

Synchrotron radiation of crystallized beams

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We study the modifications of synchrotron radiation of charges in a storage ring as they are cooled. The pair correlation lengths between the charges are manifest in the synchrotron radiation and coherence effects exist for wavelengths longer than the coherence lengths between the charges. Therefore, the synchrotron radiation can be used as a diagnostic tool to determine the state (gas, liquid, crystal) of the charged plasma in the storage ring. We show also that the total power of the synchrotron radiation is significantly reduced for crystallized beams, both coasting and bunched. This opens the possibility of accelerating particles to ultrarelativistic energies using small-sized cyclic accelerators. [S1063-651X(99)07407-3]

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I. INTRODUCTION

Ion-beam crystallization is an exciting and relatively new field of physics in which a new state of matter is sought. Namely, ions which rapidly circulate in a storage ring and are cooled are expected to form geometrically ordered structures (crystals) which have a density much smaller than normal crystalline solids [1,2]. Although great effort is currently invested in achieving such crystals [2–5], there is still no clear-cut experimental evidence for them. It is hoped, however, that crystalline beams will be produced in the near future.

Synchrotron radiation, on the other hand, is a very well-established field of physics that has been investigated continuously from the early days of particle accelerators. Many synchrotron sources are operating around the world [e.g., DESY (Hamburg, Germany), NSLS (Brookhaven, USA), KEK (Tsukuba, Japan)], and many applications already exist [6].

It is the purpose of this paper to establish a link between beam crystallization and synchrotron radiation. This link is twofold:

(i) To use synchrotron radiation and modifications thereof in order to *detect* the creation and existence of beam crystals. This is required since for fast beams direct detection methods are difficult to implement [2]. Thus synchrotron radiation can be used as an indirect diagnostic method to detect the formation of beam crystals. The diagnostic methods discussed below are also applicable to liquid and gaseous beams.

(ii) Even more importantly, once beam crystals are formed, they can be used to *modify* the synchrotron radiation with respect to the ordinary incoherent case. Therefore, one can achieve dramatic suppression and enhancement effects of synchrotron radiation using crystallized beams [7]. In particular, the total power that is radiated from an equispaced circulating chain of particles is much smaller (in the appro-

priate limits [7]) than the radiation from the same number of randomly located particles. This opens the possibility for accelerating particles to ultrarelativistic energies with little radiation loss, which is currently the main limitation of circular electron accelerators. Thus the suppression of synchrotron radiation by beam crystallization may eventually lead to the construction of smaller-sized circular electron accelerators.

In the following we shall detail the connection between beam crystals and synchrotron radiation. It is important to emphasize that currently researchers are trying to obtain beam crystals of heavy ions that can be cooled with electrons and lasers. For heavy ions, however, the synchrotron radiation is small. Even for protons, e.g., the lightest of the “heavy ions,” the synchrotron radiation is about a factor of 10^{-13} smaller than for electrons with the same energy. Thus, we expect that realistically the effects predicted in this paper will be important only for liquid or crystallized electron beams. This, however, poses the challenge of obtaining crystallized electron beams. Thus we hope that the ideas put forward in this paper will motivate experimentalists to work towards obtaining crystallized electron beams. In any case, however, the analysis presented below applies to any species of charged particles. Thus the theory can in principle be verified for ion-beam crystals. Also we stress from the outset that the effects discussed below go beyond what is known as “coherent synchrotron radiation” which is the coherent enhancement of synchrotron radiation of small electron bunches for wavelengths that are longer than the bunch size (see, e.g., [8–11] and Sec. IV below).

The paper is structured in the following way. In Sec. II we discuss the general theory that underlies the suppression and enhancement effects of synchrotron radiation. In Sec. III the theory is applied to the three phases of a coasting charged-particle beam that occur in practice: gaseous, liquid, and crystal. In Sec. IV the necessary modifications of the theory for a bunched beam are discussed. In Sec. V we present analytical and numerical results concerning the suppression of synchrotron radiation by crystalline beams, both coasting and bunched. Finite temperature effects are discussed explicitly. In Sec. VI we discuss our results and conclude the paper

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with proposals for experimental applications of the effects discussed in this paper.

II. GENERAL THEORY

We consider N charged particles with charge q circulating in a circular storage ring of radius ρ with velocity v . The charges are assumed to be coherent with a reference circulating charge, but are allowed to have constant time lags Δt_j as well as constant spatial displacements $\Delta \vec{r}_j$ from the reference orbit. According to the theory of radiation of moving sources, the total power that is emitted from the N charges is given by [12,13]

$$I^{(N)} = \sum_{n=1}^{\infty} g_n I_n^{(1)}, \quad (2.1)$$

where $I_n^{(1)}$ is the power that is emitted with frequency $\omega_n \equiv n\omega = nv/\rho$ due to a single circulating particle, and g_n is the form factor of the beam. The explicit expression for $I_n^{(1)}$ is [14]

$$I_n^{(1)} = \frac{q^2 c \beta}{2\pi \epsilon_0 \gamma^2 \rho^2} \left[\beta^2 \gamma^2 n J'_{2n}(2n\beta) - n^2 \int_0^\beta J_{2n}(2n\xi) d\xi \right], \quad (2.2)$$

where $\beta \equiv v/c$, c is the speed of light, $\gamma \equiv 1/\sqrt{1-\beta^2}$, and J_n are the ordinary Bessel functions [15]. The form factor is given by

$$g_n = \left| \sum_{j=1}^N \exp(in\phi_j) \right|^2 = \sum_{j,j'=1}^N \exp[in(\phi_j - \phi_{j'})] \quad (2.3)$$

and the angles ϕ_j are given by [16]

$$\phi_j = \omega \Delta t_j + \frac{\beta}{\rho} \vec{n} \cdot \Delta \vec{r}_j, \quad (2.4)$$

where \vec{n} is the unit vector pointing from the center of the ring to the observation point. We observe that in the form factor, the role of the time delays and the spatial displacements is equivalent. Thus we restrict ourselves hereafter to time delays only. This simplifies the calculations and yields qualitatively the same results. It is also compatible with the current experimental trend according to which linear ion crystals (one-dimensional crystals) are sought. We shall denote the phase differences in the following by $\theta_j \equiv \omega \Delta t_j$.

Suppose now that we treat the quantities θ_i as random variables distributed according to the normalized probability density $P(\theta_1, \dots, \theta_N)$. Then the expectation value of the total power is

$$\langle I^{(N)} \rangle = \sum_{n=1}^{\infty} \langle g_n \rangle I_n^{(1)}, \quad (2.5)$$

where

$$\begin{aligned} \langle g_n \rangle &= \int_0^{2\pi} d\theta_1 \cdots d\theta_N P(\theta_1, \dots, \theta_N) \\ &\times \sum_{j,j'=1}^N \exp[in(\theta_j - \theta_{j'})]. \end{aligned} \quad (2.6)$$

This can be rewritten as

$$\langle g_n \rangle = \int_{-2\pi}^{+2\pi} d\Delta e^{in\Delta} R_2(\Delta), \quad (2.7)$$

where

$$\begin{aligned} R_2(\Delta) &\equiv \int_0^{2\pi} d\theta_1 \cdots d\theta_N P(\theta_1, \dots, \theta_N) \\ &\times \sum_{j,j'=1}^N \delta[\Delta - (\theta_j - \theta_{j'})] \end{aligned} \quad (2.8)$$

is the two-point correlation function, i.e., the (non-normalized) chance of finding a pair of θ 's a distance Δ apart. Therefore, we conclude that the crucial quantity that determines the modifications of synchrotron radiation due to coherence effects is R_2 , our main object of study. The physics of the particle beam (temperature, structure) is reflected in R_2 and is consequently linked to modifications of the synchrotron radiation.

