Stability of solitary wave trains in Hamiltonian wave systems

J. M. Arnold
Department of Electronics and Electrical Engineering, University of Glasgow, Glasgow G12 8LT, Scotland, United Kingdom
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A class of Hamiltonian nonlinear wave equations possessing complex solitary waves with exponential decay is studied. It is shown that the interpulse interactions in a train of nearly identical solitary waves with large separations between the individual solitary waves are approximately described by a double Toda lattice system, with two variables at each lattice site. Under certain conditions, which are explicitly identified as Cauchy-Riemann equations, the two dynamical variables are real and imaginary parts of a single complex variable, leading to the complex Toda lattice equations, which is a discrete integrable dynamical system. This analysis generalizes to certain nonintegrable partial differential equations a recent result for the nonlinear Schrödinger equation, and is important for the study of nonlinear communications channels in optical fibers. An example, the cubic-quintic nonlinear Schrödinger equation, is worked out in detail to show that the theory can be carried through analytically. The theory is used to determine the stability of an infinite chain of nearly identical pulses separated by large time intervals. The entire theory is nonperturbative in the sense that the nonlinear wave equation need not be a weak perturbation of an integrable one. [S1063-651X(99)11705-7]

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I. INTRODUCTION

The behavior of long trains of pulses in nonlinear wave systems is of great importance in theoretical descriptions of optical communication systems, mode-locked lasers and many other technological applications. There are several respects in which trains of nonlinear pulses do not behave simply as single pulses; for example, the nonlinear interaction between the tails of neighboring pulses in the train causes the train to undergo its own dynamical motion which is quite separate from that of the individual pulses. In a communications channel, this dynamics causes the pulses to be moved away from their initially allocated positions, and causes errors in detection of the information initially impressed on the pulse train. In studying any dynamical system it is particulary important to identify the stationary states of the system and to characterize their stability, as these states predetermine the entire topology of the trajectories of the system. In communication systems, special attention attaches to periodic or quasiperiodic arrangements of pulses, since these are the basic carriers of the information and are generally stationary states in the case of nonlinear systems such as solitons on optical fibers. Indeed, one approach to the design of such communication systems involves the generation of a stable stationary state of the pulse train, and the subsequent modification of the train by small modulations of parameters such as pulse position to encode the information to be transmitted by the train [1,2].

It has recently been demonstrated that the dynamics of N-solitons of the nonlinear Schrödinger equation (NSE) are quite well approximated by complex Toda lattice equations (CTLEs) in the limit of large separation of almost identical solitons with arbitrary phases [3–5]. The method used to demonstrate this in [4,5] was the Karpman-Solov'ev (KS) perturbation theory [6], followed by some further simplifying assumptions to reduce the Karpman-Solov'ev dynamical system to the complex Toda lattice. The KS perturbation method applies to sufficiently small perturbations of an integrable system which has a known Lax pair, and involves lengthy calculations before the final CTLE appears. The Lagrangian method proposed in [4] is a direct reduction procedure which does not depend on the integrability of the original equations, and is applicable in principle to any Hamiltonian system having sufficiently well behaved solitary waves.

In this paper the Lagrangian theory of [3] is extended to show that a large class of nonintegrable Hamiltonian wave systems is reducible to a double Toda lattice in the limit of large separations of nearly identical solitary waves, generalizing the results of [3–5] quite considerably. The method used here is similar to that of Gorshkov-Ostrovsky [7], but is more explicitly variational and Hamiltonian than that used in [7]. This approach greatly simplifies the internal computations required to be made, since it completely eliminates many of the technicalities of the earlier approach, such as special orthogonality conditions, elimination of secular resonances, requirements for adjoint functions and other artefacts. Furthermore, the method of [7] was applied principally to equations with real-valued solutions such as Korteweg–de Vries (KdV), with only a cursory description of the simplest possible complex case, which is the integrable NSE. Here a complete treatment is obtained for the general nonintegrable Hamiltonian wave system with complex-valued wave functions. An extensive illustration of the application of the theory to solitary waves of a nonintegrable partial differential equation (PDE), the cubic-quintic nonlinear Schrödinger equation, is carried out, this example being chosen solely because of its relative simplicity. The interest here is focused on the infinite lattice of nearly identical solitary pulses and the dynamics specific to this lattice, representative of the situation encountered in optical communications. Gorshkov and Papko [8] have earlier obtained stability criteria for periodic lattices of solitons of KdV-type equations which have real-valued solutions, and these were also studied experimentally in electrical transmission lines. The stability of quasiperiodic lattices of solitons of the NSE, having complex-valued solutions, was obtained in [1] and [3]. In this paper a simple result for the stability of an infinite lattice of solitary pulses of nonintegrable Hamiltonian wave systems is derived which further extends the theory already known. This result shows that quasiperiodic trains of solitary pulses of a nonintegrable Hamiltonian wave equation can be stable under a wider range of conditions than those already known for the integrable NSE: nonintegrability stabilizes the pulse train.

