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Scaling theory for homogenization of the Maxwell equations

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A scaling theory for homogenization of the Maxwell equations is developed upon the representation of any field as a sum of its dipole, quadrupole, and magnetic dipole moments. This representation is exact and is connected neither with multipole expansion nor with the Helmholtz theorem. A chain of hierarchical equations is derived to calculate the moments. It is shown that the resulting macroscopic fields are governed by the homogenized Maxwell equations. Generally, these fields differ from the mean values of microscopic fields. [S1063-651X(99)01507-X]

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I. INTRODUCTION

Recent studies of artificial materials have focused on properties of composite materials inside which an electromagnetic wave interacts with inclusions in a resonant way. Examples are percolation systems [1], chiral materials [2], omega materials [3], artificially permeable materials [4–6], etc. An attempt to increase the observed effects entails working with a dense, high-loaded composite. Dealing with such ''resonant'' materials, one encounters two problems. The first one is that a dimension of an inclusion is comparable with the wavelength. The second one is that the mean distance between inclusions is comparable with their size. The first fact means that one should homogenize the Maxwell equations instead of the Laplace equation, whereas the second fact means that one has to work beyond the molecular optics approximation. Thus, many results known from literature (for example, mixing formulas [7]) become useless under these conditions.

Usually homogenization of the Maxwell equations is treated as averaging over a physically infinitesimal volume element (PIVE) [8]. For ''resonant'' materials direct averaging is not correct because currents, which build up closed contours, do not contribute to the mean macroscopic current but they cause additional dissipation of energy and generate a magnetic moment. The recipe for the correct averaging of circular currents seems to be well known from the theory of magnetic materials. It is an introduction of the magnetization $\vec{M}$ [9],

$$\vec{j}_{\mathrm{curl}}^{(\mathrm{macro})}=c\ \mathrm{curl}\,\vec{M}^{(\mathrm{macro})},$$

that yields

$$\vec{j}^{(\mathrm{macro})}=\partial\vec{P}^{(\mathrm{macro})}\partial t+c\ \mathrm{curl}\,\vec{M}^{(\mathrm{macro})} \tag{1}$$

and permits one not only to see how circular currents contribute to the macroscopic current but also to take into account the lost energy dissipation by regarding an imaginary part of $\vec{M}^{(\mathrm{macro})}$. In the framework of the molecular optics approximation, $\vec{M}^{(\mathrm{macro})}V_{\mathrm{PIVE}}$ and $\vec{P}^{(\mathrm{macro})}V_{\mathrm{PIVE}}$ are equal to sums of moments of molecules situated inside the PIVE. Such a simplicity tempts us to consider Eq. (1) as a multipole expansion [10–13], appending to Eq. (1) new multipole terms:

$$\vec{D}=\varepsilon\vec{E}+4\pi\vec{P}-4\pi\vec{\nabla}\cdot\hat{Q}+\cdots,\quad \vec{H}=\mu^{-1}\vec{B}-4\pi\vec{M}+\cdots.$$

Though the molecular optics is a well-developed area, this speculation seems to be an obscure place. Indeed, the multipole moments naturally appear while expanding any field in d/R powers where d is a source (inclusion) size and R is the distance between the source and a recorder (another inclusion). Such an expansion is useful in the ''molecular optics'' approximation as evaluating a local field which is a sum of an external field and fields induced by other molecules. In that case $d\ll R$, with R to be a mean distance between the molecules. On the contrary, the employment of the expansion is doubtful for evaluation of the mean fields and currents since this procedure demands knowledge of corresponding values inside the molecule [10,14], where the expansion is not held because $R<d$. Moreover, ''molecular optics'' ceases to be a good approximation even for calculation of a local field if we deal with high loaded composites where $d\sim R$.

Even in the frame of the molecular optics, there exists still another problem, i.e., the dependence of the multipoles upon the frame origin. It is well known (section 4.1 [10]) that only the lowest nonvanishing multipole moment does not depend on the location of the origin. All higher multipole moments depend upon the choice of the location. The problem of introducing origin-independent moments has a long history (see [10–13]) but still has no clear solution.

A relation like Eq. (1) could be formally deduced, avoiding speculation about multipole expansion, by employing the charge conservation law (see [15])

$$\frac{\partial\rho}{\partial t}+\mathrm{div}\vec{j}=0. \tag{2}$$

Taking into account the definition of polarization,

$$\rho=-\mathrm{div}\vec{P}, \tag{3}$$

one arrives at

$$\mathrm{div}\left(\vec{j}-\frac{\partial\vec{P}}{\partial t}\right)=0$$

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which yields Eq. (1) with vector $\vec{M}$, the physical sense of which is still undefined [8]. The last fact significantly complicates a consequent introduction of high-frequency permeability.

To prove that $\vec{M}$ is a magnetic moment of currents,

$$\int \vec{M} dv = \frac{1}{2c}\int [\vec{r}\times\vec{j}]dv, \tag{4}$$

one resorts to two assumptions (see [8]). First, the current is perfectly circular:

$$\vec{j} = c \operatorname{curl} \vec{M}.$$

It means that the first term in Eq. (1) could be ignored or

$$\frac{\partial}{\partial t}\vec{P} \ll c \operatorname{curl} \vec{M}. \tag{5}$$

For natural magnetic materials Eq. (5) fails at optical frequencies [8] whereas for composite material it already happens at mw. If the inequality (5) is broken but Eq. (1) is still held, one should look for any other definition of $\vec{M}$ instead of Eq. (4) (see [15–17]).

