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Theory of beam-plasma instability in a periodic plasma-filled waveguide

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The beam-plasma wave interaction in a periodic plasma-filled waveguide is treated in a mathematically correct manner on the basis of the integral equation (IE) method. It has been shown that the relevant boundary-value problem could be reduced to an IE with a singular kernel for the longitudinal component of the electric field on the waveguide axis. The regularization of the IE was performed by extracting the static part of the kernel. The resulting IE of the second kind with a regular kernel, being rather convenient for a numerical analysis, is treated in a quasistatic approximation as a spectral problem. First-order expressions for eigenfunctions, and an infinite set of dispersion relations linking a wave number and frequency of plasma oscillations which separate radial branches of plasma oscillations from axial ones, have been obtained in the closed analytical form, thus enabling us to avoid the problem with the so-called ''dense'' spectrum. The solutions of the relevant ''cold'' dispersion relations establish a periodical dependence of the frequency on the wave number over several periods within the accuracy of order of the neglected terms. In the presence of an electron beam they turn out to be unstable near frequencies providing the resonances of the beam with spatial plasma harmonics. Evaluations of the instability saturation level predict a more efficient beam-plasma wave energy transfer compared with those following from a conventional theoretical analysis based on the formulation of a dispersion relation in terms of an infinite determinant, with following truncation of the latter to the finite sized relation. [S1063-651X(99)10011-4]

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I. INTRODUCTION

To date vacuum microwave tubes with intense relativistic electron beams (REB's) remain the most attractive means for producing high power microwaves in centimeter and millimeter wave ranges [1,2]. However, increasing the output power by using more intense REB's is possible if the beam currents are well below the space-charge-limit current [3]. When the beam current becomes comparable with the space-charge-limit current, the beam-wave energy transfer essentially decreases. Therefore a further increase of output power can be achieved in the presence of background plasma in an interaction chamber, which provides space-charge field neutralization [4]. A notable enhancement of the output power due to the presence of a plasma was recently realized experimentally for several types of microwave tubes [5–9]. It has been thoroughly demonstrated that introducing a plasma with a proper density and radial profile results in an increase of the operation efficiency and frequency bandwidth, and provides the possibility to operate with significantly lower guiding magnetic fields and to control the output power and operation frequency smoothly. However, the presence of plasma in the slow wave structure can lead to crucial changes of its electrodynamic properties which are not fully studied and realized so far. Plasma influences such as a dielectric medium on the electrodynamic properties of the microwave tubes, were studied experimentally [10,11] and theoretically [12–14] in many papers; the plasma medium can support many of its own propagating waves in the range $\omega<\Omega_e$, where Ω_e is the plasma frequency, and ω is the frequency of wave, which can efficiently interact with the beam [15,16].

Unfortunately, dispersion properties of plasma waves in periodical waveguides have been studied much less than those of electromagnetic ones, in spite of experimental observations of the latter [17]. In order to interpret the results of experiment of Ref. [5], the beam excitation of plasma waves in a corrugated waveguide at a low plasma density was studied in Ref. [18] on the basis of a spatial harmonic expansion. The dispersion relation was obtained in the form of an infinite determinant, and was solved by its truncation to the finite size determinant, accounting for a finite number of spatial harmonics. Values obtained for the spatial growth rate of plasma waves turned out to be significantly less than that for the electromagnetic wave instability. Trying to explain the results of experiments with plasma-filled backward wave oscillators [6], the authors of Ref. [19] revealed that plasma waves originate in the so-called ''dense'' spectrum when an infinite number of radially and axially shifted branches are located in a finite frequency band, and separation of one branch from others is practically impossible. Moreover, numerical analysis performed for a large number of spatial harmonics showed that the solution of the dispersion relation loses the property of periodicity with respect to wave number required by the Floquet theorem [20]. On the basis of these results in Ref. [20], it was concluded that it is impossible to obtain a reasonable dispersion relation in such a way. Thus, the problems of beam-plasma instability in the periodical waveguides, and its influence on plasma-filled microwave device operation, are still open.