II. HAMILTONIAN WAVE EQUATIONS

We consider the class of Hamiltonian wave systems characterized by

$$i\partial_z\psi=\delta_{\psi^*}H(\psi,\psi^*) \tag{1}$$

in the phase space (ψ,ψ^*), where ψ is a complex-valued function of the evolution variable z and a transverse variable t which may often be time. The Hamiltonian H is a real-valued functional of this wave function ψ and its complex conjugate, along with any number of t derivatives of these functions, and is further assumed to be translationally invariant with respect to both z and t. The symbol δ represents the variational derivative. In the following we consider only the case of a scalar function ψ, but the theory is readily generalized to arbitrary n-component vector functions $\psi=(\psi_1,\ldots,\psi_n)$. Typical examples occurring in nonlinear optics are related to the nonlinear Schrödinger equation (NSE)

$$i\partial_z\psi+\tfrac{1}{2}\partial_t^2\psi+|\psi|^2\psi=0 \tag{2a}$$

for which the Hamiltonian is

$$H=\int_{-\infty}^{\infty}\frac{1}{2}\,(|\partial_t\psi|^2-|\psi|^4)dt. \tag{2b}$$

This model is integrable by the inverse scattering transform (IST). By deforming the Hamiltonian to

$$H=\int_{-\infty}^{\infty}\frac{1}{2}\,(|\partial_t\psi|^2-|\psi|^4+F)dt, \tag{3a}$$

where F is some Hamiltonian deformation which is a polynomial in ψ,ψ^* and their t derivatives, new nonintegrable PDEs are generated. The example we shall take later contains $F=-\frac{2}{3}a|\psi|^6$, which leads to a nonintegrable perturbed NSE

$$i\partial_z\psi+\tfrac{1}{2}\partial_t^2\psi+|\psi|^2\psi+a|\psi|^4\psi=0. \tag{3b}$$

We further suppose the existence of *solitary wave* solutions of the Hamiltonian wave equation with suitable boundary conditions $\psi\rightarrow 0$ as $|t|\rightarrow\infty$. Solitary waves can generally be constructed in the form

$$\psi(z,t)=e^{i\kappa z}\Psi(t-c^{-1}z) \tag{4}$$

for some real constant κ and a function of one variable Ψ, where c^{-1} is the reciprocal velocity of the solitary wave in the (z,t)-coordinate system. In the case that H is a real-valued *polynomial* functional of the phase-space variables (i.e., a superposition of homogeneous terms which scale with integer powers of $|\lambda|^2$ when $\psi\rightarrow\lambda\psi$ for constant $\lambda\in\mathbf{C}$), the asymptotic form of the solitary wave satisfying the boundary conditions $\psi\rightarrow 0$ at $t\rightarrow\pm\infty$ is obtained by neglecting all the nonlinear terms from the Hamiltonian PDE, since these vanish faster than the linear terms; this gives the linear differential equation

$$\{A(i\partial_t)-ic^{-1}\partial_t-\kappa\}\Psi\sim 0 \tag{5}$$

for the asymptotic behavior of the solitary wave function Ψ, where $A(i\partial_t)$ is a self-adjoint linear differential operator with constant coefficients. It follows from Eq. (5) that the asymptotic behavior of the solitary wave at $t\rightarrow\pm\infty$ is $\Psi\rightarrow\Sigma_j C_{j\pm}e^{-i\Omega_{j\pm}t}$, where $\Omega_{j\pm}$ belong to the set of complex characteristic roots of Eq. (5) [i.e., zeros of $A(\Omega)-c^{-1}\Omega-\kappa$] and $C_{j\pm}$ are complex constants. The characteristic roots Ω_j either are real or occur in complex conjugate pairs; to ensure exponential decay of the solitary wave it is necessary to select only those roots for which the condition $\mp\mathrm{Im}\,\Omega_{j\pm}>0$. When the differential operator A is second order in t derivatives, such as the perturbed NSE of Eq. (3a), there are only two characteristic roots $\Omega_\pm=\omega\mp i\eta$, with ω and $\eta>0$ real constants. Higher-order operators for A lead to more characteristic roots, but eventually the asymptotic behavior of the solitary wave is dominated by the roots with smallest imaginary part. We shall assume throughout that H is such a translation-invariant real-valued polynomial functional on the phase space, and it follows that solitary waves, if they exist at all, are exponentially decaying at infinity along the transverse variable t.

If $\psi=\psi_0(z,t)$ is a solitary wave, then by translation invariance in t it follows that $\psi_0(z,t-t_0)$ is also a solitary wave for an arbitrary constant translation t_0; it follows also from the homogeneity of each term in the Hamiltonian with respect to scaling of the complex field that $e^{i\theta_0}\psi_0(z,t-t_0)$ is also a solitary wave for an arbitrary real constant phase shift θ_0. These global symmetries lead to two conserved quantities, or *Noether invariants*, for the solitary wave

$$\mu_0=\frac{1}{2}\int_{-\infty}^{\infty}|\psi_0|^2dt, \tag{6a}$$

$$p_0=\frac{1}{4}\,i\int_{-\infty}^{\infty}(\psi_0\partial_t\psi_0^*-\psi_0^*\partial_t\psi_0)dt \tag{6b}$$

in addition to the conserved Hamiltonian functional H. These two invariants are the mass (μ_0) and momentum (p_0) of the solitary wave. Any solitary wave depends on these two invariants as parameters, as also do the wave number κ and the decay rate η. To denote this dependence we write the solitary wave function as $\Psi=\Psi(t;p_0,\mu_0)$.

III. LAGRANGIAN VARIATIONAL FORMULATION

We define the *Lagrangian*

$$L=\frac{1}{2}\,i\int_{-\infty}^{\infty}(\psi^*\partial_z\psi-\psi\partial_z\psi^*)dt-H \tag{7}$$

and an associated *action functional*

$$S=\int L dz. \tag{8}$$

The Euler-Lagrange equation $\delta_{\psi*}S=0$ of this Lagrangian with respect to arbitrary variations in the field ψ^* is precisely the Hamiltonian wave equation (1).