Assumption (5) is insufficient for Eq. (4) to be valid. The second assumption concerns space distribution of polarizability, namely, the magnetic moment $\vec{M}$ should be equal to zero outside the volume of integration. As it is shown in [8],

$$\frac{1}{2c}\int [\vec{r}\times\vec{j}]dv = \frac{1}{2}\int [\vec{r}\times \operatorname{curl}\vec{M}]$$
$$= \oint [\vec{r}\times[\vec{ds}\times\vec{M}]] + \int \vec{M} dv.$$

It is therefore seen that Eq. (4) is held if and only if the surface integral is equal to zero. This could be achieved by putting the surface slightly outside the considered volume, in the area where $\vec{M}=0$.

Certainly, this is an idealization that is never realized because the PIVE is a part of a large body. It means that the PIVE neighbors upon alike volume elements where magnetization is not equal to zero. To describe the situation, one should move on from a consideration of lumped-element objects to a consideration of distributed systems. One should distinguish currents with flux lines confined to the PIVE and those with lines terminated on the PIVE walls. The first ones contribute to magnetic polarization $\vec{M}^{(\mathrm{macro})}$, whereas the second ones contribute to the complex electric polarization $\vec{P}^{(\mathrm{macro})}$ and complex quadrupole density $\hat{Q}^{(\mathrm{macro})}$:

$$\vec{j}^{(\mathrm{macro})} = c \operatorname{curl} \vec{M}^{(\mathrm{macro})} + \partial \vec{P}^{(\mathrm{macro})}\partial t$$
$$- c \operatorname{div} \partial \hat{Q}^{(\mathrm{macro})}/\partial t + \cdots.$$

Generally, such a separation highly depends on the shape and size of the PIVE [8] that makes the procedure of averaging indeterminate.

To avoid all these problems, the authors [8,18] refuse to consider permeability at all, introducing a permittivity tensor: $\varepsilon_{ij}(\omega,k) = \varepsilon^{tr}(\omega)(\delta_{ij} - k_i k_j/k^2) + \varepsilon^{l}(\omega)k_i k_j/k^2$. As a penalty for this simplification, one should introduce an extra constitutive equation for the additional currents, flowing on each interface surface. Indeed, even uniform magnetization (microscopic circular currents) produces a surface current on any interface [9]. To write down the correct boundary conditions, one should introduce an additional constitutive equation for this surface current which is quite equivalent to introduction of permeability [19].

The conclusion of this brief review is that we should fix the following problems: first, why we can use a multipole expansion when $d \sim R$; second, what is the physical meaning of $\vec{M}$ and $\vec{P}$ if $\partial\vec{P}/\partial t \sim c \operatorname{curl} \vec{M}$; third, how to introduce origin-independent multipoles.

II. THE SCALING ALGORITHM OF THE HOMOGENIZATION

It seems that the key moment is the expression (4). It is Eq. (4) that gives us a reason to consider Eq. (1) as a multipole expansion. Moreover, Eq. (4) gives rise to a dependence of the moments upon the origin. Solely to prove Eq. (4), one should bound oneself by frames of the assumption (5) and the assumption that $\vec{M}$ and $\vec{P}$ are equal to zero outside the PIVE. To fix all aforementioned problems and to attribute a physical meaning to $\vec{M}$ and $\vec{P}$, we generalize Eq. (1). For this purpose we resort to the following mathematical lemma.

Lemma. Any field can be represented through its ‘‘electrical dipole,’’ ‘‘magnetic dipole,’’ and ‘‘electrical quadrupole’’ moments:

$$J_i = \frac{\partial}{\partial t}p_i + c\, e_{ijk}\frac{\partial}{\partial x_j}m_k - c\frac{\partial}{\partial x_k}\frac{\partial}{\partial t}q_{ik} = J_i^{(p)} + J_i^{(m)} + J_i^{(q)}, \tag{6}$$

where

$$m_i(x_j, J_k) = \frac{1}{2c}e_{ijk}x_j J_k, \tag{7}$$

$$\frac{\partial}{\partial t}q_{ij}(x_j, J_k) = -\frac{1}{2c}(x_i J_j + x_j J_i), \tag{8}$$

$$\frac{\partial}{\partial t}p_i(x_i, J_k) = -\left(x_i\frac{\partial}{\partial x_k}J_k\right). \tag{9}$$

This could be shown by straightforward calculation:

$$e_{ijk}\frac{\partial}{\partial x_j}m_k + \frac{\partial}{\partial x_k}\frac{\partial}{\partial t}q_{ik}$$
$$= e_{ijk}\frac{\partial}{\partial x_j}e_{kmn}x_m J_n + \frac{\partial}{\partial x_k}(x_i J_k + x_k J_i)$$
$$= e_{ijk}e_{kmn}\frac{\partial}{\partial x_j}x_m J_n + x_i\frac{\partial}{\partial x_k}J_k + J_k\frac{\partial}{\partial x_k}x_i$$
$$+ x_k\frac{\partial}{\partial x_k}J_i + J_i\frac{\partial}{\partial x_k}x_k$$
$$= \frac{\partial}{\partial x_n}x_i J_n - \frac{\partial}{\partial x_n}x_n J_i + \left(x_i\frac{\partial}{\partial x_k}J_k\right) + x_k\frac{\partial}{\partial x_k}J_i + 4J_i$$
$$= \left(2x_i\frac{\partial}{\partial x_k}J_k\right) + 2J_i. \tag{10}$$

It is worth emphasizing that Eq. (6) is neither a multipole expansion nor a representation of a field as a sum of irrotational and solenoidal parts according to Helmholtz's theorem.