Before applying the above formulas, we make some further simple manipulations. To avoid complications with the 2π periodicity, we define

$$\hat{R}_2(\Delta) \equiv 2[R_2(\Delta) + R_2(2\pi - \Delta)]. \quad (2.9)$$

Using the relation $R_2(\Delta) = R_2(-\Delta)$, easily derived from Eq. (2.8), we obtain

$$\langle g_n \rangle = \int_0^\pi d\Delta \cos(n\Delta) \hat{R}_2(\Delta). \quad (2.10)$$

This can finally be recast as

$$\langle g_n \rangle = N + \int_0^\pi d\Delta \cos(n\Delta) \tilde{R}_2(\Delta), \quad (2.11)$$

where $\tilde{R}_2(\Delta)$, defined in $[0, \pi]$, is the two-point correlator that does not include the ‘‘diagonal’’ part $2N\delta(\Delta)$, emerging from the $j=j'$ terms of R_2 .

An important special case is the case of independent particles, i.e.,

$$P(\theta_1, \dots, \theta_N) = \prod_{j=1}^N P_1(\theta_j), \quad (2.12)$$

where P_1 is the (normalized) one-point density of the particles. For this case the resulting form factor is

$$\langle g_n \rangle = N + N(N-1) \left| \int_0^{2\pi} d\theta P_1(\theta) e^{in\theta} \right|^2. \quad (2.13)$$

III. APPLICATION TO COOLED PARTICLE BEAMS

In the following we shall study a few representative situations of a particle beam as it is being cooled and crystallized. We shall qualitatively infer the form of the two-point correlator \bar{R}_2 for each of the cases, and calculate the resulting form factor of the synchrotron radiation. Thus the focus in this section is on the spectral modifications of the synchrotron radiation expressed by the behavior of $\langle g_n \rangle$. We shall show that the modifications of $\langle g_n \rangle$ as the temperature is lowered define an excellent tool for the diagnostics of the thermodynamic state of the beam. The modifications due to bunching are considered in Sec. IV. The suppression of the total emitted power is discussed in Sec. V.

We start with a very hot particle beam. In such a case, we expect the particles to be completely independent. Therefore, Eqs. (2.12) and (2.13) apply. For a particle beam that fills the whole ring (coasting beam), we expect on the basis of symmetry a uniform distribution

$$P_1(\theta) = \frac{1}{2\pi}. \quad (3.1)$$

The form factor becomes

$$\langle g_n \rangle = N, \quad n = 1, 2, \dots \quad (3.2)$$

Thus, for a hot coasting beam,

$$\langle I^{(N)} \rangle = NI^{(1)}. \quad (3.3)$$

This is what we expect from totally incoherent radiation of N particles.

For beams that are bunched, a typical shape is a Gaussian. The resulting density is

$$P_1(\theta) = \frac{1}{\sqrt{2\pi}\sigma^2} \sum_{m=-\infty}^{+\infty} \exp\left[-\frac{(\theta - \theta_0 + 2\pi m)^2}{2\sigma^2}\right], \quad (3.4)$$

where θ_0 is the location of the center of the bunch and σ is its angular width. The summation over m is to ensure the 2π periodicity. For $\theta_0 \gg \sigma$, $2\pi - \theta_0 \gg \sigma$, only the $m=0$ component in Eq. (3.4) is significant. The resulting form factor is

$$\langle g_n \rangle = N + N(N-1) \exp(-n^2\sigma^2). \quad (3.5)$$

In the bunched case, therefore, in addition to the incoherent term N , we also obtain a term that represents the coherent synchrotron radiation for low harmonics $n \lesssim 1/\sigma$. One obtains qualitatively the same results for other shapes of the bunch [17,18].

The above results are well-known and form the basis of the field of ‘‘coherent synchrotron radiation’’ in which enhancement of the radiation is predicted [8,12,16,17] and experimentally measured [9–11] due to the collection of the charges (electrons) into small bunches. In our case this ap-

plies to the limiting case of a bunched but very hot beam of particles in which the particles within the bunch are uncorrelated.

In the following we shall introduce the correlations between the particles as the beam is cooled and use the full expressions (2.5)–(2.11) rather than Eq. (2.13). These correlations are neglected in the field of coherent synchrotron radiation since the beam is assumed to be very hot. But the correlations become more and more important as the beam is being cooled. To simplify the treatment, we focus in this section on coasting beams. In Sec. IV we introduce the necessary modifications to describe bunched beams.

If the temperature of the beam is moderately high, we expect that the particles start to show repulsion from each other. That is, they will avoid the vicinity of each other due to the Coulomb repulsion and act like a nonideal ‘‘gas.’’ This can be described phenomenologically by

$$\bar{R}_2^{\text{gas}}(\Delta) = c_1 \bar{R}_2^0(\Delta) \left[1 - \exp\left(-\frac{\Delta^2}{2a^2}\right) \right], \quad (3.6)$$

where

$$\bar{R}_2^0(\Delta) = \frac{N(N-1)}{\pi}, \quad 0 \leq \Delta \leq \pi \quad (3.7)$$

is the trivial two-point correlator for a uniformly coasting beam of independent particles and a is the (angular) ‘‘hard-core’’ scale of repulsion. Interpreting Eq. (3.6), we modified \bar{R}_2^0 by a narrow ‘‘dip’’ of width a near $\Delta=0$ (the ‘‘correlation hole’’) such that $\bar{R}_2(0)=0$ (total repulsion at $\Delta=0$). The constant c_1 is for normalization. It is approximately $(1 - a/\sqrt{2\pi})^{-1}$ for small values of a ($a \ll \pi$). Actually, since we are still in the high-temperature regime, we need to assume that $a \ll 2\pi/N \equiv d_\theta$, i.e., the hard-core repulsion occurs on scales smaller than the mean distance between the particles. For $N \gg 1$, which is the interesting case here, the two conditions on a are consistent. We note that a depends on the temperature and increases as the temperature decreases. For $a \ll 1$ we obtain the following form factor:

$$\langle g_n^{\text{gas}} \rangle = N - \frac{N(N-1)a}{\sqrt{2\pi}} \exp\left(-\frac{n^2 a^2}{2}\right), \quad n = 1, 2, \dots \quad (3.8)$$

Since the form factor is a non-negative quantity, we immediately infer an upper limit on a :

$$a \lesssim \frac{\sqrt{2\pi}}{N} = \frac{d_\theta}{\sqrt{2\pi}}. \quad (3.9)$$

This result is intuitively clear since the hard core cannot be larger than the mean distance of the particles. It is also compatible with the assumptions above concerning a . Physically, we observe that there is a *suppression* of the synchrotron radiation for $n \lesssim 1/a$ (lower harmonics). We can therefore estimate the hard-core scale (and hence the temperature) from the coherent modifications of the synchrotron radiation for the gaslike state of the particle beam. We note that the overall suppression effect is small. For small n values, where