When the form (4) is introduced as a trial function in the Lagrangian (7) and the Euler-Lagrange equation obtained, the result is an ordinary differential equation (ODE) for the solitary wave function Ψ. Solutions of this ODE are a family of functions parametrized by invariants (6a) and (6b). If a more general trial function

$$\psi(z,t)=e^{i\theta_0(z)}\Psi(t-t_0(z);p_0(z),\mu_0(z)) \tag{9}$$

is introduced, with $\Psi(t;p_0,\mu_0)$ being this family of ODE solutions, and variations taken with respect to the parameters p_0,μ_0,t_0,θ_0 with fixed Ψ, then the resulting Euler-Lagrange equations are the Hamiltonian dynamical system

$$d_z t_0=\partial_{p_0}\overline{H}_0, \tag{10a}$$

$$-d_z\theta_0=\partial_{\mu_0}\overline{H}_0, \tag{10b}$$

$$-d_z p_0=\partial_{t_0}\overline{H}_0=0, \tag{10c}$$

$$d_z\mu_0=\partial_{\theta_0}\overline{H}_0=0, \tag{10d}$$

where the function $2\overline{H}_0$ is the wave Hamiltonian H evaluated on the trial function (4); the last two equations of the system (10a)–(10d) arise because p_0 and μ_0 are conserved quantities.

Next consider the superposition of solitary waves

$$\psi=\sum_k \psi_k(z,t) \tag{11}$$

with

$$\psi_k=e^{i\theta_k}\Psi(t-t_k;p_k,\mu_k) \tag{12}$$

with arbitrary z-dependent parameters p_k,μ_k,θ_k,t_k for each solitary wave. When the solitary waves are widely separated in time, so that $|t_{k+1}-t_k|\sim O(T)$ for $T\to\infty$, Eq. (11) is almost a solution of Eq. (1), and in the limit of infinitely large separations the superposition is an exact solution when each set of parameters for each index k satisfies Eq. (10) (with the subscript 0 replaced by k), on account of the exponential decay of the individidual solitary waves. The limit of infinite separation of the solitary waves is therefore a degenerate case. In order to improve the consistency of the solitary-wave superposition with the Hamiltonian wave equation for the case of finite but large separation between the solitons, the superposition (11) is used as a trial function in the Lagrangian variational principle to determine the functional variations of the parameters with z that best fit the evolution equation.

When the trial function (11) is substituted in Eq. (7) for the field ψ, and integrations over t carried out, the reduced Lagrangian has the form

$$\overline{L}=2\left\{\sum_{k\in\mathbb{Z}}(p_k d_z t_k-\mu_k d_z\theta_k)-\overline{H}\right\}, \tag{13}$$

where $\overline{H}$ is a function of all the parameters (see Sec. IV below). Variations are taken with respect to the parameters p_k,μ_k,θ_k,t_k, and the resulting Euler-Lagrange equations are a Hamiltonian set of discrete differential equations for these parameters, having the form

$$d_z t_k=\partial_{p_k}\overline{H}, \tag{14a}$$

$$-d_z\theta_k=\partial_{\mu_k}\overline{H}, \tag{14b}$$

$$-d_z p_k=\partial_{t_k}\overline{H}, \tag{14c}$$

$$d_z\mu_k=\partial_{\theta_k}\overline{H}. \tag{14d}$$

The Hamiltonian equations (14a)–(14d) admit the *complex* variables

$$s_k=\eta t_k+\mathrm{i}\theta_k, \tag{15a}$$

$$\sigma_k=p_k/\eta+\mathrm{i}\mu_k, \tag{15b}$$

where η is a real constant which will later be identified with an average asymptotic decay rate of the solitary pulses, provided that the Hamiltonian $\overline{H}$ is the real part of a holomorphic function of the complex variables. Under these conditions the Hamiltonian equations (14a)–(14d) are the real and imaginary parts of

$$d_z s_k=\partial_{\sigma_k}\overline{H}, \tag{16a}$$

$$-d_z\sigma_k=\partial_{s_k}\overline{H} \tag{16b}$$

with real-valued $\overline{H}$.

IV. CALCULATION OF THE AVERAGE LAGRANGIAN

The calculations necessary to obtain the average Lagrangian $\overline{L}$ and the Hamiltonian function $\overline{H}$ are given essentially by Gorshkov-Ostrovsky [7]. Here we summarize their execution in the context of the interacting pulse train. The analysis is here further simplified by introducing asymptotic scalings to select only the dominant terms, neglecting any which are asymptotically smaller than the dominant terms. The scalings are based on the idea that the pulse interactions are very weak if the pulses are widely separated in time. Hence we introduce the order parameter by $|t_{k+1}-t_k|=O(T)$ for large T, and a small parameter ε such that the parameters of the solitary wave depend on the scaled distance εz; this makes first derivatives with respect to z ordered at $O(\varepsilon)$ for $\varepsilon\to 0$. It is further required that $\varepsilon\to 0$ as $T\to\infty$; it transpires later that $\varepsilon=e^{-\overline{\eta}T/2}$, where $\overline{\eta}$ is the average asymptotic decay rate of the solitary pulses. It follows from this that the magnitude of the tails of the pulse centred on $t=t_k$ at the centers of its nearest neighbors $t=t_{k\pm 1}$ is $O(\varepsilon^2)$, and the effect at the centers of pulses $t=t_{k\pm j}$ is $O(\varepsilon^{2j})$. This justifies retaining only nearest-neighbor interactions between solitons.

A. Dynamical part of the Lagrangian

The dynamical part of the Lagrangian is the first term of Eq. (7), and is evaluated by substituting the trial function (11), interchanging the order of integration and summations, and integrating over the variable t, with the result

$$\overline{L_1} = -\mathrm{Im}\int_{-\infty}^{\infty}\psi^*\partial_z\psi dt$$
$$= -\mathrm{Im}\int_{-\infty}^{\infty}\sum_{k_1,k_2}\psi_{k_1}^*\partial_z\psi_{k_2}dt$$
$$= -\mathrm{Im}\sum_{k_1=k_2}\int_{-\infty}^{\infty}\psi_{k_1}^*\partial_z\psi_{k_2}dt$$
$$-\mathrm{Im}\sum_{k_1\neq k_2}\int_{-\infty}^{\infty}\psi_{k_1}^*\partial_z\psi_{k_2}dt. \tag{17}$$