It is important to emphasize that $\vec{m},\vec{p}$, and $\hat{q}$ on the right-hand sides of the expressions (7)–(9) are related with the entire current in a commonly accepted way [10] without additional assumptions.

The representation (6) is not self-consistent because, for example,

$$J_i^{(m)}(\vec{J}) \neq J_i^{(m)}(\vec{J}^{(m)}) = (1/2c)e_{ijk}\int_V \partial(e_{klm}x_l J_m^{(m)})/\partial x_j d^3 r.$$

As a consequence Eq. (6) is not unique. Along with it, any power (in operator sense) of the representation may be used. In the general case we come to the following expression:

$$J_i = \sum_{s=1} c_s\left(-x_i\frac{\partial}{\partial x_k}\delta_{kn}\cdots+\frac{e_{ijk}}{2}\frac{\partial}{\partial x_j}\times e_{klm}x_l\delta_{mn}\cdots-\frac{\partial}{\partial x_k}\frac{(x_i\delta_{kn}+x_j\delta_{in})}{2}\cdots\right)^s J_n$$

with $\Sigma_s c_s = 1$. Implying that the traditional expression (1) is held in numerous cases, we can assert that $c_1 \approx 1$. Further we shall consider $c_1 = 1$ [20].

Our consideration is also based on another mathematical formula [16], which is held for any bilinear form $L(x,J)$ such as

$$L(ax_1+bx_2,cJ_1+dJ_2) = acL(x_1,J_1)+adL(x_1,J_2)+bcL(x_2,J_1)+bdL(x_2,J_2)$$

where a,b,c, and d are real numbers, namely

$$L(J,r) = L(J-\langle J\rangle+\langle J\rangle, r-\langle r\rangle+\langle r\rangle) = L(\delta J+\langle J\rangle, \delta r+\langle r\rangle) = L(\delta J,\delta r)+L(\delta J,\langle r\rangle)+L(\langle J\rangle,r). \quad (11)$$

The brackets $\langle\ \rangle$ denote the averaging over a volume element, say PIVE.

In particular, Eq. (11) is held when L is one of the bilinear forms $p_i(x_j,J_k)$, $m_i(x_j,J_k)$, or $q_{ij}(x_j,J_k)$ defined by Eqs. (7)–(9). We can see that a moment of the PIVE can be represented as a sum of an origin-independent term and a moment of the whole PIVE when the PIVE is considered as a unitized indivisible pointlike construction. Further, the formula (11) will help us to introduce frame-independent multipole moments.

Let us imagine a composite sample as a sum of cells whose characteristic size is l. Thus, the $\langle\vec{j}\rangle_{(l)}$ varies only on scales greater than l. The Fourier expansion of $\langle\vec{j}\rangle_{(l)}$ contains terms with $k<2\pi/l$, whereas for $\delta\vec{j}$ it contains terms with $k>2\pi/l$. Applying Eq. (11) for a vector field of the form (6), we get

$$\vec{J}_1^{(\text{macro})} = \int_V[\vec{J}^{(m)}(\delta\vec{j},\delta\vec{r})+\vec{J}^{(q)}(\delta\vec{j},\delta\vec{r})+\vec{J}^{(p)}(\delta\vec{j},\delta\vec{r})+\vec{J}^{(m)}(\delta\vec{j},\langle\vec{r}\rangle)+\vec{J}^{(q)}(\delta\vec{j},\langle\vec{r}\rangle)+\vec{J}^{(p)}(\delta\vec{j},\langle\vec{r}\rangle)+\vec{J}^{(m)}(\langle\vec{j}\rangle,\vec{r})+\vec{J}^{(q)}(\langle\vec{j}\rangle,\vec{r})+\vec{J}^{(p)}(\langle\vec{j}\rangle,\vec{r})]\frac{dV}{V}.$$

The first group of three terms consists of quantities which are independent of the origin place. The second group of three terms is equal to zero as it can be seen from Eq. (10). The sum of the last three terms is equal to $\langle\vec{j}\rangle$ due to the independence of $\langle\vec{j}\rangle$ upon coordinate r inside the cell. Thus, after averaging we express a macroscopic (in the frame of the cell) value of the field $\vec{J}$ as a sum of four terms which are the mean value of $\langle\vec{j}\rangle$, curl to ''magnetic dipole'' moment, and time derivatives to the electric dipole moment and to the divergence of the electric quadrupole moment:

$$\begin{aligned}\vec{J}_{\vec{r}_1}^{(\text{macro})} &= \int_V\Big\{c\,\mathrm{curl}_{r_0}\vec{M}(\delta\vec{j}_0,\delta\vec{r}_0)-\mathrm{div}_{r_0}\partial\hat{Q}(\delta\vec{j}_0,\delta\vec{r}_0)/\partial t+\partial\vec{P}(\delta\vec{j}_0,\delta\vec{r}_0)/\partial t+\frac{1}{2}\mathrm{curl}_{r_0}[\vec{r}_0\times\langle\vec{J}\rangle_{\vec{r}_0\in V_1}]\\ &\quad+\frac{1}{2}\mathrm{div}_{r_0}(\vec{r}_0\langle\vec{J}\rangle_{\vec{r}_0\in V_1}+\langle\vec{J}\rangle_{\vec{r}_0\in V_1}\vec{r}_0)-\langle\vec{r}_0\rangle_{\vec{r}_0\in V_1}\mathrm{div}_{r_0}\langle\vec{J}\rangle_{\vec{r}_0\in V_1}\Big\}\frac{d^3r_0}{V}\\ &=\langle\vec{j}\rangle_{\vec{r}_0\in V_1}+\left\langle c\,\mathrm{curl}_{r_0}\vec{M}(\delta\vec{j},\delta\vec{r})-\mathrm{div}_{r_0}\frac{\partial}{\partial t}\hat{Q}(\delta\vec{j},\delta\vec{r})+\frac{\partial}{\partial t}\vec{P}(\delta\vec{j},\delta\vec{r})\right\rangle_{\vec{r}_0\in V_1}\\ &=\langle\vec{j}\rangle_{\vec{r}_0\in V_1}+c\,\mathrm{curl}_{r_1}\langle\vec{M}(\delta\vec{j},\delta\vec{r})\rangle_{\vec{r}_0\in V_1}-\mathrm{div}_{r_1}\left\langle\frac{\partial}{\partial t}\hat{Q}(\delta\vec{j},\delta\vec{r})\right\rangle_{\vec{r}_0\in V_1}+\frac{\partial}{\partial t}\langle\vec{P}(\delta\vec{j},\delta\vec{r})\rangle_{\vec{r}_0\in V_1}.\end{aligned} \quad (12)$$

All the quantities are independent of the location of the frame origin and are constant within the cell. Certainly, they fluctuate when moving on from cell to cell. To take into account that the four ''currents'' ($\vec{J}^{(m)}$, $\vec{J}^{(q)}$, $\vec{J}^{(p)}$, $\langle\vec{J}\rangle$) could fluctuate on scale $l'>l$, we shall behave in the spirit of the renormalization group [21]. We should group the primary cells into cells of size $l'=nl$ $(n>1)$ and average these four ''currents'' over new, large cells employing expression (12). Each of the four ''currents'' calculated on scale l will contribute to four currents'' on scale l'. It is obvious that we arrive at the following equation for ''current'' moments:

$$\vec{J}_N = \langle\vec{J}_{N-1}\rangle, \quad (13)$$

$$\vec{M}_N=\langle\vec{M}_{N-1}\rangle+\int_{V_N}\vec{m}(\delta\vec{j}_{N-1},\delta\vec{r}_{N-1})\frac{d^3r_{N-1}}{V_N}+\int_{V_N}\vec{m}(c\;\delta\,\text{curl}\,\vec{M}_{N-1},\delta\vec{r}_{N-1})\frac{d^3r_{N-1}}{V_N}+\int_{V_N}\vec{m}\left(\delta\frac{\partial}{\partial t}\vec{P}_{N-1},\delta\vec{r}_{N-1}\right)\frac{d^3r_{N-1}}{V_N}-\int_{V_N}\vec{m}\left(\delta\,\text{div}\,\frac{\partial}{\partial t}\hat{Q}_{N-1},\delta\vec{r}_{N-1}\right)\frac{d^3r_{N-1}}{V_N},\quad(14)$$

$$\vec{P}_N=\langle\vec{P}_{N-1}\rangle+\int_{V_N}\vec{p}(\delta\vec{j}_{N-1},\delta\vec{r}_{N-1})\frac{d^3r_{N-1}}{V_N}+\int_{V_N}\vec{p}(c\;\delta\,\text{curl}\,\vec{M}_{N-1},\delta\vec{r}_{N-1})\frac{d^3r_{N-1}}{V_N}+\int_{V_N}\vec{p}\left(\delta\frac{\partial}{\partial t}\vec{P}_{N-1},\delta\vec{r}_{N-1}\right)\frac{d^3r_{N-1}}{V_N}-\int_{V_N}\vec{p}\left(\delta\,\text{div}\,\frac{\partial}{\partial t}\hat{Q}_{N-1},\delta\vec{r}_{N-1}\right)\frac{d^3r_{N-1}}{V_N},\quad(15)$$

$$\hat{Q}_N=\langle\hat{Q}_{N-1}\rangle+\int_{V_N}\hat{q}(\delta\vec{j}_{N-1},\delta\vec{r}_{N-1})\frac{d^3r_{N-1}}{V_N}+\int_{V_N}\hat{q}(c\;\delta\,\text{curl}\,\vec{M}_{N-1},\delta\vec{r}_{N-1})\frac{d^3r_{N-1}}{V_N}+\int_{V_N}\hat{q}\left(\delta\frac{\partial}{\partial t}\vec{P}_{N-1},\delta\vec{r}_{N-1}\right)\frac{d^3r_{N-1}}{V_N}-\int_{V_N}\hat{q}\left(\delta\,\text{div}\,\frac{\partial}{\partial t}\hat{Q}_{N-1},\delta\vec{r}_{N-1}\right)\frac{d^3r_{N-1}}{V_N}.\quad(16)$$

Here the functions $\vec{m}$, $\hat{q}$, and $\vec{p}$ are defined by Eqs. (7)–(9). The equations allow us to calculate origin-independent moments on the Nth level if we know the distribution of the same moments on the $(N-1)$th level [22]. We have to go on until we reach a certain level, where the diameter of the volume of averaging becomes equal to the correlation length L_c of inhomogeneities. At this step the moments cease to depend upon the spatial variables. Thus, we arrive at the following determination of the macroscopic value of the field $\vec{J}$:

$$\vec{J}^{(\text{macro})}=c\,\text{curl}\,\vec{M}_J^{(\text{macro})}-c\,\text{div}\frac{\partial}{\partial t}\hat{Q}_J^{(\text{macro})}+\frac{\partial}{\partial t}\vec{P}_J^{(\text{macro})}+\langle\vec{J}\rangle.\quad(17)$$