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In this paper an approach to the analysis of the beam-plasma instability is suggested. It is based on the integral equation (IE) method [21,22], which seems to be the most feasible one for an analysis of multiwave and multimodal regimes. Therefore, it can be expected that it also will be fruitful for the treatment of systems with ''dense'' spectra to which, in principle, all plasma-filled periodical structures should refer.

It is shown that the dispersion relation obtained previously in the form of an infinite determinant [16-18] can be represented in terms of a homogeneous singular IE for the unknown total longitudinal electric field on the waveguide axis. It is well known from the general theory of the IE that the formulation of spectral problems in such a manner is quite reasonable. Moreover this formulation is frequently used in the theory and computations of various elements of microwave technique such as thin-film periodic structures [23], groove waveguides [24], microstrip lines [25], etc. Due to the ability to take the multimodal content of the fields accurately into account, in our case such a formulation allows us to obtain not only more precise results, but gives us the opportunity to obtain more insight about the dispersion properties of the plasma waves in periodical waveguides. In particular it gives us the unique chance to avoid the problem of the ''dense'' spectrum, allowing us to separate the radial plasma modes from the axial ones. Our approach also promotes an understanding of numerical troubles mentioned in Ref. [20], and finding a way to overcome them.

The initial singular IE after regularization was treated analytically in a quasistatic approximation. From the requirement of periodicity of the general solution, an infinite set of dispersion relations was obtained. Each of these corresponds to the certain axially shifted branch of radial plasma modes. Any of the dispersion relations obtained can be considered independently, providing complete information about the dispersion properties of plasma waves, the distribution of fields within the waveguide, and the growth rate of the beam-plasma instability. The results of our analysis indicate that the instability of plasma waves in the case of a low density plasma (no beam resonance with the lowest spatial plasma harmonic) can be more efficient than that predicted in the framework of the conventional analysis [18]. It can lead to a more effective beam-plasma wave energy transfer and to a widening of the region of unstable frequencies and wave numbers.

The remainder of the paper is organized as follows. In Sec. II our theoretical model is described with an example of a planar periodic waveguide filled with a longitudinally magnetized plasma and driven by a thin sheet electron beam. It is shown that the initial IE for a longitudinal electric field on the waveguide axis has a singular kernel, and the simplest method of its regularization is proposed. The resulting IE of the second kind is treated in the quasistatic approximation in Sec. III. The first-order closed form solution and the general dispersion relation in the quasistatic approximation is derived in Sec. IV. The infinite set of ''cold'' dispersion relations and their solutions, as well as a procedure of avoiding the ''dense'' spectrum, are described in Sec. V. Instabilities of Cherenkov type are found and analyzed in Sec. VI. They can be interpreted as resonant interactions of the beam with the highest spatial harmonics of the plasma wave field. Conclusions and remarks are contained in Sec. VII.

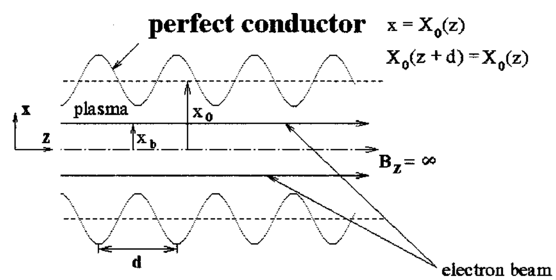

FIG. 1. Geometry of the problem.

II. THEORETICAL MODEL

Normally the model of an infinitely long periodic axisymmetric waveguide is used for a consideration of wave processes in high-power plasma-filled devices such as BWO's, TWT's and some others. Since this paper is devoted, first, to exploring the general qualitative issues concerning the behavior of systems with a ''dense'' spectrum, we restrict ourselves to the case of a planar geometry, which gives us a fine chance to obtain a number of results in an analytical form which greatly promotes the development of a correct qualitative analysis of such an intricate question.