The first term in $\overline{L_1}$ represents a summation over 1-pulse contributions which persist in the limit of infinite separation between the pulses; the second term represents contributions due to 2-pulse interactions, and tends to vanish when the pulses become infinitely widely separated. The result for the first term consisting of 1-pulse contributions is

$$\overline{L_{11}} = -\mathrm{Im}\sum_k\int_{-\infty}^{\infty}\psi_k^*[i(d_z\theta_k)\psi_k-(d_zt_k)(\partial_t\psi_k)]dt$$
$$=2\sum_k(p_kd_zt_k-\mu_kd_z\theta_k). \tag{18}$$

Corrections to the 1-pulse Lagrangian to account for the 2-pulse interactions can be expressed in terms of modified parameters, so that the entire dynamical part $\overline{L_1}$ has the form

$$\overline{L_1}=2\sum_k(p_k'd_zt_k-\mu_k'd_z\theta_k), \tag{19}$$

where

$$\mu_k'=\mu_k+\frac{1}{2}\sum_{j\neq k}\int_{-\infty}^{\infty}\psi_j^*\psi_kdt, \tag{20a}$$

$$p_k'=p_k+\frac{1}{2}\,\mathrm{Im}\sum_{j\neq k}\int_{-\infty}^{\infty}\psi_j^*\partial_t\psi_kdt. \tag{20b}$$

It will be shown later that under the asymptotic scaling in force here the second terms on the right-hand sides of Eqs. (20a) and (20b) are negligible, and $\mu_k'\sim\mu_k$ and $p_k'\sim p_k$.

B. Hamiltonian part of the Lagrangian

The Hamiltonian part of the Lagrangian $\overline{H}$ also decomposes into two parts, one which persists when the pulses are infinitely separated and one which vanishes when the separations are increased. This decomposition is written as

$$\overline{H}=\sum_k\overline{H}_{0k}+\overline{V}, \tag{21}$$

where $\overline{V}\to 0$ as $\varepsilon\to 0$, and $\overline{H}_{0k}$ is the Hamiltonian of a single solitary pulse $\psi=\psi_k$.

1. 1-pulse self-terms

The term $\overline{H}_{0k}(p_k,\mu_k)$ is the residual Hamiltonian for an isolated solitary pulse at $t=t_k$ after all the others have been removed to infinity to reduce the interaction to zero. In general the Hamiltonian $\overline{H}_{0k}$ may be a rather complicated function of the parameters of each solitary pulse. However, under the conditions which are of interest here, of nearly identical solitary pulses with large separations in time, it can be approximated in a quite general way. Introduce the asymptotic scalings $p_k\sim\overline{p}+O(\varepsilon)$, $\mu_k\sim\overline{\mu}+O(\varepsilon)$, where $\overline{p}=N^{-1}\Sigma_{k=1}^N p_k$ and $\overline{\mu}=N^{-1}\Sigma_{k=1}^N\mu_k$ are mean values over the N solitary pulses (with a limit taken if $N\to\infty$) and ε is a small quantity. It is shown later that the average quantities $\overline{p}$ and $\overline{\mu}$ are conserved by the dynamics of the lattice. With this choice of reference, the Taylor expansion of the unperturbed part of $\overline{H}$, $\Sigma_k\overline{H}_{0k}$, is

$$\sum_k\overline{H}_{0k}(p_k,\mu_k)\sim N\overline{H}_0(\overline{p},\overline{\mu})+\sum_k\{\tfrac{1}{2}\overline{H}_{0pp}(p_k-\overline{p})^2$$
$$+\tfrac{1}{2}\overline{H}_{0\mu\mu}(\mu_k-\overline{\mu})^2$$
$$+\overline{H}_{0p\mu}(p_k-\overline{p})(\mu_k-\overline{\mu})\}+O(\varepsilon^3), \tag{22}$$

where

$$H_{0pp}=\partial_p^2\overline{H}_0(p,\mu)|_{p=\overline{p},\mu=\overline{\mu}}, \tag{23a}$$

$$H_{0\mu\mu}=\partial_\mu^2\overline{H}_0(p,\mu)|_{p=\overline{p},\mu=\overline{\mu}}, \tag{23b}$$

$$H_{0p\mu}=\partial_p\partial_\mu\overline{H}_0(p,\mu)|_{p=\overline{p},\mu=\overline{\mu}}. \tag{23c}$$

Here the linear parts of the Taylor expansion have summed to zero over the sum variable k, and the leading-order variation in $\overline{H}_0$ is therefore quadratic in the perturbations and ordered at $O(\varepsilon^2)$. For the purpose of later calculations, we here define the parameters

$$\rho_\pm=\tfrac{1}{2}(\overline{\eta}^2H_{0pp}\pm H_{0\mu\mu}), \tag{24a}$$

$$\rho_0=\overline{\eta}H_{0p\mu}. \tag{24b}$$

The first term in the Taylor expansion (22), $N\overline{H}_0(\overline{p},\overline{\mu})$, is independent of the dynamical variables, and can therefore be subtracted off the Hamiltonian with no effect on the dynamics. This also deals with the potential problem of nonfiniteness of this term when the number of lattice sites N becomes infinite.

