$, but the relative magnitude of L and ξ is arbitrary. Substitution of $1/\tau_a$ by $-i\delta\omega$ gives the correlator

$$C(\delta\omega)=\frac{l\sqrt{-i\tau_D\delta\omega}}{NL\sinh\sqrt{-i\tau_D\delta\omega}}\exp\left(-\delta_{\beta,1}\frac{L}{2Nl}\right), \quad (5)$$

where $\tau_D=L^2/D$ is the diffusion time. The range of validity of Eq. (5) is $L/\xi\ll\sqrt{\tau_D\delta\omega}\ll L/l$, or equivalently $D/\xi^2\ll\delta\omega\ll c/l$. In the diffusive regime, for $L\ll\xi$, the correlator (5) reduces to the known result [6] from perturbation theory.

For $\max(D/L^2,D/\xi^2)\ll\delta\omega\ll c/l$ the decay of the absolute value of the correlator is a stretched exponential,

$$|C|=\frac{2l}{NL}\sqrt{\tau_D\delta\omega}\exp\left(-\sqrt{\tfrac{1}{2}\tau_D\delta\omega}-\delta_{\beta,1}\frac{L}{2Nl}\right). \quad (6)$$

In the localized regime, when ξ becomes smaller than L, the onset of this tail is pushed to higher frequencies, but it retains its functional form. The weight of the tail is reduced by a factor $\exp(-L/2Nl)$ in the presence of time-reversal symmetry. There is no reduction factor if time-reversal symmetry is broken.

To test our analytical findings we have carried out numerical simulations. The disordered medium is modeled by a two-dimensional square lattice (lattice constant a, length L, width W). The (relative) dielectric constant ε fluctuates from site to site between $1\pm\delta\varepsilon$. The multiple scattering of a scalar wave Ψ (for the case $\beta=1$) is described by discretizing the Helmholtz equation $[\nabla^2+(\omega/c)^2\varepsilon]\Psi=0$ and computing the transmission matrix using the recursive Green function technique [15]. The mean free path l is determined from the average transmission probability $\langle\mathrm{Tr}\,tt^\dagger\rangle=N(1+L/l)^{-1}$ in the diffusive regime [8]. The correlator C is obtained by averaging $t_{nm}(\omega+\delta\omega)t^*_{nm}(\omega)$ over the mode indices n,m and over different realizations of the disorder. We choose $\omega^2=2(c/a)^2$, $\delta\varepsilon=0.4$, leading to $l=22.1a$. The width $W=11a$ is kept fixed (corresponding to $N=5$), while the length L is varied in the range (400–1600)a. These waveguides are well in the localized regime, L/ξ ranging from 3 to 12. A large number (some 10^4–10^5) of realizations were needed to average out the statistical fluctuations, and this restricted our simulations to a relatively small value of N. For the same reason we had to limit the range of $\delta\omega$ in the data set with the largest L.

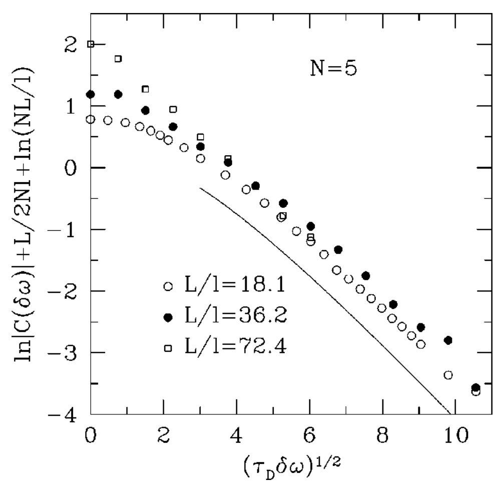

FIG. 1. Frequency dependence of the logarithm of the absolute value of the correlator $C(\delta\omega)$. The data points follow from a numerical simulation for $N=5$, the solid curve is the analytical high-frequency result (6) for $N\gg 1$ (with $\beta=1$). The decay of the correlator is given by the diffusive time constant $\tau_D=L^2/D$ even if the length L of the waveguide is greater than the localization length $\xi=6l$. The offset of about 0.6 between the numerical and analytical results is probably a finite-N effect.

Results for the absolute value of the correlator are plotted in Fig. 1 (data points) and are compared with the analytical high-frequency prediction for $N\gg 1$ (solid curve). We see from Fig. 1 that the correlators for different values of L/ξ converge for large $\delta\omega$ to a curve that lies somewhat above the theoretical prediction. The offset is about 0.6, and could be easily explained as an $\mathcal{O}(1)$ uncertainty in the exponent in Eq. (1) due to the fact that N is not $\gg 1$ in the simulation. Regardless of this offset, the simulation confirms both analytical predictions: The stretched exponential decay $\propto\exp(-\sqrt{\tau_D\delta\omega/2})$ and the exponential suppression factor $\exp(-L/2\xi)$. We emphasize that the time constant $\tau_D=L^2/D$ of the high-frequency decay is the diffusion time for *the entire length* L of the waveguide, even though the localization length ξ is up to a factor of 12 smaller than L.

We can summarize our findings by the statement that the correlator of the transmission amplitudes *factorizes* in the high-frequency regime: $C\to f_1(\delta\omega)f_2(\xi)$. The frequency dependence of f_1 depends on the diffusive time through the waveguide, even if it is longer than the localization length.

Localization has no effect on f_1, but only on f_2. We can contrast this factorization with the high-frequency asymptotics $K \rightarrow f_3(\delta\omega)$ of the correlator of the reflection amplitudes. In the corresponding absorbing problem the high-frequency regime corresponds to an absorption length smaller than the localization length, so it is obvious that K becomes independent of ξ in that regime. The factorization of C is less obvious. Since the localized regime is accessible experimentally [16], we believe that an experimental test of our prediction should be feasible.

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