Broad-band colored noise: Digital simulation and dynamical effects

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We propose a broad-band colored noise, which allows the transition between ''red'' and ''green'' noises. A double-integral algorithm for solving the Langevin equation with this noise is developed. The steady currents of an overdamped particle moving in the correlation and diffusion ratchets are calculated. It is shown that the flow reverses sign when the external noise is filtered off in the region of low frequencies to a sufficient extent. The present noise is also compared with the Ornstein-Uhlenbeck colored noise and the harmonic noise. [S1063-651X(99)12112-3]

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Investigations on the influence of Gaussian colored noise on nonlinear dynamical systems are usually carried out by the consideration of the Ornstein-Uhlenbeck (OU) process [1] or the harmonic (narrow-band quasimonochromatic) noise [2–4] as an external fluctuation force. Several interesting phenomena result from the spectral density $S(\omega)$ of the noise being to decay with at least the inverse second power of the frequency ω^{-2}. Now a challenge comes from a unified describing for the noise source with different colors, in which the power spectrum may not fall to zero as $\omega\to\infty$. In fact, applying a Gaussian white noise to a low-pass filter plus a high-pass one yields a colored noise with the required properties. The inverse spectrum of the noise can be written as $a_{-2}\omega^{-2}+a_0+a_2\omega^2$. The interesting material is to study the statistical properties of this broad-band colored noise, because it allows the transition from a ''red'' noise to a ''green'' one. Recently, the transport of a Brownian particle moving in a one-dimensional ratchet potential driven by noise force has been investigated in some detail [5]. This is a good and important example for studying the dynamical effects of external noise with different colors. The aims of this paper are to show the features of a broad-band colored noise, as well as to compare this noise with the OU colored and harmonic noises.

The equation of motion of a Brownian particle in a nonlinear potential $V(x)$ driven by an external noise with intensity D reads

$$\dot{x}(t)=f(x)+\sqrt{2D}\epsilon(t), \tag{1}$$

$$\dot{y}(t)=\epsilon(t), \tag{2}$$

$$\dot{\epsilon}(t)=-[\tau_1^{-1}+\tau_2^{-1}(1+R_2/R_1)]\epsilon(t) -(\tau_1\tau_2)^{-1}y(t)+\tau_2^{-1}\eta(t), \tag{3}$$

and $f(x)=-dV(x)/dx$. Where $\tau_1=R_1C_1$ and $\tau_2=R_2C_2$ are two time constants of a low-pass filter plus a high-pass one, the input Gaussian white noise $\eta(t)$ satisfies $\langle\eta(t)\rangle=0$ and $\langle\eta(t)\eta(t')\rangle=\delta(t-t')$, the final output signal $\epsilon(t)$ is the voltage of the resistance R_1 (see Fig. 1).

Let $R_2\ll R_1$ and the solutions of Eqs. (2) and (3) are written as

$$y(t)=-\sum_{i=1}^{2}\tau_i\Psi_i(t),\quad \epsilon(t)=\sum_{i=1}^{2}\Psi_i(t), \tag{4}$$

where

$$\Psi_i(t)=a_i\exp(-t/\tau_i)+m_i\int_0^t\exp[(s-t)/\tau_i]\eta(s)ds, \tag{5}$$

in which

$$a_1=\frac{y(0)+\tau_2\epsilon(0)}{\tau_2-\tau_1},\quad a_2=\frac{y(0)+\tau_1\epsilon(0)}{\tau_1-\tau_2},$$
$$m_1=\frac{1}{\tau_2-\tau_1},\quad m_2=\frac{\tau_1}{\tau_2(\tau_1-\tau_2)}. \tag{6}$$

If the noise $\epsilon(t)$ is a stationary process, its correlation function depends only on $|t-t'|$. After assuming the initial distribution of ϵ and y to be the Gaussian-type with the variations determined by

$$\langle\epsilon^2(0)\rangle=\frac{\tau_1}{2\tau_2(\tau_1+\tau_2)},$$
$$\langle y^2(0)\rangle=\frac{\tau_1^2}{2(\tau_1+\tau_2)}, \tag{7}$$
$$\langle\epsilon(0)y(0)\rangle=0,$$

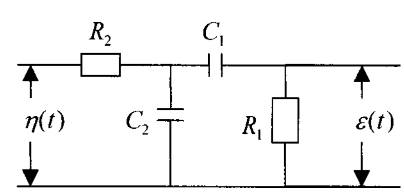

FIG. 1. Schematic image of an electrical filter.