Let us consider the algorithm with the intent of employing it on a computer. At the first step we have a system of elementary cells building up the whole body under consideration. For simplicity we shall consider the cells to be cubes with side a_0. We know the current $\vec{j}$ in each cell. The first step consists of two substeps. At the first substep we join together eight neighborhood cells producing a new set of large cells. For each of such cells we can calculate the mean current $\langle\vec{j}\rangle$, distribution of the fluctuating part of the current $\delta\vec{j}$, and magnetic and quadrupole moments of $\delta\vec{j}$: $\vec{m}=\Sigma[\delta\vec{r}\times\delta\vec{j}]/8$, $q_{ij}=\Sigma(\delta r_i\delta j_j+\delta r_j\delta j_i)/16$, where summation is performed over eight primary cells constituting a new secondary cell and δr_i is a position of the primary cell with regard to the center of the secondary cell. At the next substep we join eight secondary cells into a new cell that will play the role of the primary cell at the next step. Computing mean current $\vec{j}^{(\text{mean})}=\langle\langle\vec{j}\rangle\rangle$, magnetic current $\vec{j}^{(m)}=c\ \text{curl}\,\vec{m}$, quadrupole current $\vec{j}^{(q)}=-c\ \text{div}\ \partial\hat{q}/\partial t$, and polarization current $\vec{j}^{(p)}=\langle\partial\vec{p}/\partial t\rangle=-\Sigma[\delta\vec{r}\text{div}(\delta\vec{j})/8]$, we arrive at the primordial situation: the body is split into cells which are cubes but with side $4a_0$. The only difference is that instead of dealing with a single quantity, namely with the current $\vec{j}$, we should treat four quantities: $\vec{j}^{(\text{mean})}$, $\vec{j}^{(m)}$, $\vec{j}^{(q)}$, and $\vec{j}^{(p)}$. As a result, after two substeps, we obtain sixteen quantities in each cell of the third level (with side $16a_0$): $\vec{j}_{\text{mean}}^{(\text{mean})}$, $\vec{j}_{\text{mean}}^{(m)}$, $\vec{j}_{\text{mean}}^{(q)}$, $\vec{j}_{\text{mean}}^{(p)}$ produced from $\vec{j}^{(\text{mean})}$, $\vec{j}_m^{(\text{mean})}$, $\vec{j}_m^{(m)}$, $\vec{j}_m^{(q)}$, $\vec{j}_m^{(p)}$ produced from $\vec{j}^{(m)}$, $\vec{j}_q^{(\text{mean})}$, $\vec{j}_q^{(m)}$, $\vec{j}_q^{(q)}$, $\vec{j}_q^{(p)}$ produced from $\vec{j}^{(q)}$, and $\vec{j}_p^{(\text{mean})}$, $\vec{j}_p^{(m)}$, $\vec{j}_p^{(q)}$, $\vec{j}_p^{(p)}$ produced from $\vec{j}^{(p)}$. Going on, we construct new quantities

$$\vec{j}^{(\text{mean})}=\vec{j}_{\text{mean}}^{(\text{mean})}+\vec{j}_m^{(\text{mean})}+\vec{j}_q^{(\text{mean})}+\vec{j}_p^{(\text{mean})},\quad(18)$$

$$\vec{j}^{(m)}=\vec{j}_{\text{mean}}^{(m)}+\vec{j}_m^{(m)}+\vec{j}_q^{(m)}+\vec{j}_p^{(m)},\quad(19)$$

$$\vec{j}^{(p)}=\vec{j}_{\text{mean}}^{(p)}+\vec{j}_m^{(p)}+\vec{j}_q^{(p)}+\vec{j}_p^{(p)},\quad(20)$$

$$\vec{j}^{(q)}=\vec{j}_{\text{mean}}^{(q)}+\vec{j}_m^{(q)}+\vec{j}_q^{(q)}+\vec{j}_p^{(q)}.\quad(21)$$

Thus we have completed the step and are ready to do the next one applying the proposed algorithm.

Moreover, taking into account that $\langle f'\rangle=\langle f\rangle'$ for any function and differentiation operation denoted by a prime (brackets denote averaging over a cell), we obtain Eqs. (13) and (16) from Eqs. (18) and (21).

III. THE HOMOGENIZED MAXWELL EQUATION FOR COMPOSITE MATERIALS

In the case of composite materials instead of the Lorentz equations valid in vacuum, one should average the Maxwell material equations:

$$\text{curl}\,\vec{E}=\frac{i\omega}{c}\mu\vec{H},\quad \text{curl}\,\vec{H}=-\frac{i\omega}{c}\varepsilon\vec{E},$$

where ε and μ are complex functions depending on ω and r. Employing Eq. (17) for homogenization yields

$$\text{curl}\,\vec{E}^{(\text{macro})}=\frac{i\omega}{c}(\mu\vec{H})^{(\text{macro})},$$

$$\text{curl}\,\vec{H}^{(\text{macro})}=-\frac{i\omega}{c}(\varepsilon\vec{E})^{(\text{macro})}\quad(22)$$

with a new couple of fields,

$$(\varepsilon\vec{E})^{(\text{macro})}=c\,\text{curl}\,\vec{I}^{(\text{macro})}+ic\omega\,\text{div}\,\hat{Q}^{(\text{macro})}+\langle\varepsilon\vec{E}\rangle,\quad(23)$$

$$(\mu\vec{H})^{(\mathrm{macro})}=c\,\mathrm{curl}\,\vec{L}^{(\mathrm{macro})}+ic\,\omega\,\mathrm{div}\,\hat{Z}^{(\mathrm{macro})}+\langle\mu\vec{H}\rangle. \tag{24}$$