Thus we, consider a planar symmetrical metallic waveguide with arbitrary periodic walls filled with the ''cold'' collisionless homogeneous plasma (see Fig. 1) and driven by an infinitesimally thin sheet electron beam located symmetrically at distances $x=\pm x_b$ from the waveguide axis. An infinitely large magnetic field is applied along the z axis. All wave perturbations assumed to be of TM polarization and into symmetrical with respect to the z axis $[E_z(x)=E_z(-x)]$, allowing one to consider only the region $x>0$.

Following the Floquet theorem, we can represent the fields in the waveguide as a superposition of spatial harmonics,

$$\mathbf{A}(x,z,t)=\sum_{n=-\infty}^{\infty} A_n(x)\exp(ih_n z-i\omega t), \qquad (1)$$

where $\mathbf{A}=\{E_x,H_y,E_z\}$, and $h_n=k_z+nk_0$, k_z,ω are the wave number and frequency of perturbations, and $k_0=2\pi/d$, d is the period of the structure. Substituting Eq. (1) into the Maxwell equations and applying the boundary condition on the ideal wall $E_\tau(x,z)|_{x=X_0(z)}=0$, where E_τ is the tangential component of the electric field, and $x=X_0(z)$ is the equation of the waveguide boundary, after some mathematical manipulations described in many papers (see, for example, Ref. [26]) we obtain the equation

$$\sum_{n=-\infty}^{\infty} a_n e^{ih_n z}\left(1-\frac{ih_n\varepsilon}{\kappa_n^2}\frac{d}{dz}\right) \times\left[\cos(\kappa_n X_0(z))+\frac{2\pi c^2(k^2-h_n^2)\nu}{(\omega-h_n v)^2\kappa_n} \times\cos(\kappa_n X_0(z))\sin(\kappa_n(X_0(z)-x_b))\right]=0, \qquad (2)$$

where

$$a_n=\frac{1}{d}\int_{-d/2}^{d/2}E_z(0,z)e^{-ih_nz}dz,$$

$$\kappa_n=\sqrt{\varepsilon(k^2-h_n^2)},\quad \varepsilon=1-\Omega_e^2/\omega^2,\quad \nu=2I_b/\beta\gamma^3I_A, \tag{3}$$

$$I_A=mc^3/e,\quad k=\omega/c,\quad \gamma=(1-v^2/c^2)^{-1/2},$$

$E_z(0,z)$ is the z component of the electric field on the waveguide axis, I_b is the beam current per unit length in the transverse direction, I_A is the Alfven current, v is the speed of the beam, c is the speed of light, and e and m are the charge and mass of in electron, respectively.

The traditional approach to the analysis of Eq. (2) [18–20] provides the expansions $\cos(\kappa_nX_0(z))$ and $\sin(\kappa_n(X_0(z)-x_b))$ into the Fourier series, resulting in the equation

$$\sum_{m=-\infty}^{\infty}e^{imk_0z}\sum_{n=-\infty}^{\infty}D_{mn}(\omega,k_z)a_n=0,$$

from which the dispersion relation in the form of the infinite determinant

$$\det||D_{mn}(\omega,k_z)||=0, \tag{4}$$

linking ω and k_z, follows. However, attempts to analyze such a relation numerically by truncating the infinite matrix to some reasonable size for the frequency within the region of plasma wave existence ($\omega<\Omega_e$) was not successful [19,20], since the solution obtained did not satisfy the requirement of periodicity with respect to wave number according to the Floquet theorem. [27].

Below we offer an alternative formulation of the dispersion relation which provides the passage from unknown Fourier coefficients a_n to the unknown periodical function $\Psi(z)$, linked with the total longitudinal electric field on the waveguide axis by the simple relation $\Psi(z)=E_z(0,z)e^{-ik_zz}$ and the derivation of the IE for it. Substituting Eq. (3) into Eq. (2), and changing the order of summation and integration, we obtain the following IE over the structure period:

$$\int_{-d/2}^{d/2}G(z,z')\Psi(z')dz'=0,\quad z\in(-d/2,d/2), \tag{5a}$$

where $G(z,z')=G_e(z,z')+G_b(z,z')$:

$$G_e(z,z')=\sum_{n=-\infty}^{\infty}G_{en}(z,z'), \tag{5b}$$

$$G_{en}(z,z')=e^{ink_0(z-z')}\left(1-\frac{ih_n\varepsilon}{\kappa_n^2}\frac{d}{dz}\right)\cos(\kappa_nX_0(z)),$$

$$G_b(z,z')=\sum_{n=-\infty}^{\infty}\frac{2\pi c^2(k^2-h_n^2)\nu}{(\omega-h_nv)^2\kappa_n}\times e^{ink_0(z-z')}\cos(\kappa_nX_0(z))\sin[\kappa_n(X_0(z)-x_b)]. \tag{5c}$$

Equations (5b), and (5c) allow us to explore in detail the analytical properties of the kernel $G(z,z')$. Evaluating $G_e(z,z')$ at large n as

$$G_{en}(z,z')=\frac{1}{2}[(1+X'(z))e^{ik_zX(z)}e^{ink_0(z+X(z)-z')}+(1-X'(z))e^{-ik_zX(z)}e^{-ink_0(z-X(z)-z')}], \tag{6}$$

where $X(z)=|\varepsilon|^{1/2}X_0(z)$, we can conclude that the sum for $G_e(z,z')$ diverges at the points $z'=z\pm X(z)$. At these points $G_e(z,z')$ cannot be approximated by the partial finite sums $G_{eN}(z,z')=\Sigma_{n=-N}^{N}G_{en}(z,z')$, regardless on the number N. Thus the truncation of the infinite determinant (4), which is equivalent to approximating the singular kernel of IE (5a) by finite sums having regular values in the vicinity of the singular points mentioned above seems not to be correct, and according to the theory of singular IE's can lead to rough mistakes in the final results. Meanwhile it should be pointed out that the beam part of the kernel $G_b(z,z')$ is regular about both variables.

In order to circumvent these difficulties, we separate the static part of the kernel, which just contains all singularities inherent to the general kernel,

$$G_e(z,z')=G_e^{\rm st}(z,z')+G_r(z,z'), \tag{7a}$$

where

$$G_e^{st}(z,z')=\sum_{n=-\infty}^{\infty}e^{ink_0(z-z')}\left(1-\frac{i}{h_n}\frac{d}{dz}\right)\cos(h_nX(z)), \tag{7b}$$

$$G_r(z,z')=\sum_{n=-\infty}^{\infty}e^{ink_0(z-z')}\left[\cos(\kappa_nX_0(z))-\cos(h_nX(z))-i\frac{d}{dz}\left(\frac{\varepsilon h_n}{\kappa_n^2}\cos(\kappa_nX_0(z))+\frac{1}{h_n}\cos(h_nX(z))\right)\right]. \tag{7c}$$

It can be shown that $G_r(z,z')$ is regular about both variables.

Representing $\cos(h_nX(z))$ and $\sin(h_nX(z))$ in terms of exponents, and using relation

$$\sum_{n=-\infty}^{\infty}\delta(x+nd)=\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}e^{ink_0x},$$

we can take the integrals associated with the static part of the kernel in a closed analytical form:

$$\int_{-d/2}^{d/2}G_e^{st}(z,z')\Psi(z')dz'=\frac{1}{2}[e^{ik_zX(z)}\Psi(z+X(z))(1+X'(z))+e^{-ik_zX(z)}\Psi(z-X(z))(1-X'(z))]. \tag{8}$$

Such a transformation is fully correct if the functions $f_{\pm}(z)=z\pm X(z)$ are monotonous, i.e., $|X'(z)|\leqslant 1$.

Thus, finally, we have an IE of the second kind with the regular kernel, and shifts in the arguments of the unknown function:

$$\frac{1}{2}[e^{ik_zX(z)}\Psi(z+X(z))(1+X'(z))+e^{-ik_zX(z)}\Psi(z-X(z)) \times(1-X'(z))]+\int_{-d/2}^{d/2}[G_r(z,z') + G_b(z,z')]\Psi(z')dz'=0. \tag{9}$$

Thus, the obtained IE [Eq. (9)] is mathematically rigorous if the condition $|X'(z)|\leqslant 1$ holds (also see Ref. [27]). It contains all information about spectral properties of the system considered. Also, having solved it, we will be able to calculate the field distribution over the entire waveguide.

