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Nonequilibrium first-order phase transition induced by additive noise

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We show that a nonequilibrium first-order phase transition can be induced by additive noise. As a model system to study this phenomenon, we consider a nonlinear lattice of overdamped oscillators with both additive and multiplicative noise terms. Predictions from mean field theory are successfully confirmed by numerical simulations. A physical explanation for the mechanism of the transition is given. [S1063-651X(99)51912-0]

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Among all the counterintuitive phenomena observed in nonlinear systems with noise (such as stochastic resonance [1], noise-induced transport [2], coherence resonance [3], resonant activation [4], etc.) an important place is occupied by noise-induced transitions. Discovered in the 1980s [5] and confirmed by numerous experiments (see, for example, [6]), noise-induced transitions have attracted intensive attention due to the surprising ability of noise to produce order in the system. These transitions can be characterized by a qualitative change in the probability distribution of the system (e.g., by a change in the number of maxima). In the 1990s other kinds of transitions were found, such as those giving rise to noise-induced oscillations in single nonlinear oscillators [7,8]. On the other hand, systems of spatially coupled overdamped oscillators have been recently shown to display noise-induced *phase* transitions. In this case, contrary to the previous phenomena, the system exhibits ergodicity breaking and the transition can be characterized by standard tools in equilibrium statistical mechanics [9]. Several models have exhibited so far the existence of noise-induced second-order (continuous) phase transitions leading to the creation of a nonzero mean field [9–12]. In [13] it was shown that noise-induced phase transitions can also be of first order (discontinuous).

In the majority of the above-mentioned studies, phase transitions are induced by multiplicative noise. However, recent results [14–16] have shown that additive noise can play a crucial role in this phenomenon, and even induce a transition by itself. Such an influence has been observed both in oscillatory [14] and in nonoscillatory (overdamped) systems [15,16]. The present Rapid Communication shows that additive noise can also induce *first-order* nonequilibrium transitions in spatially extended systems. These are *pure* noise-induced phase transitions, in the sense that they do not exist in the system in the absence of noise. The study is performed on a nonlinear lattice of coupled stochastic overdamped oscillators introduced in [11] and further studied in [15,16,18,19]. It is described by the following set of Langevin equations:

$$\dot{x}_i = f(x_i) + g(x_i)\xi_i(t) + \frac{D}{2d}\sum_j (x_j - x_i) + \zeta_i(t), \tag{1}$$

where $x_i(t)$ represents the state of the ith oscillator, and the sum runs over all nearest neighbors of cell i. The strength of the coupling is measured by D, and d is the dimension of the lattice, which has $N = L^d$ elements. The noise terms $\xi_i(t)$ and $\zeta_i(t)$ are mutually uncorrelated, Gaussian distributed, with zero mean and white in both space and time,

$$\langle \xi_i(t)\xi_j(t')\rangle = \sigma_\xi^2 \delta_{i,j}\delta(t-t'), \tag{2}$$

$$\langle \zeta_i(t)\zeta_j(t')\rangle = \sigma_\zeta^2 \delta_{i,j}\delta(t-t'). \tag{3}$$

For the sake of simplicity, the functions $f(x)$ and $g(x)$ are taken to be of the form [11]

$$f(x) = -x(1+x^2)^2, \quad g(x) = a^2 + x^2, \tag{4}$$

so that two different sources of additive noise can be considered to exist in this system: the first one, controlled by σ_ζ, is completely uncorrelated with the multiplicative noise; the second one, controlled by a, is strongly correlated with it.

The behavior of this system can be analytically studied by means of a standard mean-field theory (MFT) procedure [9]. The mean-field approximation consists of replacing the nearest-neighbor interaction by a global term in the Fokker-Planck equation corresponding to Eq. (1). In this way, one obtains the following steady-state probability distribution $w_{\rm st}$:

$$w_{\rm st}(x,m) = \frac{C(m)}{\sqrt{\sigma_\xi^2 g^2(x) + \sigma_\zeta^2}} \exp\left(2\int_0^x \frac{f(y) - D(y-m)}{\sigma_\xi^2 g^2(y) + \sigma_\zeta^2}\,dy\right), \tag{5}$$

where $C(m)$ is a normalization constant and m is a mean field, defined by the equation

$$m = \int_{-\infty}^{\infty} x\, w_{\rm st}(x,m)\,dx. \tag{6}$$

By solving Eq. (6) self-consistently with respect to the variable m, one can find transitions between ordered ($m \neq 0$) and disordered ($m = 0$) phases. As shown in [11], for $a = 1$ and $\sigma_\zeta = 0$ the system exhibits a disorder-order phase transition, followed by a reentrant transition back to disorder, both induced by multiplicative noise. When a or σ_ζ are used to control the system, additive noise is seen to lead to similar transitions [15,16]. In all cases, the transition (which exists only in the presence of noise) is of second order. But when the complete system is analyzed more carefully, new aspects