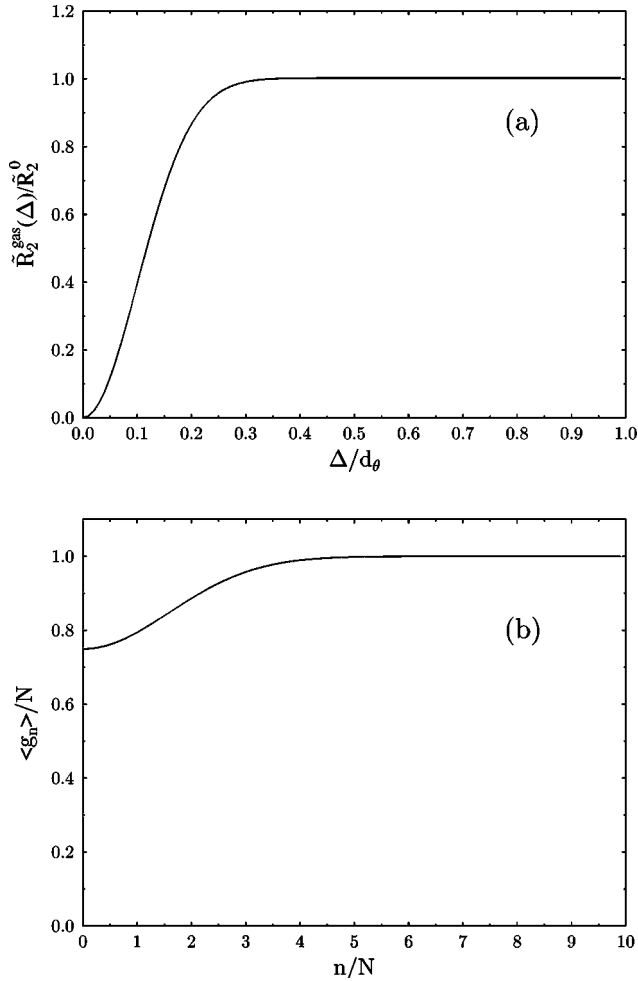


FIG. 1. The two-point correlator (a) and the form factor (b) for the “gaseous” state of the beam, described by Eqs. (3.6) and (3.8), respectively. We used the parameters $a = d_\theta/10$, $N = 100$.

the reduction of the emitted power is largest, the relative suppression with respect to the incoherent case amounts to only

$$\left| \frac{\langle g_n^{\text{gas}} \rangle - N}{N} \right| \approx \frac{\sqrt{2\pi a}}{d_\theta} \ll 1. \quad (3.10)$$

The situation is illustrated in Fig. 1.

When the temperature becomes smaller such that the Coulomb energy is comparable to the thermal energy, we expect the particle beam to become somewhat ordered and to form a liquidlike plasma. The partial order is a precursor to crystallization. In particular, the (angular) distance on which the repulsion between particles is manifest is d_θ , and the order effects should persist over a few mean distances. The two-point correlator is qualitatively given by

$$\tilde{R}_2^{\text{liq}}(\Delta) = c_2 \tilde{R}_2^0(\Delta) \left[1 - \frac{\sin^2(\pi\Delta/d_\theta)}{(\pi\Delta/d_\theta)^2} \right], \quad (3.11)$$

where $c_2 \approx (1 - 1/N)^{-1}$ is a normalization factor. As before, we assume $N \gg 1$. The above two-point correlator displays a strong repulsion for small distances [$\tilde{R}_2^{\text{liq}}(0) = 0$] as well as oscillations that persist for a few mean distances. It eventu-

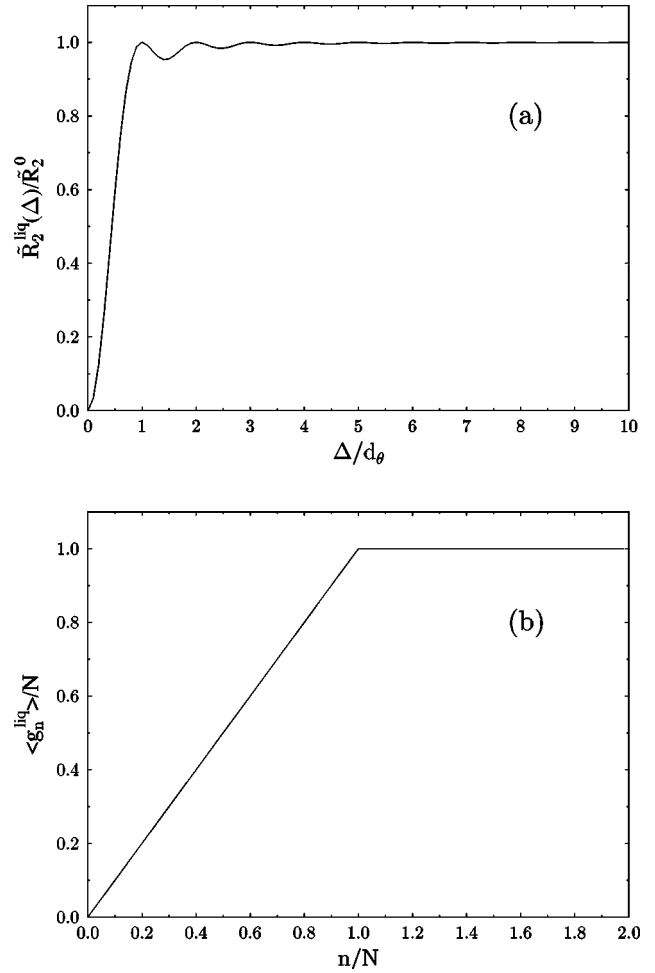


FIG. 2. The two-point correlator (a) and the form factor (b) for the “liquid” state of the beam, described by Eqs. (3.11) and (3.12).

ally reaches the asymptotic limit of uncorrelated particles. Thus it represents an intermediate situation (“liquid”) between the slight mutual repulsion (“gas”) treated above and long-range order (“crystal”) discussed below. The above expression was obtained as a result of an exact calculation for a one-dimensional chain of particles with logarithmic repulsion by Dyson in the context of random matrix theory [19]. The resulting form factor is

$$\langle g_n^{\text{liq}} \rangle = \min(n, N). \quad (3.12)$$

Thus, the suppression effect is very prominent in this situation, and there is effectively complete suppression of the synchrotron radiation for small values of n (see also Fig. 2). In terms of a wavelength, the suppression is felt for wavelengths that are comparable to or longer than the mean distance between the particles. Comparing Eqs. (3.8) and (3.12), we conclude that as the order becomes more manifest (temperature decreases), the suppression effect becomes more prominent, but the onset of suppression is shifted to longer wavelengths.