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we obtain the correlation function of ϵ with a stationary form, i.e.,

$$\langle \epsilon(t)\epsilon(t')\rangle = \frac{\tau_1}{2(\tau_1^2-\tau_2^2)}\left\{-\exp\left(-\frac{|t-t'|}{\tau_1}\right) + \frac{\tau_1}{\tau_2}\exp\left(-\frac{|t-t'|}{\tau_2}\right)\right\}. \tag{8}$$

The corresponding dimensionless spectrum is

$$S_\epsilon(\omega) = 2\int_0^\infty \langle \epsilon(t)\epsilon(t+\lambda)\rangle \cos(\omega\lambda)d\lambda = \frac{\tau_1^2\omega^2}{(1+\tau_1^2\omega^2)(1+\tau_2^2\omega^2)}. \tag{9}$$

It is noticed that the correlation function (8) shows anticorrelation whereas its spectrum flows from a broad band. When $\tau_1 \to \infty$, $S_\epsilon(\omega) = (1+\tau_2^2\omega^2)^{-1}$ as well as $S_\epsilon(\omega) = (\tau_1\omega)^2/(1+\tau_1^2\omega^2)$ if $\tau_2 \to 0$. Namely, the spectrum (9) can be recovered as the ones of a low-frequency (OU) red noise [1] and a high-frequency green noise [7], respectively. Moreover, the white noise appears in the limits of both $\tau_1\to\infty$ and $\tau_2 \to 0$.

We now develop a double-integral algorithm for solving a set of Langevin equations (1)–(3). First, the exact integral-closed algorithm for simulating ϵ and y can be obtained through

$$\Psi_i(t+\Delta t) = \exp(-\Delta t/\tau_i)\Psi_i(t) + m_i\omega_i \quad (i=1,2). \tag{10}$$

The second step is to expand the term $f[x(t)]$ to first order within the time interval $[t, t+\Delta t]$, and then to integrate Eq. (1) with the initial condition $x(t')|_{t'=t} = x(t)$ [8,9] where we have

$$x(t+\Delta t) = x(t) - \frac{f[x(t)]}{f'[x(t)]}[1-\exp(f'(x(t))\Delta t)] + \sqrt{2D}Z(t), \tag{11}$$

with

$$Z(t) = \int_t^{t+\Delta t}\exp\{f'[x(t)](t+\Delta t - s)\}\epsilon(s)ds = \sum_{i=1}^{2}\left\{\frac{\exp\{f'[x(t)]\Delta t\} - \exp(-\Delta t/\tau_i)}{f'[x(t)]+\tau_i^{-1}}\Psi_i(t) - \frac{m_i\omega_i}{f'[(x(t))]+\tau_i^{-1}}\right\} + \frac{f'[x(t)]\tau_2^{-1}}{\{f'[x(t)]+\tau_1^{-1}\}\{f'[x(t)]+\tau_2^{-1}\}}\omega_3. \tag{12}$$

In Eqs. (10) and (12)

$$\omega_i = \int_t^{t+\Delta t}\exp[(t+\Delta t - s)/\tau_i]\eta(s)ds, \tag{13}$$

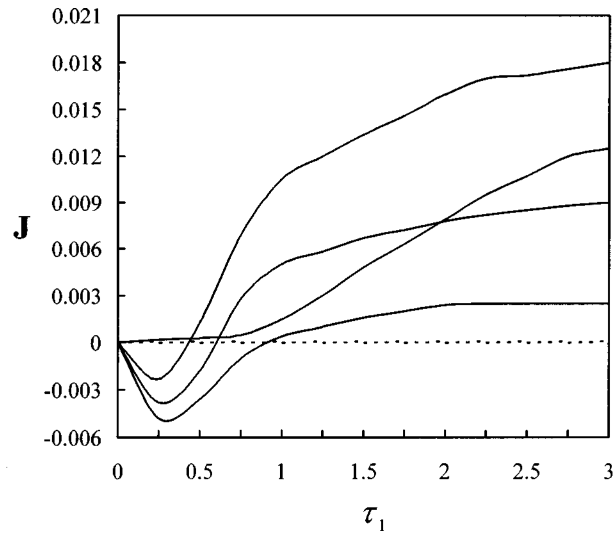

FIG. 2. The current as a function of τ_1 when $D=0.3$ and for four values of $\tau_2 = 0.09$, 0.7, 0.05, and 0.02 from top to bottom.