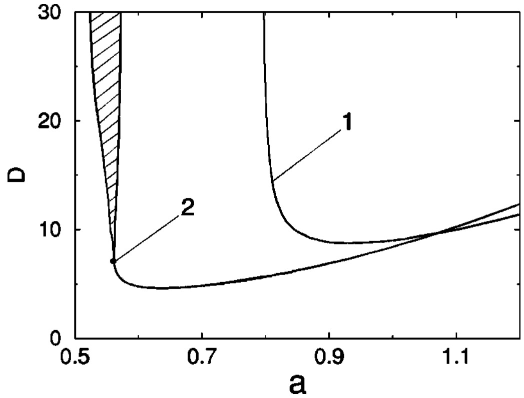

FIG. 1. Phase transition boundaries on the plane (a,D) for $\sigma_\zeta=0$ and two different intensities of the multiplicative noise (curve 1, $\sigma_\xi^2=1.6$; curve 2, $\sigma_\xi^2=3.0$). The dashed region (starting with the dot) corresponds to the coexistence of the disordered and ordered phases.

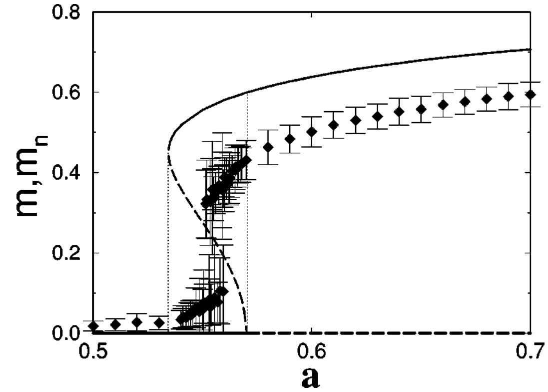

FIG. 2. First-order phase transition induced by additive noise. Order parameters m, m_n vs a for $D=20$, $\sigma_\xi^2=3.0$ and $\sigma_\zeta^2=0.0$. MFT predictions (solid line) and numerical simulations (diamonds) are presented. The dotted line delimits the coexistence region exhibited by MFT. The unstable state is plotted by the dashed line.

arise. Figure 1 shows order-disorder transition lines in the plane (a,D), for $\sigma_\zeta=0$ and two different values of the multiplicative noise intensity σ_ξ^2. Curve 1 separates regions of disorder (below the curve) and order (above the curve) for small multiplicative noise intensity. In this case, the ordered region is characterized by three self-consistent solutions of Eq. (6), one of them unstable $(m=0)$ and the other two stable and symmetrical. These new solutions appear continuously from $m=0$ in the course of the transition. Hence, curve 1 corresponds to a *second-order* phase transition from disorder to order as a increases, followed by a reentrant transition back to disorder (also of second order).

The situation changes noticeably when the multiplicative noise intensity increases. In that case (curve 2 in Fig. 1), a region appears where Eq. (6) has five roots, three of which ($m=0$ and two symmetrical points) are stable. This region is shown as dashed in the figure. Thus, for large enough values of D, a region of coexistence appears in the transition between order and disorder. This region is limited by discontinuous transition lines between $m=0$ and a nonzero, finite value of m. Hence, additive noise is seen to induce a *first-order* phase transition in this system for large enough values of the coupling strength and multiplicative noise intensity. The reentrant transition is again of second order.

When the first-order phase transition appears, hysteresis can be expected to occur in the coexistence region (if a certain algorithm is applied [17]). The dependence of the order parameter m on the control parameter a as predicted by MFT is shown in Fig. 2 by a solid line. The region of possible hysteresis is bounded by dotted lines.