As crystallization takes place, long-range order effects become important. We consider in the following the simplest crystal, namely the linear chain. To describe the situation, we use a distribution function that corresponds to a thermal dis-

tribution of small displacements around the crystalline state with only nearest-neighbor interactions taken into account for simplicity:

$$P(\theta_1, \dots, \theta_N) = c_3 \exp \left[-\eta \sum_{j=1}^N (\varphi_{j+1} - \varphi_j)^2 \right]. \quad (3.13)$$

Here c_3 is a normalization constant, $\eta \equiv (q^2 N^2)/(16\pi^3 \epsilon_0 d k_B T)$, T is the temperature, and $d \equiv 2\pi\rho/N$ is the mean distance between the charges. The small displacements φ_j are defined as follows:

$$\varphi_1 \equiv \theta_1, \quad (3.14)$$

$$\varphi_j \equiv (\theta_j - \theta_1) - (j-1)d\theta, \quad j=2,3,\dots,N, \quad (3.15)$$

$$\varphi_{N+1} \equiv \varphi_1. \quad (3.16)$$

The exponent η can be rewritten as

$$\eta = \left(\frac{N}{2\pi} \right)^2 \left(\frac{\text{typical potential energy}}{\text{typical kinetic energy}} \right) = \left(\frac{N}{2\pi} \right)^2 \Gamma, \quad (3.17)$$

where Γ is the plasma parameter in one dimension [2],

$$\Gamma = \frac{q^2}{4\pi\epsilon_0 d k_B T}. \quad (3.18)$$

We note that the assumption of only nearest-neighbor interactions is not severe since for small displacements the interaction with the n th neighbor reduces as $1/n^3$.

In this case it is easier to obtain the form factor directly, without explicitly calculating the two-point correlator. A straightforward but lengthy calculation gives the following result for the form factor:

$$\langle g_n^{\text{cry}} \rangle = N + 2 \sum_{l=1}^{N-1} (N-l) \cos \left(\frac{2\pi n l}{N} \right) \exp \left[-\frac{n^2 \pi^2 l(N-l)}{\Gamma N^3} \right]. \quad (3.19)$$

In order to obtain the above result, we assumed $\Gamma \gg 1$, i.e., a cold beam. This is a necessary condition for crystallization. In order to interpret this result, we consider two limiting cases. If the maximal exponent in Eq. (3.19) (as a function of l), given by $n^2 \pi^2 / (4\Gamma N)$, is much smaller than 1, we can replace the exponential in Eq. (3.19) with 1 and get

$$\langle g_n^{\text{cry}} \rangle \approx \begin{cases} N^2, & N \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases} \quad (3.20)$$

This means that for very cold crystals there is a total suppression of the radiation for all harmonics, except the ones that are divisible by the number of particles N . For these special harmonics we get total constructive interference. The suppression of the leading harmonics results in an enormous reduction of the *total* power emitted by the synchrotron radiation (see Sec. V). In case crystallized electron beams can be produced, this effect gives rise to the possibility of significantly reducing the synchrotron radiation, currently the main limitation for circular electron accelerators. We men-

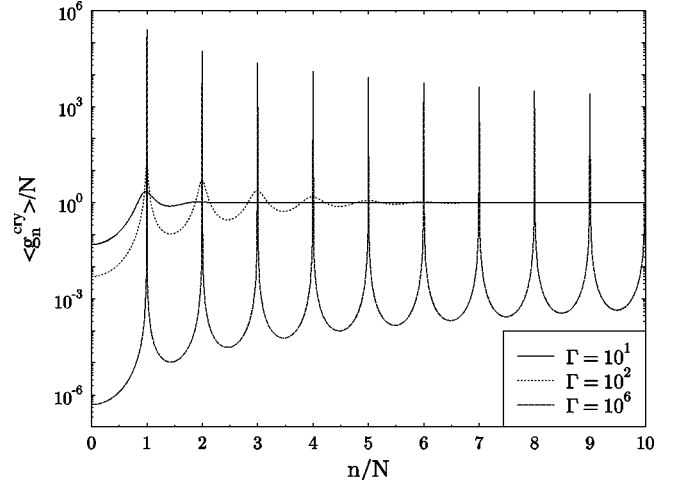


FIG. 3. The form factor for the crystalline state of the beam, described by Eq. (3.19). We considered the cases $N=10^6, \Gamma=10^1, 10^2, 10^6$.

tion in passing that the result (3.20) can also be obtained directly from calculating the form factor for a completely frozen crystal [7,12,13]. The other limit of Eq. (3.19) is for the first exponential factor ($l=1$) to be already small, such that only the first term needs to be considered. That is, for $n^2 \pi^2 / (\Gamma N^2) \gg 1$, we obtain

$$\langle g_n^{\text{cry}} \rangle \approx N + 2N \cos \left(\frac{2\pi n}{N} \right) \exp \left(-\frac{n^2 \pi^2}{\Gamma N^2} \right), \quad (3.21)$$

which describes small ‘ripples’ over the incoherent radiation, with a decaying amplitude that has oscillations with period N . In Fig. 3 we plot the numerically computed form factor (3.19) for the specific cases $N=10^6, \Gamma=10^1, 10^2, 10^6$. For $\Gamma=10^6$ we see a series of sharp peaks located at $n/N=1, 2, \dots$. This is expected since in this case Γ is very large and thus Eq. (3.20) holds approximately for the range of n shown in Fig. 3. For smaller values of Γ we observe a transition from sharp peaks to decaying ripples. Even in the case $\Gamma=10^6$ the sharp peaks will eventually die away.

The depression of g_n^{cry} at $n \approx 1$ can be computed analytically. Expanding the exponential factor in Eq. (3.19) to first order in $1/\Gamma$, we obtain

$$\frac{1}{N} g_1^{\text{cry}} \approx \frac{1}{2\Gamma}. \quad (3.22)$$

This is in perfect agreement with the results displayed in Fig. 3.

The above results concerning the crystalline state indicate that the plasma parameter Γ can be determined from the form factor of the synchrotron radiation (provided N is known). This defines a useful *diagnostic* tool for measuring the temperature of the crystal. We also conclude that crystalline beams can be *applied* to selectively suppress and enhance harmonics of the radiation, achieving up to total suppression ($g_n=0$) or total constructive interference ($g_n=N^2$).

To summarize this section, we have shown that the synchrotron radiation and its modifications with respect to the

incoherent state are strongly connected with the physical state of the beam. The form factor reflects the important scales and can be used to diagnose the state of the beam (“gas,” “liquid,” “solid”) as well as its temperature.

IV. MODIFICATIONS FOR BUNCHED BEAMS

Experimentally it is sometimes useful to work with bunched beams in which the particles occupy only a small fraction of the ring. Thus we consider in this section the modifications of the above theory for bunched beams. These modifications are straightforward. It turns out that only the lowest harmonics [up to $n \approx 2\pi/(\text{bunch angular length})$] are affected. Qualitatively this can be understood by examining Eq. (2.6), since the bunching will be felt only for values of n such that $n(\theta_j - \theta_{j'}) \lesssim 2\pi$. This yields the above estimate. In the following we detail the theory quantitatively.