III. QUASISTATIC APPROXIMATION

In the general case the IE (9) should be analyzed by numerical methods. However, in order to obtain more insight into the properties of its solutions, here we consider the quasistatic approximation, allowing us to greatly simplify its kernel.

The relevant quasistatic equation can be easily obtained from Eq. (9), when the speed of light tends into infinity, and assuming that the beam is nonrelativistic. This yields

$$G_r(z,z')\rightarrow 0,$$

$$G_b(z,z')\rightarrow G_b^{st}(z,z')=-\sum_{n=-\infty}^{\infty}\left(1-\frac{i}{h_n}\frac{d}{dz}\right) \times e^{ink_0(z-z')}\frac{2\pi c^2h_n\nu}{(\omega-h_nv)^2|\varepsilon|^{1/2}} \times\cos(h_n|\varepsilon|^{1/2}x_b)\sin[h_n|\varepsilon|^{1/2}(X_0(z)-x_b)]. \tag{10}$$

The static part of the beam fraction of the kernel $G_b^{st}(z,z')$ can be calculated in a closed analytical form. For simplicity, assuming $x_b=0$ and representing $\sin(h_nX(z))$ in Eq. (10) in terms of exponents, we rewrite $G_b^{st}(z,z')$ in the form

$$G_b^{st}(z,z')=\frac{i\pi c^2\nu}{|\varepsilon|^{1/2}d}[e^{ik_zX(z)}F(z+X(z)-z')(1+X'(z)) +e^{-ik_zX(z)}F(z-X(z)-z')(1-X'(z))], \tag{11}$$

where

$$F(x)=\sum_{n=-\infty}^{\infty}\frac{h_n}{(\omega-h_nv)^2}e^{ink_0x}.$$

The expression for $F(x)$ can be reduced to a combination of the table sums (see the Appendix), and over the interval $x\in(-d,d)$ can be represented as

$$F(x)=\frac{1}{k_0v^2}\left[\frac{\omega}{k_0v}\frac{\partial}{\partial\Delta}+1\right]f(\Delta,x), \tag{12a}$$

where

$$f(\Delta,x)=-\pi\frac{\exp[i\Delta(k_0x-\pi\mathrm{sgn}(x)]}{\sin\pi\Delta}, \tag{12b}$$

$$\Delta=\frac{\omega-k_zv}{k_0v},\quad \mathrm{sgn}(x)=\begin{cases}1, & x>0\\ -1, & x<0.\end{cases}$$

Note that $F(x)$ has a jump of the first kind at $x=0$. The accurate method of integration accounting this has been shown in the Appendix. Making use (12a) and (12b) the integral term in Eq. (9) can be transformed to the (details can be found in the Appendix)

$$\int_{-d/2}^{d/2}G_b^{\mathrm{st}}(z,z')\Psi(z')dz' =-\frac{\pi i\nu}{\beta^2|\varepsilon|^{1/2}}\left(1+\frac{\omega}{k_0v}\frac{\partial}{\partial\Delta}\right)\frac{e^{i\Delta k_0z}}{\sin\pi\Delta} \times\{\cos\pi\Delta[C_+(1+X'(z))e^{i(\omega/v)X(z)} -C_-(1-X'(z))e^{-i(\omega/v)X(z)}]+i\sin\pi\Delta[\varphi_+(z) \times(1+X'(z))e^{i(\omega/v)X(z)}-\varphi_-(z) \times(1-X'(z))e^{-i(\omega/v)X(z)}]\}, \tag{13}$$

where

$$C_{\pm}=\int_{-d/2\pm s}^{d/2\pm s}\exp(-i\Delta z)\Psi(z)dz,$$

$$\varphi_{\pm}(z)=\int_{z\pm X(z)}^{d/2\pm s}\exp(-i\Delta z')\Psi(z')dz' -\int_{-d/2\pm s}^{z\pm X(z)}\exp(-i\Delta z')\Psi(z')dz',s =X(d/2).$$

Thus the static part of the beam fraction of the kernel $G_b^{st}(z,z')$ is reduced to a degenerated Volterra-type kernel, allowing us, in principle, to transform the relevant IE into an ordinary differential one.