The second-order terms in the Taylor expansion (22) can be transformed using Eqs. (24a) and (24b) to the expression

$$\tfrac{1}{2}\rho_-(\overline{\eta}^{-2}\Delta p_k^2-\Delta\mu_k^2)+\rho_0\overline{\eta}^{-1}\Delta p_k\Delta\mu_k$$
$$+\tfrac{1}{2}\rho_+(\overline{\eta}^{-2}\Delta p_k^2+\Delta\mu_k^2),$$

where $\Delta p_k = p_k - \bar{p}$ and $\Delta\mu_k = \mu_k - \bar{\mu}$. If the complex variable $\sigma_k = \bar{\eta}^{-1}\Delta p_k + \mathrm{i}\Delta\mu_k$ is introduced, then the first two terms of this expression are

$$\tfrac{1}{2}\rho_-(\bar{\eta}^{-2}\Delta p_k^2 - \Delta\mu_k^2) + \rho_0\bar{\eta}^{-1}\Delta p_k \Delta\mu_k = \tfrac{1}{2}\,\mathrm{Re}[\sigma_k^2(\rho_- - \mathrm{i}\rho_0)]$$

from which it follows that if and only if $\rho_+ = 0$ then the second-order part of the Hamiltonian (22) is the real part of a holomorphic function of the complex variables σ_k. Although the parameter ρ_0 has been included here for mathematical completeness, when time reversal symmetry applies, it generally happens that $\rho_0 = 0$.

2. 2-pulse interaction terms

If the original wave Hamiltonian H is polynomial in ψ, its t derivatives and the corresponding complex conjugates, as assumed at the outset, then $\bar{V}$ will be a superposition of terms of the form

$$\Delta\bar{V}_{n,r} = C_{n,r}\sum_{k_1,\ldots,k_n}{}' \int_{-\infty}^{\infty}\phi_{k_1}^{(1)*}\cdots\phi_{k_r}^{(r)*}\phi_{k_{r+1}}^{(r+1)}\cdots\phi_{k_n}^{(n)}dt, \tag{25}$$

where $\phi_k^{(j)}$ denotes ψ_k or any of its t derivatives, $C_{n,r}$ are complex constants, and $1 \leqslant r \leqslant n$. The prime on the summation denotes the omission of the term for which $k_1 = k_2 = \cdots = k_n$. In the case where the original wave field ψ is a scalar, then the phase invariance property of the original Hamiltonian requires that n be an even integer, and $r = \frac{1}{2}n$. The functions $\phi_k^{(j)}$ are all localized in t, around $t = t_k$, and decay exponentially rapidly for large $|t - t_k|$ with a decay rate $\eta_k \sim \bar{\eta}$. It follows that the dominant $\Delta\bar{V}_{n,r}$ for $n > 2$ are those for which $n-1$ of the n indices $k_1, \ldots, k_n$ are identical and only one is different from the rest. Hence we can express the dominant integrals in the form

$$\Delta\bar{V}_n \sim C_n\sum_{k_1,k_2}{}' \int_{-\infty}^{\infty}\phi_{k_1}^*\Phi_{k_2}dt + C_n^*\sum_{k_1,k_2}{}' \int_{-\infty}^{\infty}\phi_{k_1}\Phi_{k_2}^*dt, \tag{26}$$

where ϕ stands for ψ or any of its derivatives, and Φ_k is a function localized around $t = t_k$ and decaying exponentially with an approximate decay rate of $(n-1)\bar{\eta}$, since Φ_k is composed of a product of $n-1$ functions each decaying with an approximate decay rate $\bar{\eta}$. The integrals are then evaluated approximately by substituting for ϕ_{k_1} its exponential approximation near $t = t_{k_2}$

$$\phi_{k_1} \sim A_+ e^{-\eta_{k_1}(t - t_{k_1}) + i\theta_{k_1}} \quad (t_{k_2} \gg t_{k_1}) \tag{27a}$$

$$\sim A_- e^{+\eta_{k_1}(t - t_{k_1}) + i\theta_{k_1}} \quad (t_{k_2} \ll t_{k_1}), \tag{27b}$$

where $A_\pm(p_{k_1}, \mu_{k_1})$ are complex numbers. The integral over t is now carried out, noting that the integration variable t can be replaced by $t - t_{k_2}$. The end result of the integration is that the dominant contribution to $\bar{V}$ has the form

$$\bar{V} = \sum_k \{V_1(p_k, p_{k+1}, \mu_k, \mu_{k+1})e^{-\eta_k(t_{k+1}-t_k)}e^{-i(\theta_{k+1}-\theta_k)} + V_2(p_k, p_{k+1}, \mu_k, \mu_{k+1})e^{-\eta_{k+1}(t_{k+1}-t_k)}e^{i(\theta_{k+1}-\theta_k)}\}. \tag{28}$$

The two coefficients V_1 and V_2 absorb all the scale factors and t integrals.

If all the solitary wave parameters are nearly identical, and we *choose* the asymptotic scale parameter to be $\varepsilon = e^{-\bar{\eta}T/2}$, where $T = \min_k|t_{k+1} - t_k|$, then all the terms in $\bar{V}$ are already $O(\varepsilon^2)$, and the k-dependent parameters appearing as arguments of $V_{1,2}$ can be replaced by their average values, simplifying $\bar{V}$ to

$$\bar{V} \sim -\sum_k e^{-\bar{\eta}(t_{k+1}-t_k)}\mathrm{Re}\{K(\bar{p},\bar{\mu})e^{-i(\theta_{k+1}-\theta_k)}\} \tag{29}$$

with $\bar{\eta} = N^{-1}\Sigma_{k=1}^N \eta_k$. Here $K(\bar{p},\bar{\mu})$ is a k-independent constant; this constant may be complex if the solitary wave function Ψ is chirped. The negative sign of the interaction potential in Eq. (29) has been chosen purely for convenience later. Now both parts of $\bar{H}$, Eqs. (22) and (29), are $O(\varepsilon^2)$ in the variations about the mean values of the parameters.