We start with the gaseous phase and consider a narrow bunch of N particles with an (effective) angular width $\sigma \equiv 2\pi/Q$, $Q \gg 1$. In order to be specific, we shall assume that the shape of the bunch is a Gaussian, and that the (one-point) charge density is given by Eq. (3.4) above. In the absence of correlations, the two-point correlation function of the Gaussian bunch (GB) reads

$$\bar{R}_2^{\text{GB}}(\Delta) = \frac{N(N-1)}{\sqrt{\pi\sigma^2}} \sum_{m=-\infty}^{+\infty} \exp\left[-\frac{(\Delta + 2\pi m)^2}{4\sigma^2}\right], \quad (4.1)$$

from which we calculate the form factor (3.5). In order to include the hard-core repulsion between the charges, we operate as in the coasting case and modify \bar{R}_2^{GB} with a narrow dip,

$$\bar{R}_2^{\text{gas, bunch}}(\Delta) = c_4 \bar{R}_2^{\text{GB}}(\Delta) \left[1 - \exp\left(-\frac{\Delta^2}{2a^2}\right)\right]. \quad (4.2)$$

As before, $c_4 \approx 1$ to leading order in N . When calculating the form factor, the first term in the brackets gives Eq. (3.5). For the second term, we can use $\bar{R}_2^{\text{GB}} \approx 1$ since we assumed $a \ll \sigma$. Hence, we obtain

$$\begin{aligned} \langle g_n^{\text{gas, bunch}} \rangle &= N - \frac{N^2 a}{\sqrt{2\pi}} \exp\left(-\frac{n^2 a^2}{2}\right) \\ &\quad + N(N-1) \exp(-n^2 \sigma^2) \\ &= \langle g_n^{\text{gas}} \rangle + N(N-1) \exp(-n^2 \sigma^2). \end{aligned} \quad (4.3)$$

That is, the form factor of the gaseous coasting beam contains an additional enhancement feature for low harmonics, $n \lesssim Q$. This is suggestive, because of the scale separation between the length of the bunch and the hard-core scale, $a \ll \sigma$. For the liquid phase, a similar analysis applies. We need to replace the term \bar{R}_2^0 in Eq. (3.11) with \bar{R}_2^{GB} , and similar considerations will lead to the conclusion that we get the same type of enhancement of the low harmonics due to bunching,

$$\langle g_n^{\text{liq, bunch}} \rangle = \langle g_n^{\text{liq}} \rangle + N(N-1) \exp(-n^2 \sigma^2). \quad (4.4)$$

For the crystalline state (linear chain) with finite temperature we model the bunch by adding two limiting “fictitious” charges at both ends of the bunch. These charges are stationary with respect to the bunch’s reference frame. They do not radiate and serve only for confinement. To make the calculations tractable, we assume only nearest-neighbor interactions. Lengthy but straightforward calculation yields the form factor

$$\begin{aligned} \langle g_n^{\text{cry, bunch}} \rangle &= N + 2 \sum_{l=1}^{N-1} (N-l) \cos(nld_\theta) \\ &\quad \times \exp\left[-\frac{n^2 d_\theta^2 l(N+1-l)}{4\Gamma(N+1)}\right]. \end{aligned} \quad (4.5)$$

For small values of n we replace the exponents by 1 and obtain

$$\langle g_n^{\text{cry, bunch}} \rangle \approx \frac{\sin^2(\pi n/Q)}{\sin^2[\pi n/(QN)]}, \quad (4.6)$$

which is the form factor of a frozen linear crystalline bunch. In particular, it exhibits an enhancement for the low harmonics $n \lesssim Q$. If n is so large that only the first term is significant, we essentially recover the result (3.21). Results for the case $N = 5\sqrt{2} \times 10^3$, $Q = 100\sqrt{2}$, and $\Gamma = 10^1, 10^2$ are shown in Fig. 4. The parameters were chosen such that d_θ is the same as for the coasting case. We observe that significant enhancement indeed occurs for the lower harmonics, which is essentially independent of the temperature as suggested by Eq. (4.6). Otherwise, the form factor (normalized by the number of charges) is the same as for the coasting case.

To summarize this section, we investigated the modifications that result from the bunching of the particle beam. In all cases, we found that a significant enhancement occurs for the low harmonics $n \lesssim Q$. Otherwise, we get qualitatively the same results as for a coasting beam.

V. TOTAL POWER

In Secs. III and IV we concentrated on a discussion of the form factor g_n of the beam. We showed that important information on the thermodynamic state of the beam is already contained in g_n . The total emitted power, however, the subject of this section, depends on the interplay between g_n and the partial power levels $I_n^{(1)}$ of a single radiating charge [see Eq. (2.1)]. The total power $I^{(1)}$ of a single radiating charge is given by

$$I^{(1)} = \sum_{n=1}^{\infty} I_n^{(1)} = \frac{q^2 c}{6\pi\epsilon_0 \rho^2} \beta^4 \gamma^4. \quad (5.1)$$

This result agrees with Larmor’s well-known formula for the total radiated power of a single charge in the nonrelativistic limit [13]. We introduce the parameter $s = \beta\gamma$. It characterizes the three relativistic regimes important for the discussion in this paper: Nonrelativistic ($s \ll 1$), relativistic ($s \approx 1$), and ultrarelativistic ($s \gg 1$). With the help of the total power (5.1) we define the normalized power levels

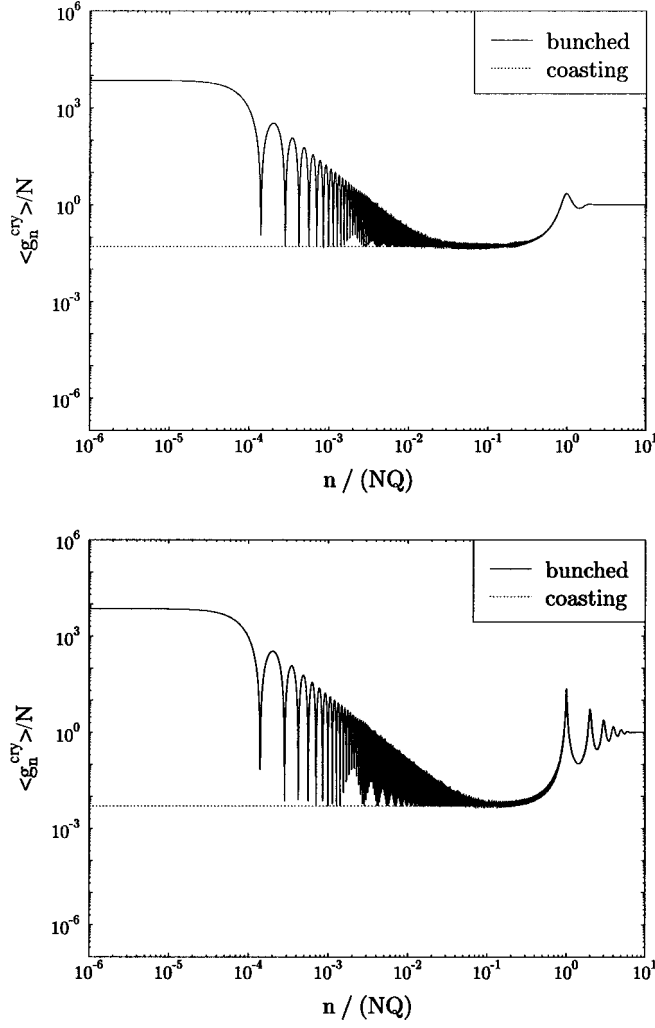


FIG. 4. The form factor for the crystalline state of a bunched beam described by Eq. (4.5). We considered the cases $N=5\sqrt{2} \times 10^3$, $Q=100\sqrt{2}$, $\Gamma=10^1$ (upper plot), $\Gamma=10^2$ (lower plot).

$$\tilde{I}_n^{(1)} \equiv \frac{I_n^{(1)}}{I^{(1)}}. \quad (5.2)$$

Since the purpose of this section is to discuss suppression effects in the total emitted synchrotron-radiation power, we define the suppression factor

$$\alpha(N, \beta) \equiv \frac{I^{(N)}}{NI^{(1)}} = \frac{1}{N} \sum_{n=1}^{\infty} g_n \tilde{I}_n^{(1)}. \quad (5.3)$$

In the case of N incoherently radiating charges, we have $\alpha(N, \beta) = 1$. A suppression effect corresponds to $\alpha(N, \beta) < 1$. Enhancement of synchrotron radiation corresponds to $\alpha(N, \beta) > 1$.