In order to contrast the previous MFT results, we have performed simulations of the complete model (1)–(4) using the numerical methods described in [9,18]. The order parameter m_n is computed as

$$m_n=\left\langle\left|\frac{1}{L^2}\sum_{i=1}^{N}x_i\right|\right\rangle,$$

where $\langle\,\rangle$ denotes time average. Results for a two-dimensional lattice with lateral size $L=32$ are shown with diamonds in Fig. 2. Analyzing this figure one can observe that MFT overestimates the size of the coexistence region. This effect, analogous to what was observed for multiplicative-noise-induced transitions [11], can be explained in terms of an ‘‘effective potential’’ derived for the system at short times (see the discussion below). For instance, as a increases the system leaves the disordered phase not when this state becomes unstable but earlier, when the potential minima corresponding to the ordered states become much lower than the minimum corresponding to the state $m=0$. It should also be mentioned that the numerical simulations did not show hysteresis, because in the coexistence region the system occupied any of the three possible states, independently of the initial conditions. This fact can be explained by the small size of the simulated system, which permits jumps between steady states when the system is sufficiently perturbed (e.g., by slightly changing the parameter a).

Now we consider the second kind of additive noise present in the system, namely, the one uncorrelated with the multiplicative noise ($a=0$ and $\sigma_\zeta^2\neq 0$). MFT results are presented in the phase diagram of Fig. 3, which shows transitions lines in the plane (σ_ξ^2,D) for three different values of the additive noise intensity σ_ζ^2. A coexistence region is again found in the disorder-order transition (left) branch for all three values of σ_ζ^2. For points in the dashed region (inset plot in Fig. 3), the system is in a disordered phase for small and large values of σ_ζ^2, and in an ordered phase for intermediate values of this parameter. Hence, in that region additive noise is able to induce two consecutive phase transitions from disorder to order and back to disorder. The character of the first transition is very sensitive to the parameter values: as can be clearly seen in Fig. 4 (curves 1 and 2), for very close values of D and σ_ξ^2, additive noise can induce either a second- or a first-order phase transition. Note also that, if we consider the multiplicative noise intensity as a control parameter, the width of the coexistence region as predicted by MFT decreases with an increase of the additive noise intensity.

Numerical simulations for this kind of additive noise are shown as diamonds in Fig. 4, again for a two-dimensional lattice with $L=32$. MFT overestimates once more the loca-

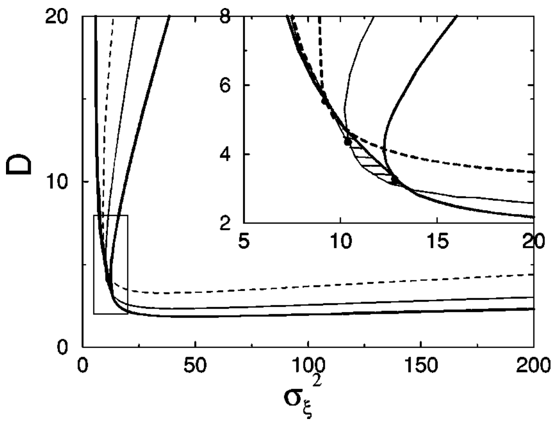

FIG. 3. Phase diagram in the plane (σ_ξ^2, D) for $a=0$ and three different values of the additive noise intensity: $\sigma_\zeta^2 = 0.3$ (thick solid line), 0.5 (thin solid line), and 1.0 (dashed line). For large coupling D additive noise shrinks the region of coexisting solutions, whereas its left boundary coincides for different σ_ζ^2 and remains unaffected. The inset plot shows peculiarities of the transition lines in the small box. Inside the dashed region an increase of additive noise induces disorder-order and the reentrant transition (see the text and Fig. 4).

tion of the transition, hence if, according to MFT, a transition is observed for $D=4.15$, in numerical simulations it occurs for $D=6.5$. The region of possible hysteresis for this set of parameters is too thin to be shown in Fig. 4; this fact is also confirmed by numerical simulations. But if we slightly increase D, hysteresis appears [17]. For example, for $D=7.0$ the hysteresis region spans even from $\sigma_\xi^2=0.0$ to 0.2.

We have thus seen so far that numerical simulations qualitatively confirm the existence of a first-order phase transition induced by additive noise in this system, as predicted by MFT. We note that in the two limiting cases of correlation between multiplicative and additive noise, the transition occurs. We also note that variation of both the multiplicative noise intensity and the coupling strength can change the order of this transition.