The results (22) and (29) can be used to show that the average values $\bar{p}$ and $\bar{\mu}$ are conserved by the dynamics. The only part of the Hamiltonian $\bar{H}$ dependent on the variables t_k and θ_k is the interaction potential $\bar{V}$, up to the order $O(\varepsilon^2)$. It follows from the last two equations of (14a)–(14d) that

$$d_z\bar{p} = N^{-1}\sum_{k=1}^N d_z p_k = -N^{-1}\sum_{k=1}^N \partial_{t_k}\bar{V} = 0, \tag{30}$$

$$d_z\bar{\mu} = N^{-1}\sum_{k=1}^N d_z\mu_k = N^{-1}\sum_{k=1}^N \partial_{\theta_k}\bar{V} = 0. \tag{31}$$

3. Exceptional case: n=2

The interaction integrals with only two functions $\phi^{(j)}$ in the integrand are special cases, leading to secular asymptotics with respect to the small parameter ε at leading order, of the form $O(\varepsilon^2\ln\varepsilon)$. The principal types of these integrals are reducible to

$$I_{jk}^{(1)} = \int_{-\infty}^{\infty}\psi_j^*\psi_k dt, \tag{32a}$$

$$I_{jk}^{(2)} = i\int_{-\infty}^{\infty}(\psi_j^*\partial_t\psi_k - \psi_k\partial_t\psi_j^*)dt, \tag{32b}$$

$$I_{jk}^{(3)} = \int_{-\infty}^{\infty}\partial_t\psi_j^*\partial_t\psi_k dt. \tag{32c}$$

It is not possible here to replace one of the factors of the integrand by an exponential approximation, as the resulting integral may not converge. If each ψ_k is a solitary wave with exponential decay, and the two functions have the same decay rates, then in the time interval lying between the two pulse centers $t_k < t < t_{k+1}$ one pulse is exponentially decreasing and the other is exponentially increasing, so their product is constant. Integrating this constant over the range of separation of the two pulses leads to a factor $O(T)\sim O(\ln\varepsilon)$ in the asymptotic value of the integral. Further, since this integral arises from a region at the midpoint of which both factors in the integrand are themselves exponentially small, the integral also has a factor $O(e^{-\bar{\eta}T})\sim O(\varepsilon^2)$. These two asymptotic factors in composition give a total asymptotic order of $O(\varepsilon^2\ln\varepsilon)$ for the integrals (32a)–(32c).

This estimate can be applied directly to the remainder terms in the modified invariants of Eqs. (20a) and (20b), and the asymptotic estimates

$$\mu_k' = \mu_k + O(\varepsilon^2\ln\varepsilon), \tag{33}$$

$$p_k' = p_k + O(\varepsilon^2\ln\varepsilon) \tag{34}$$

for $\varepsilon\to 0$ are thereby established, as claimed after Eqs. (20a) and(20b).

C. Canonical Hamiltonian

After discarding all terms which are to be neglected under these rules, the result for the reduced Hamiltonian $\bar{H}$ has the form

$$\bar{H} = \sum_{k\in\mathbb{Z}} \{\bar{H}_{0k} - \mathrm{Re}[Ke^{-\bar{\eta}(t_{k+1}-t_k)}e^{-i(\theta_{k+1}-\theta_k)}]\} \tag{35}$$

with a single k-independent complex constant K.

It is evident from Eq. (35) that the complex variables $s_k = \bar{\eta}t_k + i\theta_k$ can be introduced in the exponentials, and the Hamiltonian is the real part of a holomorphic function of these variables. Under certain conditions, noted after Eqs. (15a) and (15b), the Hamiltonian may also be the real part of a holomorphic function of the complex conjugate momenta, $\sigma_k = \bar{\eta}^{-1}p_k + i\mu_k$; this requires that

$$\bar{H}_{0\mu\mu} = -\bar{\eta}^2\bar{H}_{0pp} \tag{36}$$

for each isolated solitary pulse, which follows directly from the Cauchy-Riemann equations applied to each of the complex variables σ_k. In the case where the original Hamiltonian wave equation is the NSE [from Eq. (2b) in Sec. II above] this condition is satisfied [3], and the resulting discrete dynamical system is the complex Toda Lattice, which is integrable by the inverse scattering transform [3,5]. When the PDE is nonintegrable then the holomorphicity condition (36) may not be satisfied, and the resulting discrete dynamical system is not a complex Toda lattice; we refer to such a system as a double Toda lattice, since it has two degrees of freedom at each lattice point, parametrized by t_k and θ_k. An interesting conjecture to explore would be that it is precisely the integrable PDEs that reduce to the holomorphic complex Toda lattice system, but this is not established so far.

Several scaling transformations can be made on the general system which, in combination, reduce it to a canonical form. By means of transformations

$$\Psi\to e^{-i\omega t}\Psi, \quad t_k\to t_k-\omega z \tag{37a}$$

for some constant ω the system can be reduced to one with $\bar{p}=0$. By scaling the time variable

$$\bar{\eta}t_k\to t_k, \quad \bar{\eta}^{-1}p_k\to p_k \tag{37b}$$

the system can be reduced to one with $\bar{\eta}=1$. By scaling the distance

$$\rho_- z\to z, \quad K\to\rho_- K, \quad \rho_+/\rho_- = \rho \tag{37c}$$

the coefficient ρ_- can be reduced to $\rho_-=1$. After a second length scaling

$$\lambda z\to z, \quad p_k\to\lambda p_k, \quad \nu_k\to\lambda\nu_k, \quad K\to\lambda^2 K \tag{37d}$$

the system reduces to one with arbitrary $|K|$. The phase transformation

$$\theta_k\to\theta_k + k\arg K \tag{37e}$$

reduces the system to one with $\arg K=0$. The net effect of all these transformations is to bring the general nonholomorphic Hamiltonian dynamical system resulting from the perturbation theory to a canonical form with $\bar{p}=0$, $\bar{\eta}=1$, $\rho_-=1$, $\arg K=0$, having the general form

$$\bar{H} = \sum_{k\in\mathbb{Z}} \{\tfrac{1}{2}\rho(p_k^2+\nu_k^2) + \mathrm{Re}[\tfrac{1}{2}(p_k+i\nu_k)^2 - Ke^{-(t_{k+1}-t_k)}e^{-i(\theta_{k+1}-\theta_k)}]\} \tag{38}$$

for conjugate variable pairs (t_k,p_k) and (θ_k,ν_k) with $\mu_k = \bar{\mu}+\nu_k$, ρ a constant. K is an arbitrary real positive constant; $K=4$ turns out to be a convenient value. Here we have set $\rho_0=0$ assuming time-reversal symmetry of the wave Hamiltonian, but the theory is easily extended for the case $\rho_0\neq 0$.