where $(i=1,2,3)$ and $\tau_3 = -f'[x(t)]^{-1}$. The correlations for the Gaussian stochastic variables ω_i are determined by

$$\langle \omega_i\omega_j\rangle = \frac{\tau_i\tau_j}{\tau_i+\tau_j}[1-\exp(-\Delta t/\tau_i - \Delta t/\tau_j)] \quad (i,j=1,2,3). \tag{14}$$

In this study, the potential is taken to be a smooth ratchetlike periodic one with period $L=1$,

$$V(x) = -[\sin(2\pi x) + 0.25\sin(4\pi x)]/(2\pi). \tag{15}$$

The objective of the simulations is to determine the dimensionless average velocity and current of the particle in the stationary state. We solve the Eqs. (1)–(3) M times for computing and plotting [10]:

$$J = \frac{1}{L}\langle \dot{x}(t)\rangle_{st} = \frac{1}{LM}\sum_{m=1}^{M}\left(\frac{x_m(t)-x(0)}{t}\right).$$

In the following calculations, $M=1000$, $\Delta t = 0.02$, and $t=3000$.

The current as a function of τ_1 for fixed $D=0.3$ and different τ_2 is shown in Fig. 2. When τ_1 becomes large, the present noise reduces to the OU colored one thus the current is positive for the potential (15). Moreover, the asymptotical value of the positive current is optimized for a finite τ_2. This is in agreement with the previous result [6]. On the other hand, the noise ϵ shows greenish for both small-to-moderate τ_1 and small τ_2; the average motion of the particle is along the direction of the left. It is observed that the negative current increases with increasing of τ_1 starting from zero, and then its absolute value achieves a maximum. However, the current occurs reversal when τ_1 increases continuously. This can be understood from the following facts. The sum of spectrum between red and green noises is equal approximately to a constant (i.e., the one of white noise). It is known that there is no net current for any ratchet system driven only by the Gaussian white noise. This means that the red noise offsets the influence of the green one on the transport flow.

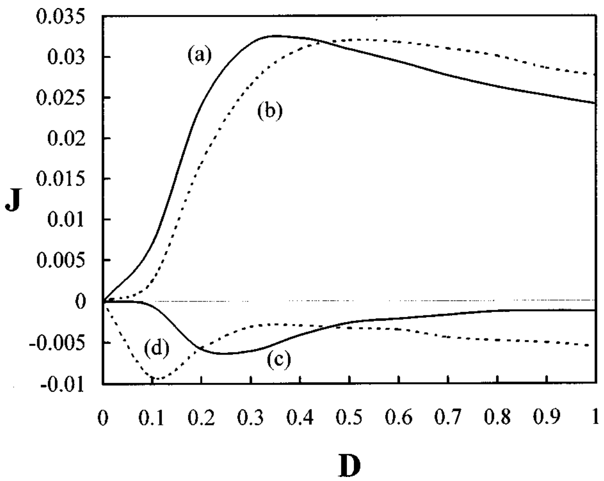

FIG. 3. The current as a function of D for two kinds of colored noises. The solid and dashed lines correspond to the proposed noise and the harmonic noise, respectively. The parameters are: (a) $\tau_1 = 3.0$, $\tau_2 = 0.2$; (b) $\Gamma = 20$, $\Omega_0^2 = 100$; (c) $\tau_1 = 0.35$, $\tau_2 = 0.005$; and (d) $\Gamma = 10$, $\Omega_0^2 = 1000$.