Here $\hat{Z}^{(\mathrm{macro})}$ is a quadrupole moment of the ''magnetic current'' $\mu\vec{H}$; $\vec{I}^{(\mathrm{macro})}$ is related to the macroscopic magnetization $\vec{M}^{(\mathrm{macro})}$ caused by fluctuations of the total current $(\omega\varepsilon\vec{E}/4i\pi)$. Indeed,

$$\vec{M}\sim[\delta\vec{r}\times\delta\vec{j}]=(\omega/4ic\pi)[\delta\vec{r}\times\delta(\varepsilon\vec{E})]\sim\vec{I}.$$

More exactly, $\vec{M}^{(\mathrm{macro})}=-(i\omega/4\pi)\vec{I}^{(\mathrm{macro})}$. This term should be calculated employing Eq. (13) where $\vec{m}$ should be taken from Eq. (7) with $\vec{j}=(\omega\varepsilon\vec{E}/4i\pi)$.

The vector $\vec{L}^{(\mathrm{macro})}$ has been phenomenologically introduced in [23]. In the present consideration it appears while homogenizing the magnetic field $\vec{B}$. It is a ''magnetic,'' in the sense (6), part of $\vec{B}^{(\mathrm{macro})}=(\mu\vec{H})^{(\mathrm{macro})}$. This term comes from Eq. (13), where $\vec{m}$ should be taken from Eq. (7) with $\vec{j}=(i\omega\mu\vec{H}/4\pi)$. Thus, $\vec{L}$ is proportional to ''anapolization'' [24] or density of anapole dipole moments. The anapole or toroidal pole moments were introduced in nuclear physics [25] to describe objects without electric and magnetic moments. An electrodynamical anapole could be thought of as a toroidal solenoid with poloidal currents. (For more details, see [24].)

It is convenient to introduce new fields $\vec{E}_0=\vec{E}^{(\mathrm{macro})}-4\pi\vec{L}^{(\mathrm{macro})}$ and $\vec{H}_0=\vec{H}^{(\mathrm{macro})}-4\pi\vec{M}^{(\mathrm{macro})}$ (see [23]). These fields are introduced, much as the field $\vec{H}=\langle\vec{h}\rangle-4\pi\vec{M}$ is traditionally introduced. Here $\langle\vec{h}\rangle$ is the mean microscopic magnetic field usually denoted as $\vec{B}$ and called the magnetic induction.

Employing the new fields $\vec{E}_0$ and $\vec{H}_0$, we can recast Eqs. (22) as

$$\mathrm{curl}\,\vec{E}_0=\frac{i\omega}{c}\mu_{\mathrm{eff}}\vec{H}_0,\quad \mathrm{curl}\,\vec{H}_0=-\frac{i\omega\varepsilon_{\mathrm{eff}}}{c}\vec{E}_0, \tag{25}$$

where constitutive parameters $\varepsilon_{\mathrm{eff}}$, μ_{eff} are defined by the equations

$$\varepsilon_{\mathrm{eff}}\vec{E}_0=\langle\varepsilon\vec{E}\rangle+i\omega c\,\mathrm{div}\,\hat{Q}^{(\mathrm{macro})}, \tag{26}$$

$$\mu_{\mathrm{eff}}\vec{H}_0=\langle\mu\vec{H}\rangle+i\omega c\,\mathrm{div}\,\hat{Z}^{(\mathrm{macro})}, \tag{27}$$

which differ from the custom ones $\varepsilon_{\mathrm{eff}}\langle\vec{E}\rangle=\langle\varepsilon\vec{E}\rangle$, $\mu_{\mathrm{eff}}\langle\vec{H}\rangle=\langle\mu\vec{H}\rangle$ (see [7]).

The fields $\vec{E}_0$ and $\vec{H}_0$ are governed by the usual Maxwell equations. As a consequence, the usual boundary conditions (continuity of tangential components of $\vec{E}_0$ and $\vec{H}_0$ on any interface surface) should connect fields outside the medium with the ''subzero'' fields. The fields $\vec{E}^{(\mathrm{macro})}$, $\vec{H}^{(\mathrm{macro})}$ change across the boundary. To understand this fact, it is worth referring to the traditional homogenization procedure where $\vec{H}^{(\mathrm{macro})}$ corresponds to $\vec{B}$ and $\vec{H}_0=\vec{H}^{(\mathrm{macro})}-4\pi\vec{M}$ does the same to $\vec{H}$. As it follows from Eq. (25), the boundary condition for $\vec{H}^{(\mathrm{macro})}$ reminds us of the one for $\vec{B}$ in traditional consideration (see Chap. 19 [8]):

$$\vec{n}\times(\vec{H}_2^{(\mathrm{macro})}-\vec{H}_1^{(\mathrm{macro})})=4\pi\vec{n}\times(\vec{M}_2^{(\mathrm{macro})}-\vec{M}_1^{(\mathrm{macro})}).$$