The behavior of $\tilde{I}_n^{(1)}$ as a function of n is the key for understanding the suppression effect of the total emitted synchrotron power. It is qualitatively different in the three relativistic regimes (see Fig. 5). For $s \ll 1$ we have $\beta \ll 1$ and $\tilde{I}_n^{(1)}$ decays exponentially in n . This is illustrated in Fig. 5(a). It shows $\tilde{I}_n^{(1)}$ as a function of n for $s=0.1$. Expanding Eq. (2.2) to leading order in β , we obtain

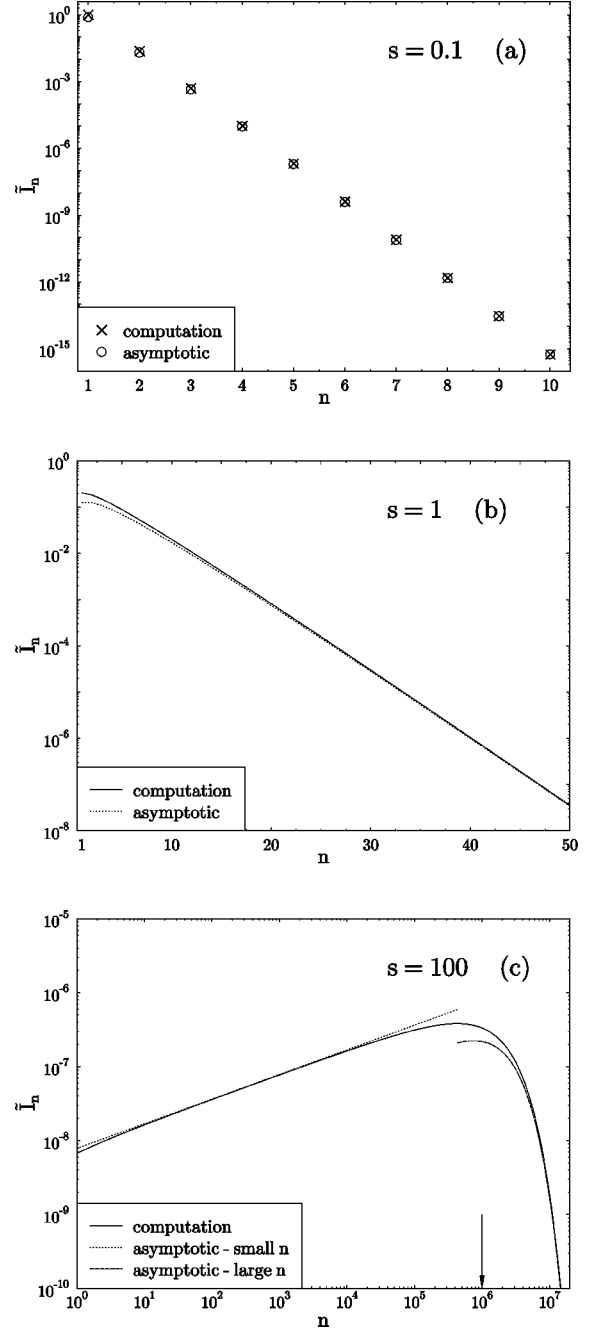


FIG. 5. The normalized partial powers $\tilde{I}_n^{(1)}$ as a function of n for (a) $s=0.1$, (b) $s=1$, and (c) $s=100$. The analytical asymptotic results (5.5), (5.6), (5.7), and (5.8) are also shown in the respective panels. The arrow in (c) points at $n = \gamma^3$.

$$\tilde{I}_n^{(1)} \approx \frac{3(n+1)n^{2n+1}}{(2n+1)(2n)!} \beta^{2n-2}, \quad \beta \ll 1. \quad (5.4)$$

We verify that $\tilde{I}_1^{(1)} \approx 1$ in this limit. Using Stirling's formula, we obtain

$$\tilde{I}_n^{(1)} \approx \frac{3(n+1)\sqrt{n}}{2(2n+1)\sqrt{\pi}\beta^2} \left(\frac{e\beta}{2}\right)^{2n}, \quad \beta \ll 1, \quad n \gg 1, \quad (5.5)$$

which proves the exponential decay of $\tilde{I}_n^{(1)}$ for large n . The result (5.5) is also shown in Fig. 5(a). The exponential decay

for large n persists in the case $s \approx 1$, albeit with a much smaller decay constant. This is illustrated in Fig. 5(b). In this case we also have an analytical approximation. It is given by [14]

$$\tilde{T}_n^{(1)} \approx \frac{3\sqrt{n}}{4\sqrt{\pi}\beta^2\gamma^{9/2}} \left(\frac{\beta\gamma e^{1/\gamma}}{1+\gamma} \right)^{2n}, \quad 1 \lesssim \gamma, n \gg \gamma^3. \quad (5.6)$$

The analytical approximation (5.6) is shown as the dashed line in Fig. 5(b). It describes the numerical data very well. The same figure also shows that a qualitative change with respect to the nonrelativistic case [Fig. 5(a)] occurs only for small n , where $\tilde{T}_n^{(1)}$ starts with a near-zero slope. In the ultrarelativistic case ($s \gg 1$) the behavior of $\tilde{T}_n^{(1)}$ changes qualitatively. For small n it shows an initial power-law increase according to [14]

$$\tilde{T}_n^{(1)} \approx 0.78\gamma^{-4}n^{1/3}, \quad \gamma \gg 1, 1 \leq n \leq \gamma^3. \quad (5.7)$$

At $n \approx 0.29\gamma^3$ it reaches a maximum and then decays exponentially according to [14]

$$\tilde{T}_n^{(1)} \approx \frac{3\sqrt{n}}{4\sqrt{\pi}\gamma^{9/2}} \exp\left(-\frac{2n}{3\gamma^3}\right), \quad \gamma \gg 1, n \gg \gamma^3. \quad (5.8)$$

This behavior is illustrated in Fig. 5(c) for the case $s=100$ (full line). The analytical results (5.7) and (5.8) (dashed lines) are also shown in Fig. 5(c). They compare well with the data in the appropriate limits. We now show that the behavior of $\tilde{T}_n^{(1)}$ in conjunction with the behavior of g_n leads to substantial suppression of synchrotron radiation for cold beams.

We first discuss the case of a coasting crystallized linear chain at $T=0$. It consists of N equispaced particles according to $\theta_j = 2\pi j/N$, $j=1,2,\dots,N$. For g_n we have the result (3.20). For the suppression factor α we obtain in this case

$$\alpha(N,\beta) = N \sum_{m=1}^{\infty} \tilde{T}_{mN}^{(1)}. \quad (5.9)$$

We saw above that independently of s the normalized partial powers $\tilde{T}_n^{(1)}$ always decay exponentially for large enough n . Thus, there is always an N_0 such that $\tilde{T}_n^{(1)}$ decays exponentially for $n > N_0$ and, therefore, for $N > N_0$,

$$\alpha \approx N\tilde{T}_N \quad (5.10)$$

to a very good approximation. But since $\tilde{T}_n^{(1)}$ decays exponentially for $N > N_0$, α is exponentially small. In other words, for large enough particle number we obtain exponential suppression of synchrotron radiation independently of the relativistic regime of the beam. This result is illustrated in Fig. 6 ($\Gamma = \infty$ case). It shows the suppression factor for $s=0.1$, 1, and 100 as a function of the particle number N . In all three cases we indeed obtain exponential suppression as predicted from the structure of Eq. (5.9).