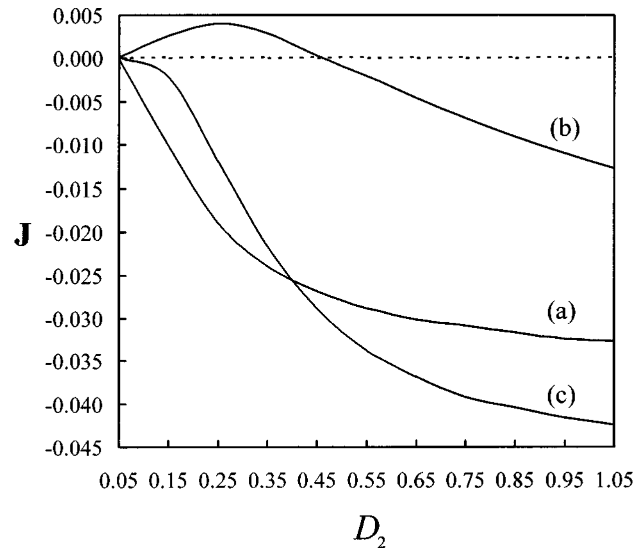

FIG. 4. The current induced by three kinds of noises as a function of D_2 in the diffusion ratchet for the fixed $D_1 = 0.05$ and $t_p = 1$. (a) The white noise ($\tau_1 = 5.0$, $\tau_2 = 0.005$); (b) the OU red noise ($\tau_1 = 5.0$, $\tau_2 = 0.2$); (c) the green noise ($\tau_1 = 0.3$, $\tau_2 = 0.005$).

Thus the balance between reddish and greenish of the colored noise discussed here can be realized for a group of the critical parameters (τ_{1p}, τ_{2p}).

The comparison of the proposed noise with the harmonic noise is performed, and the dependence of the current on the intensity D of noise is plotted in Fig. 3. The harmonic noise $h(t)$ is defined through the stochastic differential equations [3]: $\dot{h} = u$, $\dot{u} = -\Gamma u - \Omega_0^2 h + \Omega_0^2 \eta(t)$, in which Γ and Ω_0 are damping and frequency coefficients of the white noise (η)-driven oscillator. It can see that the present noise is the velocity of an overdamped harmonic noise. The dimensionless spectral density of $h(t)$ reads as $S_h(\omega) = \Omega_0^2/[\Gamma^2\omega^2 + (\Omega_0^2 - \omega^2)^2]$, which does not allow the high-frequency green noise. It is found that the positive and negative maxima of the current occur, respectively, in the cases of $\tau_1 \to \infty$ and $\tau_2 \to 0$ for the proposed noise. The corresponding cases appear, respectively, at $\Gamma \to \infty$, $\Omega^2 \to \infty$ ($\Gamma/\Omega_0^2 = \tau =$ finite) and $\Omega_0^2 \gg \Gamma^2$ ($\tau \to 0$, adiabatic limit) for the harmonic noise. Obviously, our noise has nice and direct properties of the parameters comparing with the harmonic noise.

In Fig. 4, we calculate the current of the particles moving in a diffusion ratchet [11]. The equation of motion of the particle is $\dot{x}(t) = f(x) + \sqrt{2D(t)}\epsilon(t)$. Here the time-dependent intensity $D(t)$ of noise changes periodically between two values D_1 and D_2 with temporal period $2t_p$. The noise source $\epsilon(t)$ is taken to be the Gaussian white noise, OU colored noise and green noise, respectively. They can be produced by the present broad-band colored noise. It is observed that the absolute values of asymptotic current induced by OU red noise is lowest, and the one driven by the green noise is greatest, so that the net flow can be increased when color of the noise varies.

In summary, an approach for generating a broad-band colored noise is proposed by using a white noise to drive a low-pass filter plus a high-pass one. It can reduce simply to the Ornstein-Uhlenbeck colored noise and the high-frequency green noise. The current of a particle moving in a static ratchet reverses sign when the external broad-band noise is filtered off in the region of low frequencies to a sufficient extent. It is also found that the current along the hard-side of the ratchet can be optimized when the proposed noise transits the green one. Moreover, the present broad-band noise has nice properties compared with other colored noises, and it can be applied to control the direction and magnitude of the particles' flow.

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