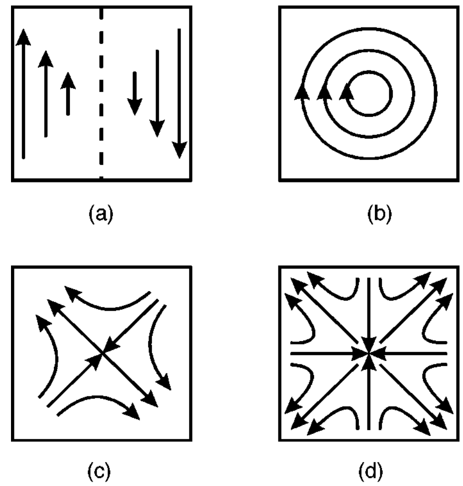

FIG. 1. Possible patterns of fluctuating current.

Repeating the same speculations, it is easy to show that for $\vec{E}^{(\mathrm{macro})}$ one has

$$\vec{n}\times(\vec{E}_2^{(\mathrm{macro})}-\vec{E}_1^{(\mathrm{macro})})=4\pi\vec{n}\times(\vec{L}_2^{(\mathrm{macro})}-\vec{L}_1^{(\mathrm{macro})}).$$

In the vicinity of the interface outside the composite, say in vacuum, there could exist a lot of evanescent waves. This could result in nonzero anapolization of vacuum. Hence one should deal with the macroscopic fields instead of mean fields in vacuum. This is a typical situation if we deal with inhomogeneous systems [26–29].

IV. CONCLUSION AND DISCUSSION

To repel an accusation in abstract contemplation, let us consider examples of the ''multipole'' media. Permeable composites made of nonpermeable ingredients are well known. The simplest example is a composite loaded with highly conducting spherical inclusions. Due to eddy currents, there appears a magnetic moment of the inclusion. The composite being placed, say, in a microwave field exhibits properties of diamagnetic material [23]. The inclusions of more complicated structure can exhibit resonant excitation resulting in induced magnetic moment. Examples of such inclusions are open rings [4], dielectric spheres [5], helices, and bihelices [2,6]. In this case we can observe either diamagnetism or paramagnetism, depending upon the relation between the working and resonant frequencies. Q medium is a smarter system. As a composite made of identical dielectric spheres is permeable, the material made of different sized spheres may be nonpermeable. The concentrations and radii may be chosen so that one part of the spheres is excited in a diamagnetic mode and the other in a paramagnetic one. Such a system should be described by its quadrupole moment.

Obviously we can design a still smarter system with inclusions whose electric and magnetic dipole as well as electric quadrupole moments are equal to zero. A corresponding distribution of current is presented in Fig. 1(d). At first sight, it seems that following the proposed procedure, one cannot treat such systems in a proper way since formulas (14) and (15) yield a zero value for $\vec{M}$ and $\hat{Q}$. Here we must say that, generally speaking, on the right sides of equations like Eqs.

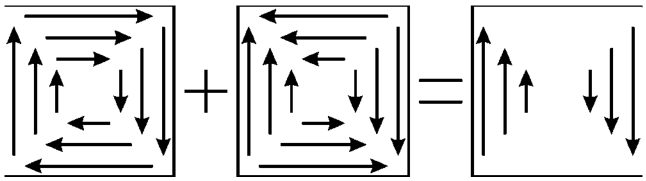

FIG. 2. A graphical analog of Eq. (6); the pattern shown in Fig. 1(a) could be presented as a combination of the currents shown in Figs. 1(b) and 1(c).

(14)–(16) the macroscopic values, say $\vec{M}_{N-1}^{(\mathrm{macro})}$, should stay instead of the mean values. The macroscopic value is connected with the corresponding mean value through Eq. (17),

$$\vec{M}_{N-1}^{(\mathrm{macro})}=c\,\mathrm{curl}\,\vec{T}_{N-1}^{(\mathrm{macro})}+ic\,\omega\,\mathrm{div}\,\hat{W}_{N-1}^{(\mathrm{macro})}+\langle\vec{M}_{N-1}\rangle. \tag{28}$$

This leads to the introduction of additional fields. In the example (28) these are an anapolization $\vec{T}$ and a tensor field $\hat{W}$, describing densities of higher moments. Certainly, one has to write down the equations for these fields, introducing a new set of fields *ad infinitum*. To cut off the chain of the equations one should assume macroscopic values to be equal to mean ones for any field set. This implies straightforward averaging of the quantities over the PIVE of size L_c, avoiding an iteration procedure. In doing so, one should calculate moments relative to the PIVE center, but to sum moments calculated relative to a local position inside the PIVE, as it is done at the first step of the iteration procedure.

The cutoff problem is tightly connected with the averaging procedure. In Sec. II we concluded that the iteration procedure must be stopped after arriving at the scale L_c. At this scale there are no fluctuations. Nevertheless, if we deal with alternating in time fields, say with a plane wave, there is still a spatial dependence of the fields exp(*ikr*). It is natural to expect that the averaging would not disturb the dependence. For uniform space fields the averaging over the PIVE of radius L_c does not change the values of the fields, whereas for a plane wave the procedure changes the amplitude by a factor $\sin(kL_c)/(kL_c)$. In other words, there is no reason to consider terms higher than $O((kL_c)^2)$ while passing from microscopic to macroscopic Maxwell equations without significantly changing the averaging procedure. It is necessary to mention that one cannot completely disregard effects of the fields retardation because they may be of lower order in (kL_c). In particular, the retardation is responsible for chirality (see [30,2]).