Let us now present a possible physical mechanism behind this effect. In [16,18] it was argued that the short-time evolution of the average value of the local field can be described by the equation

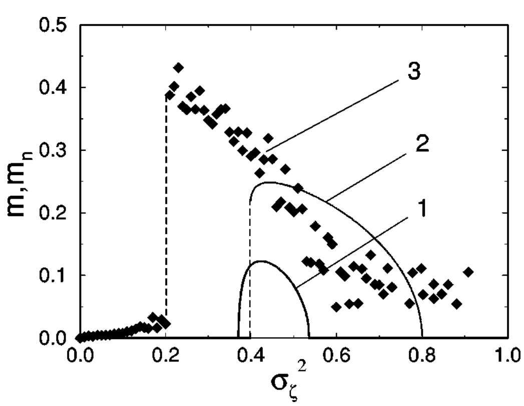

FIG. 4. First- and second-order phase transitions induced by uncorrelated additive noise. Curves 1 ($D=3.5$, $\sigma_\xi^2=12.0$) and 2 ($D=4.15$, $\sigma_\xi^2=11.0$) correspond to MFT results, diamonds to numerical simulations ($D=6.5$, $\sigma_\xi^2=11$).

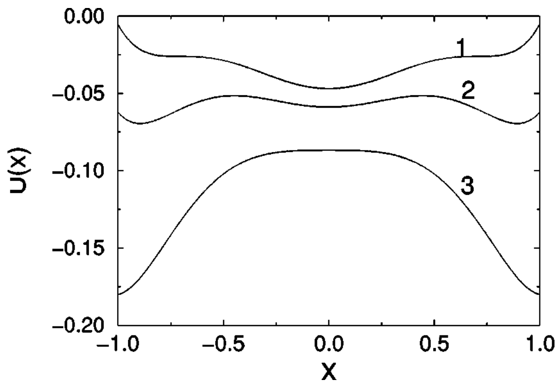

FIG. 5. ''Effective'' potential for the short-time evolution of m for $a^2=0.25$ (curve 1), 0.28 (curve 2), and 0.34 (curve 3). Other parameters are $\sigma_\xi^2=3.0$ and $\sigma_\zeta^2=0.0$.

$$\dot{\overline{x}} = f(\overline{x}) + \frac{\sigma_\xi^2}{2} g(\overline{x}) g'(\overline{x}), \qquad (7)$$

for which an ''effective'' potential can be derived. It is described by $U(x)=U_0(x)+U_{\text{noise}} = -\int f(x)dx - \sigma_\xi^2 g^2(x)/4$, where U_{noise} represents the influence of the multiplicative noise. We can trace the behavior of this potential in the presence of multiplicative noise, for the case $\sigma_\zeta^2=0$ and nonzero a. Its evolution for increasing a is shown in Fig. 5. This approach can be clearly seen to successfully explain the mechanism of the first-order transition: first, only the zero state is stable (curve 1), then there is a region where three stable states coexist (curve 2), and finally, the disordered state becomes unstable (curve 3). This approach also explains why a variation of the multiplicative noise intensity influences the order of the transition: for another (lower) σ_ξ^2 there is no region where ordered and disordered phases simultaneously exist. We emphasize that the ''effective'' potential is derived only for short-time evolution, and should not be confused with the ''stochastic'' potential [5], which for this system remains always monostable. For the other case of correlation between multiplicative and additive noise, in the region of additive noise induced transition, the ''effective'' potential always has three minima (two symmetric minima are lower than the central one). Overcritical additive noise causes an escape from zero state and leads to the transition. Hence, the ''effective'' potential approximation does not explain all results of MFT: it explains well the transition but not an existence of threshold in the additive noise intensity. It is important to add that the transition under consideration has much in common with the phenomenon of stochastic resonance: in both cases there is a multistability, and there exists an optimal value of the additive noise intensity for which the ordering is the most effective one. This similarity is limited by the fact that here the multistability is induced only in short-time terms, and there is no external signal to be synchronized with (see also [16]).

In conclusion, we have reported the existence of nonequilibrium first-order phase transitions induced by additive noise. Such a phenomenon can be expected to be experimentally observed [18] in systems exhibiting shifts in a transition induced by multiplicative noise. Possible candidates could be photosensitive chemical reactions [20,21], liquid crystals

[22,23], and Rayleigh-Bénard convection under a fluctuating temperature gradient [24]. It should also be mentioned that another form of coupling, Swift-Hohenberg, is possible in the presented model. In that case, one can observe ordered spatial patterns appearing as a result of a first-order phase transition induced by additive noise.

The results presented here open up several questions. First, it should be determined whether the behaviors reported are universal. Second, one should investigate the translation of these effects into other phenomena, such as globally synchronized oscillations in subexcitable media [25], transport properties in coupled ratchets [26], and relation between noise-induced transitions and stochastic resonance in systems with external forcing. Finally, these results could be of relevance for the stochastic modeling of transitions and irregular oscillations that have been explained in the frames of deterministic theory [8,14,27].

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