Using Eq. (5.10) and the above expressions for \tilde{T}_n in the relevant relativistic regimes, we obtain explicit analytical formulas for $\alpha(N,\beta)$:

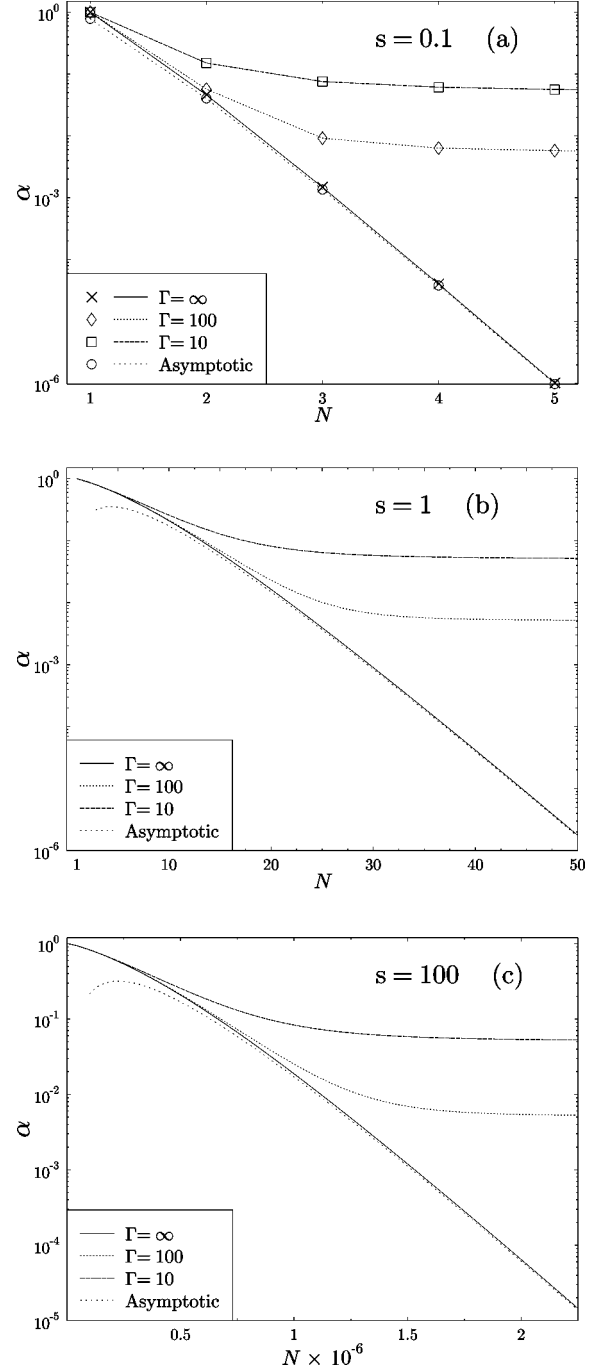


FIG. 6. Suppression factors for the crystallized chain for three different plasma parameters ($\Gamma = 10, 100, \infty$) in the three relativistic regimes: (a) $s=0.1$, (b) $s=1$, and (c) $s=100$. The asymptotic theoretical curves correspond to Eqs. (5.11), (5.12), and (5.13).

$$\alpha(N,\beta) \approx \frac{3N^{3/2}}{4\sqrt{\pi}\beta^2} \left(\frac{e\beta}{2} \right)^{2N}, \quad s \leq 1, N \gg 1, \quad (5.11)$$

$$\alpha(N,\beta) \approx \frac{3N^{3/2}}{4\sqrt{\pi}\beta^2\gamma^{9/2}} \left(\frac{\beta\gamma e^{1/\gamma}}{1+\gamma} \right)^{2N}, \quad s \leq 1, N \gg \gamma^3, \quad (5.12)$$

$$\alpha(N,\beta) \approx \frac{3N^{3/2}}{4\sqrt{\pi}\beta^2\gamma^{9/2}} \exp\left(-\frac{2N}{3\gamma^3}\right), \quad s \gg 1, N \gg \gamma^3. \quad (5.13)$$

Figure 6 shows that the analytical formulas are very good approximations of the numerical data in their respective ranges of validity.

Next we consider the linear chain at finite temperature. In this case the form factor (3.19) applies. Because of the structure of Eq. (3.19) and the asymptotic exponential decay of $\tilde{T}_n^{(1)}$ for large n , we can compute the asymptotic behavior of $\alpha(N, \beta)$ for large N . Using Eq. (3.22), we obtain

$$\begin{aligned} \alpha(N, \beta) &= \frac{1}{N} \sum_{n=1}^{\infty} g_n^{\text{cry}} \tilde{T}_n^{(1)} \approx \frac{1}{N} g_1^{\text{cry}} \sum_{n=1}^{\infty} \tilde{T}_n^{(1)} = g_1^{\text{cry}}/N \\ &\approx \frac{1}{2\Gamma}, \quad N \gg \gamma^3. \end{aligned} \quad (5.14)$$

Thus, for large N and in all three relativistic regimes, the asymptotic suppression is independent of N and saturates at $\alpha = 1/(2\Gamma)$. This behavior is clearly illustrated in Fig. 6, which shows the suppression factor for $\Gamma = 10, 100$, and ∞ for all three values of s considered. The onset of saturation in the vicinity of some $N = N_c$ is physically clear because of the following reason. Finite Γ corresponds to a finite temperature, which furthermore corresponds to a finite correlation length of the particles in the linear chain. But since the suppression of the synchrotron radiation is a coherent process, it is intuitively clear that no further suppression can be achieved once the total particle number exceeds the correlation length. Consequently the suppression effect has to saturate.

In Sec. III we pointed out that measuring the depth of the correlation hole in g_n^{cry} for small values of n defines an experimental method for measuring the plasma parameter of the beam. Since the saturation value of α depends only on Γ , measuring the suppression factor of coasting beams for large N defines yet another experimental procedure for measuring Γ .

We now turn to the case of bunched beams, which is very important experimentally. We shall consider short bunches of angular width $\sigma = 2\pi/Q$, $Q \gg 1$, which consist of N charges. The distinctive feature of short bunched beams is that for lower modes there is an enhancement of the synchrotron radiation which is due to the short length of the bunch, and is independent of the internal structure. This can be seen, e.g., by comparing Eqs. (3.5) and (4.6) for the hot and frozen cases, respectively: In both cases there is an enhancement $O(N^2)$ for $n \leq O(Q)$. This renders the task of observing significant suppression of the synchrotron radiation more difficult. Thus we carefully consider in the following a few parameter regimes in order to identify the ones in which the suppression effect is prominent. It is helpful to first analyze the frozen ($T=0$, maximally correlated) compared to the very hot ($T \rightarrow \infty$, uncorrelated) case since suppression effects are maximally manifest in the frozen case. Finite temperature interpolates between the two extremes. In the subrelativistic case $\beta \ll 1$, essentially all of the radiation power is concentrated in the first harmonic [see Eq. (5.5) and Fig. 5(a)]. Thus, for $Q \gg 1$ there is an almost complete constructive interference and consequently

$$\alpha_{\text{bunch}}(N, Q; \beta, \Gamma) \approx N, \quad \beta \ll 1. \quad (5.15)$$