Disregarding the effects higher than $O(a^2)$ means that it is possible to take for $\vec{M}^{(\mathrm{macro})},\vec{L}^{(\mathrm{macro})}$ the corresponding mean values. Indeed, in any primary cell the fluctuating part $\vec{\delta j}$ of any field has one of the distribution patterns presented in Fig. 1 (see Fig. 2). Certainly Fig. 1(a) presents a common situation corresponding to the existence of a nonzero spatial derivative of the field inside the cell. The contribution of other patterns is $O(a_0^3)$ compared to $1a$-cell contribution (it is assumed that there is a finite number of cells with such singular patterns). Thus we can avoid consideration of the cells without loss of accuracy.

Confining to the first two terms of the Taylor expansion of $\vec{j}$: $j_i=j_0+d_{ik}\delta r_k$, we can see that inside the $1a$ cell there exists a "zero" plane such that $\vec{\delta j}=\vec{0}$ on it and that $\delta j_i=d_{ik}\delta r_k$ [31] increases linearly while coming outward from the plane, where $d_{ik}\sim j/a$. Only magnetic dipole and electric quadrupole moments of such a current distribution have nonzero values. All other multipoles are equal to zero. The magnetic dipole density $m_i=(\int_{V_{\mathrm{cell}}}e_{ijk}d_{il}\delta r_l\delta r_k)/V_{\mathrm{cell}}\sim a^2j/a\sim O(a)$. Contribution of the quadrupole moment is of the same order [32], whereas the disturbance of a plane wave is o small from these values. Hence, one should take $\vec{M}$ and $\hat{Q}$ into account. The next terms in the Taylor expansion of current give rise to higher moments. The density of the $2l$-pole magnetic and the $2(l+1)$-pole electric moment are $O(a^l)$ [see [10] formula (9.9)]. Even for $l=2$ (magnetic quadrupole moment) the contribution is $O(a^2)$, which is of the same order as the correction to the wave amplitude due to averaging and hence it must be abandoned [33].

From what has been said, it follows that high-multipole media cannot be described in terms of local constitutive equations for just the averaging procedure to become nonlocal.

There is another method of averaging which is employed in so-called strong fluctuation theory (SFT) [34]. Unfortunately, this method has an even more limited field of application than averaging over the PIVE. The method implies that the value of the field is completely determined by the value of the local constitutive parameters. As a consequence, first, despite considering the correlation function one does not really take into account that the field inside an inclusion depends upon its membership of a certain cluster. The percolation theory is best suited to illustrate this statement for a static limit. The SFT, which is equivalent to the Bruggeman approximation with neglecting correlations, says that, depending upon the component concentration, an inclusion can exist in only one of two possible positions: below the percolation threshold each inclusion is surrounded by a matrix material, while above the threshold all inclusions belong to the sole infinite single-connected cluster. The correlation in the inclusion distribution can change the value of the percolation threshold. Obviously it is a rough approximation because the formation of finite clusters, "dead ends," and parallel paths of the infinite cluster existing in real composites are ignored.

Second, the SFT ignores the dynamic effect: the field distribution inside inclusions. Consideration of a correlation, say in a bilocal approximation, does not improve the situation. Indeed, the correction to the static value of permittivity describes reradiation of the incident wave into a noncoherent component [35,34]. To describe effects connected with the retardation of fields on the scale of inclusions (skin effect [23], artificial magnetic properties [4–6], etc.) one needs a more sophisticated theory.

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$$J_i=\frac{\partial P_i}{\partial t}-\frac{\partial}{\partial x_j}\frac{\partial}{\partial t}\frac{Q_{ij}}{2}+e_{ijk}\frac{\partial M_k}{\partial x_j}+\frac{\partial^2}{\partial x_j \partial x_k}\frac{\partial}{\partial t}\frac{Q_{ij}}{6}-e_{ijl}\frac{\partial^2}{\partial x_j \partial x_k}\frac{M_{lk}}{2},$$

where

$$\frac{\partial}{\partial t}P_i=\frac{1}{4}r_i r_j\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}J_k;\ \ \frac{\partial}{\partial t}Q_{ij}=-\frac{3}{2}(r_iJ_j+r_jJ_i);$$

$$\frac{\partial}{\partial t}Q_{ijk}=-\frac{1}{2}(r_ir_jJ_k+r_kr_jJ_i+r_kr_iJ_j);$$

$$M_i=e_{ijk}r_jJ_k;\quad M_{ij}=\frac{1}{6}r_je_{ikl}r_kJ_l.$$

Unfortunately, contrary to Eqs. (6)–(9), some of the terms have no clear physical sense.

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[32] The patterns in Figs. 1(b) and 1(c) are of the same order. One can comprise the $1a$ pattern employing $1b$ and $1c$ patterns (see Fig. 2), which is a graphical analog of Eq. (6): Fig. 1(a) =Fig. 1(b) + Fig. 1(c). So, in some sense, Eq. (6) is the presentation of the field as the sum of a part symmetric in δr and a part that is antisymmetric.

[33] The restriction to Eq. (17) or even to Eq. (1) does not mean that the fields generated by the current from the PISVE have dipole-quadrupole character. As is shown in [10], multipole expansion of these radiated fields in a source-free region contains, in the general case, all multipole contributions (see Chap. 16.2 in [10]).

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