If $\rho=0$ in Eq. (38) then the complex variables $s_k=t_k+i\theta_k$, $\sigma_k=p_k+i\nu_k$ can be introduced, and the real Hamiltonian system (14a)–(14d) is compatible with the complex Hamiltonian system (16a) and (16b). [It is unimportant whether the momentum conjugate to θ_k is defined as ν_k or $\mu_k=\bar{\mu}+\nu_k$ because both give rise to the same Hamiltonian system (14a)–(14d) when $\bar{\mu}$ is a constant.] If, on the other hand, $\rho\neq 0$, then there are no complex variables for which the two systems are compatible. It is noteworthy that the only remnant in the canonical system (38) of the parameters of the original wave system is the single dimensionless real parameter ρ, which represents the deviation of the canonical system from holomorphicity. The holomorphic case, leading to the integrable complex Toda lattice, is $\rho=0$, and arises when the original PDE is the NSE. It is not known at present if the general nonholomorphic Toda lattice system is integrable; however, it is a significant conclusion of the theory described here that the only possible deviation from the holomorphic case is the perturbation $\Sigma_{k\in\mathbb{Z}}\frac{1}{2}\rho(p_k^2+\nu_k^2)$ in the Hamiltonian of the discrete dynamical system.

V. EXAMPLE: CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION

We consider now the application of this theory to the perturbed NSE, whose Hamiltonian is given by Eq. (3a). We choose the perturbation

$$F=-\tfrac{2}{3}a|\psi|^6, \tag{39}$$

where a is a real constant. Physically relevant cases occur commonly when $a<0$, representing the effect of a saturating nonlinearity [9,10]. The Euler-Lagrange equation resulting from the variations of the Lagrangian with respect to the field ψ is

$$i\partial_z\psi+\tfrac{1}{2}\partial_t^2\psi+|\psi|^2\psi+a|\psi|^4\psi=0, \tag{40}$$

which is generally known as the *cubic-quintic* NSE. Equation (40) has solitary wave solutions obtained by the method outlined in Eq. (4) [11], of the form

$$\Psi(t;p,\mu)=\eta e^{-i\omega t}\frac{\operatorname{sech}(\eta t)}{\{(1-2b)+b\operatorname{sech}^2(\eta t)\}^{1/2}}, \tag{41}$$

with the relations

$$2\kappa=\eta^2+\omega^2, \tag{42a}$$

$$c^{-1}=-\omega \tag{42b}$$

required to satisfy Eq. (5). In addition, the constant b satisfies the quadratic equation

$$b(1-b)=\tfrac{2}{3}\eta^2 a. \tag{43}$$

Here we choose the lower branch of b satisfying $2b<1$, which contains $b=0$ at $a=0$, corresponding to the unperturbed NSE. The solitary wave (41) degenerates along this branch of b to the one-soliton of the NSE

$$\Psi=\eta e^{-i\omega t}\operatorname{sech}\eta t$$

when $a=0$.

In the following expressions the *unscaled* parameters, prior to the scalings (37a)–(37e), are computed for this model. Also the subscript 0 on the dynamical variables is dropped, for better appearance of the complicated expressions. The mass μ and momentum p are given for $b>0$, corresponding to $a>0$, by

$$\mu=\frac{1}{2}\int_{-\infty}^{\infty}|\Psi|^2dt=\eta\frac{1}{\sqrt{g}}\operatorname{arctanh}\sqrt{g}, \tag{44a}$$

$$p=\frac{1}{4}i\int_{-\infty}^{\infty}(\Psi\partial_t\Psi^*-\Psi^*\partial_t\Psi)dt=-\omega\mu, \tag{44b}$$

$$\overline{H}_0=\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{2}(|\partial_t\Psi|^2-|\Psi|^4-\tfrac{2}{3}a|\Psi|^6)dt$$
$$=-\frac{\mu^3}{2g'}\left(1-\frac{1}{\sqrt{g'}}\tanh\sqrt{g'}\right)+\frac{1}{2\mu}p^2, \tag{44c}$$

where $g=\frac{8}{3}a\eta^2$ and $g'=\frac{8}{3}a\mu^2$, with analytic continuations in the case $a<0$.

The Cauchy-Riemann factors in Eq. (36) can be computed from Eq. (44c) as

$$-\eta^2\partial_p^2\overline{H}_0(p,\mu)=-\mu\left(\frac{\tanh\sqrt{g'}}{\sqrt{g'}}\right)^2, \tag{45a}$$

$$\partial_\mu^2\overline{H}_0(0,\mu)=-\mu\left(\frac{\tanh\sqrt{g'}}{\sqrt{g'}}\right)\operatorname{sech}^2\sqrt{g'} \tag{45b}$$

from which it follows that the Cauchy-Riemann condition (36) can only be satisfied for $a=0\,(g'=0)$. This is precisely the case for which the NSE perturbation vanishes.