Therefore, the subrelativistic case is irrelevant as far as suppression is concerned. We now turn to the ultrarelativistic case $\gamma \gg 1$. We first consider $Q \gg \gamma^3 \gg 1$. Since, as we saw above, most of the radiation power is concentrated in the harmonics $1 \leq n \leq O(\gamma^3)$, we obtain also in this regime almost constructive interference, and consequently

$$\alpha_{\text{bunch}}(N, Q; \beta, \Gamma) \approx N, \quad Q \gg \gamma^3 \gg 1, \quad (5.16)$$

meaning no suppression in this case. Also the case $\gamma^3 \gg NQ \gg 1$ is not interesting from the suppression point of view. Considering the form factor of the frozen bunch, Eq. (4.6), it can be approximated by rectangles of height N^2 and width Q near the harmonics $n = NQp$, where p is an integer. Since $\gamma^3 \gg NQ$, the function \tilde{T}_n varies only mildly from NQp to $NQ(p+1)$, and therefore the form factor can be replaced by its average N . But this average is the same as for the extremely hot case, and hence in both extremes we get almost the same suppression factor:

$$\alpha_{\text{bunch}}(N, Q; \beta, \Gamma) \approx 1, \quad \gamma^3 \gg NQ \gg 1. \quad (5.17)$$

We are left with the regime $NQ \gg \gamma^3 \gg Q \gg 1$, in which suppression effects are possible at all. Simple considerations (as were used above) can be used to decompose α in this regime as follows:

$$\alpha_{\text{frozen-bunch}}(N, Q; \beta) \approx \alpha_{\text{low}}(N, Q; \beta) + \alpha_{\text{frozen-crystal}}(NQ; \beta), \quad (5.18)$$

where α_{low} originates from the contributions of the lower modes $1 \leq n \leq Q$ and $\alpha_{\text{frozen-crystal}}$ is given by Eq. (5.13). Since in the regime considered $\alpha_{\text{frozen-crystal}}(NQ; \beta) \ll 1$, the overall suppression factor will be much smaller than 1 only if $\alpha_{\text{low}} \ll 1$. Using the asymptotic expression (5.7) for \tilde{T}_n in the lower regime, this is equivalent to requiring

$$\frac{NQ^{4/3}}{\gamma^4} \ll 1. \quad (5.19)$$

Assembling all the requirements, we arrive at the following conditions under which suppression is important for bunched beams:

$$\left(\frac{\gamma^3}{Q}\right)^{4/3} \gg N \gg \frac{\gamma^3}{Q} \gg 1, \quad Q \gg 1. \quad (5.20)$$

From these conditions it is evident that a fairly large γ is required in order for the suppression effect to be significant. In Fig. 7 we computed α_{bunch} as a function of N for $s = 100$, $Q = 10(\sqrt{5} - 1)$, and various values of Γ . (The choice of Q was such as to make it a ‘‘generic’’ irrational thus preventing number-theoretical peculiarities.) It is evident from the figure that indeed the maximal suppression occurs where predicted by the theory, and that this suppression is quite large. It should be noted that in experiments one may start with a hot bunch and cool it down, keeping the angular width constant (e.g., by strong bunching). In that case, the *observed* suppression factor will be $\alpha_{\text{bunch}}/\alpha_{\text{hot-bunch}}$. But since in the interesting domain $\alpha_{\text{hot-bunch}} \approx 1$, the results presented in the figure apply also to the actual experimental situation.

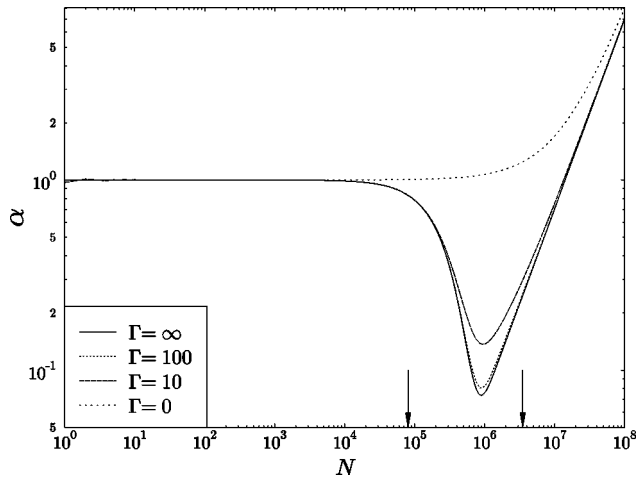


FIG. 7. Suppression factor for the crystallized bunched beam [$Q=10(\sqrt{5}-1)$] for four different plasma parameters ($\Gamma=0,10,100,\infty$) in the ultrarelativistic regime $s=100$. The arrows point to $N=\gamma^3/Q$ and $N=(\gamma^3/Q)^{4/3}$.

In this section we dealt exclusively with the crystalline state. It is also clear by inspection of Fig. 5 and of Fig. 2(b) that for large enough N , substantial suppression of synchrotron radiation can be achieved for *liquid* beams. This, again, is important since modern electron coolers are close to providing a liquid beam of electrons. Thus it may soon be possible to check our theory with the help of liquid electron beams.

VI. DISCUSSION, SUMMARY, AND CONCLUSIONS

The suppression of the radiation of geometrically ordered charges was first noticed by Thomson [20]. He employed this effect for motivating the stability of atoms, which, according to classical theory, should radiate and decay. Suppression of synchrotron radiation in the context of accelerators was first noted by Schiff [12]. But in Schiff's time, a mechanism for establishing the order in a beam of charged particles was not available. Only recently, with progress in the cooling of beams by electrons and lasers, was it possible to envision the production of crystallized beams whose syn-

chrotron radiation is exponentially small. It should be borne in mind, however, that synchrotron radiation is not very important for heavy ion beams that can easily be cooled with electrons and lasers. Dramatic effects are expected to occur only for crystallized electrons where the synchrotron radiation is orders of magnitude stronger. The drawback is that electrons cannot be cooled directly with lasers. We hope, however, that this paper will stimulate experimentalists to develop cooling schemes for electron beams. One possibility would be to use sympathetic cooling of electrons with a beam of heavy ions that can be cooled by lasers.

In this paper we dealt only with one-dimensional crystals, the simplest crystalline structures, and assumed that the storage ring under consideration supports such a crystal. For a given storage ring with fixed confining forces, however, there will be a transition to more complicated crystalline structures as the number of particles grows. For example, Hasse and Schiffer [21] predict that the one-dimensional chain will be transformed into a two-dimensional zigzag crystalline structure as the density grows, and a further transition to three-dimensional helical crystals will follow (see also [22] for computations with realistic storage-ring parameters). These expectations were verified by experiments with a mini-storage-ring in which the ions are essentially stationary [23]. Dealing with structures more complicated than the one-dimensional crystalline chain goes beyond the scope of the current study and requires further work.

This paper discusses various forms of ordered beams that may occur in practice: Gaseous, liquid and crystalline, coasting, and bunched. It is pointed out that the suppression effect occurs on two levels: In the form factor of the beam and in the total radiated power. While the modifications in the form factor may be used as a diagnostic tool for inferring the thermodynamic state of the beam, the suppression of the total power may eventually lead to the construction of small-sized cyclic electron accelerators.

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