The coefficient K in the potential $\overline{V}$ can also be computed analytically for this example. The result is

$$K(p,\mu)=4\eta^3(I_1+a\eta^2I_2), \tag{46}$$

where

$$I_n=\frac{1}{\sqrt{1-2b}}\int_{-\infty}^{\infty}e^{-t}\frac{\operatorname{sech}^{2n+1}t}{(1-2b+b\operatorname{sech}^2t)^{n+1/2}}dt. \tag{47}$$

These integrals can in turn be transformed to

$$I_n=2\int_0^{\infty}\frac{u^{n-1}}{\{(u+1)^2-g\}^{n+1/2}}du, \tag{48}$$

which evaluate to

$$I_1=\frac{2}{g}\{(1-g)^{-1/2}-1\}, \tag{49a}$$

$$I_2=\frac{2}{3}(1-g)^{-3/2}\left\{1+\frac{4}{g^2}\left[1-\frac{3g}{2}-(1-g)^{3/2}\right]\right\}. \tag{49b}$$

Here the coefficient is expressed explicitly as a function of the decay rate η, but Eq. (44a) permits a direct relationship between η and μ

$$\eta=\mu\frac{1}{\sqrt{g'}}\tanh\sqrt{g'}. \tag{50}$$

These expressions degenerate when $a=0\,(b=0, g=0, g'=0)$ to

$$\mu=\eta, \tag{51a}$$

$$p=-\omega\eta, \tag{51b}$$

$$\overline{H}_0(p,\mu)=-\frac{1}{6}\eta^3+\frac{p^2}{2\eta}, \tag{51c}$$

$$K(p,\mu)=4\eta^3, \tag{51d}$$

which are already known from previous studies of the NSE [3,12].

VI. STABILITY OF AN INFINITE QUASIPERIODIC TRAIN

We return now to the scaled canonical dynamical lattice (38), and investigate the stability of its stationary states. After eliminating the conjugate momenta p_k and μ_k, the remaining variables t_k and θ_k satisfy the differential equations

$$d_z^2 t_k = (1+\rho)\mathrm{Re}\{K[e^{-(t_{k+1}-t_k)}e^{-i(\theta_{k+1}-\theta_k)} - e^{-(t_k-t_{k-1})}e^{-i(\theta_k-\theta_{k-1})}]\}, \tag{52a}$$

$$d_z^2 \theta_k = (1-\rho)\mathrm{Im}\{K[e^{-(t_{k+1}-t_k)}e^{-i(\theta_{k+1}-\theta_k)} - e^{-(t_k-t_{k-1})}e^{-i(\theta_k-\theta_{k-1})}]\}. \tag{52b}$$

A stationary solution, for which $d_z^2 t_k = 0$, $d_z^2\theta_k = 0$ for all $k \in \mathbb{Z}$, is given by

$$e^{-(t_{k+1}-t_k)}e^{-i(\theta_{k+1}-\theta_k)} - e^{-(t_{k+1}-t_k)}e^{-i(\theta_k-\theta_{k-1})} = 0, \tag{53}$$

which reduces to

$$s_{k+1} - 2s_k + s_{k-1} = 0, \tag{54}$$

where $s_k = t_k + i\theta_k$. The solution of this difference equation is

$$s_k = Ak + B \tag{55}$$

with arbitrary complex coefficients A and B. This means that the particles are placed at periodic points $t_k = kT$ with a uniform phase increment $\theta_k = k\alpha$, where $T = \mathrm{Re}\,A$ and $\alpha = \mathrm{Im}\,A$. The constant B can be set to zero without loss of generality.

The stability of these stationary states can be determined by linearizing the system of equations (52a) and (52b) about its fixed points. Let $t_k = kT + q_k$ and $\theta_k = k\alpha + \delta_k$, where q_k and δ_k are assumed to be infinitesimally small. Introduce complex variables

$$\xi_k = q_k + i\delta_k \tag{56}$$

and search for solutions of the form

$$\xi_k = \Xi_+ e^{ik\beta}e^{i\lambda z} + \Xi_- e^{-ik\beta}e^{-i\lambda^* z} \tag{57}$$

with complex amplitudes $\Xi_\pm$ and complex eigenvalue λ to be determined. Here $-\pi \leq \beta \leq \pi$ is the Floquet phase of the particular Fourier mode, to be distinguished from the mean interparticle phase α of the stationary lattice. Solving the linear algebraic system that results from this substitution yields the eigenvalue

$$\lambda^2 = -\lambda_0^2 \sin(\beta/2)(\cos\alpha \pm \sqrt{\rho^2 - \sin^2\alpha}) \tag{58}$$

with $\lambda_0 = 2\sqrt{|K|}e^{-T/2}$. Stability of the lattice requires that $\mathrm{Im}\,\lambda = 0$ for all four values of λ corresponding to a particular β, over all values of $-\pi \leq \beta \leq \pi$. This leads to the condition

$$\pi - |\arcsin\rho| \leq \alpha \leq \pi + |\arcsin\rho|. \tag{59}$$

The result (59) is very significant. When the original Hamiltonian wave equation is the NSE [example (2b) in Sec. II], then we have $\rho = 0$, leading from Eq. (59) to $\alpha = \pi$ as the only interparticle (intersoliton) phase for which the stationary lattice is stable. This result has been derived several times before [1,4]. However, when the NSE Hamiltonian (2b) is deformed by a perturbation, as for example occurs in the Hamiltonian for the cubic-quintic NSE Eq. (3a) with Eq. (39), then $\rho \neq 0$ and a band of stable interparticle phases may open around the single value $\alpha = \pi$, rendering the lattice of nonintegrable solitary pulses more stable than the lattice of integrable solitons.

VII. CONCLUSIONS

In conclusion, it is shown that the dynamics of multiple solitary waves of a class of nonintegrable Hamiltonian wave systems can be reduced to a nonholomorphic version of the complex Toda lattice, generalizing recent results [3–6] for solitons of an integrable wave system. This result applies in the limit of nearly identical solitary pulses separated by large intervals of time, which happens to be a case of particular practical interest in applications of nonlinear optical pulses to communication systems. This lattice model has been used to determine the stability of quasiperiodic lattices, which are stationary states. The analysis given here assumes the stability of the individual solitary wave to be guaranteed, which may not actually be the case for the solitary waves of some Hamiltonians.

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