Here we proceed more simply, assuming that the beam weakly modifies the temporal and spatial dependencies of plasma perturbations compared to the ''cold'' case, resulting in only slow changes of their amplitudes and phases. The most efficient interaction between the beam and plasma waves in such a case should be expected when $\Delta\simeq m$, i.e., $k_z+mk_0=\omega/v$, that corresponds to the resonance of the beam with the mth spatial harmonic of the plasma wave field. Following this argument we can neglect the term which is proportional to $\sin\pi\Delta$ in figure brackets in the right-hand side of Eq. (13). Taking into account that for plasma-filled devices the unequality $\omega/k_0v\ll 1$ normally holds, we can leave only the largest term (proportional to $\sin^{-2}\pi\Delta$), after taking the partial derivation with respect to Δ in Eq. (13). As a result we obtain the equation

$$\begin{aligned} &e^{ik_zX(z)}(1+X'(z))\Psi(z+X(z))+e^{-ik_zX(z)}(1-X'(z))\\ &\quad\times\Psi(z-X(z))\\ &\quad=\frac{i\nu}{\beta^2|\varepsilon|^{1/2}}\left(\pi\cot\pi\Delta-\frac{\omega}{k_0\nu}\frac{\pi^2}{\sin^2\pi\Delta}\right)\\ &\quad\times[C_+(1+X'(z))e^{i(\omega/v)X(z)}\\ &\quad-C_-(1-X'(z))e^{-i(\omega/v)X(z)}]. \end{aligned} \tag{14}$$

Equation (14) hardly can be solved without further simplifications, even without the beam $(\nu=0)$, since the unknown function $\Psi(z)$ has shifts in the argument. In Sec. IV one partial case will be considered when the approximate analytical solution can be obtained. This will enable us to obtain some insight about the properties of the solution, which can be very useful for developing a more rigorous analysis.

IV. FIRST-ORDER CLOSED FORM SOLUTION

Obviously the Trivelpiece-Gould waves in the smooth waveguide become electrostatic at frequencies near Ω_e, i.e., when $|\varepsilon|\ll 1$. In this region the condition $|X'(z)|\ll 1$ is satisfied even for comparatively large ripples. Shifts in the arguments of the unknown function $\Psi(z)$ on the left-hand side of Eq. (14) can be much less than its period d. This gives us the opportunity to expand $\Psi(z\pm X(z))$ into the Taylor series, thereby eliminating the shifts in its argument. So, after expansion in the first-order approximation, we have the first-order inhomogeneous differential equation

$$\Psi'(z)-if_0(z)\Psi(z)=f_b(z), \tag{15}$$

where

$$f_0(z)=\frac{\cos k_zX(z)+iX'(z)\sin k_zX(z)}{X(z)(\sin k_zX(z)-iX'(z)\cos k_zX(z))},$$

$$f_b(z)=\frac{i\nu}{\beta^2|\varepsilon|^{1/2}}\left(\pi\cot\pi\Delta-\frac{\omega}{k_0\nu}\frac{\pi^2}{\sin^2\pi\Delta}\right)\frac{e^{i\Delta k_0z}[C_+(1+X'(z))e^{i(\omega/v)z}-C_-(1-X'(z))e^{-i(\omega/v)z}]}{X(z)(i\sin k_zX(z)+X'(z)\cos k_zX(z))}.$$

The general solution of Eq. (15) looks like

$$\Psi(z)=e^{i\phi(z)}\left[C_1+\int_0^z f_b(z')e^{-i\phi(z')}dz'\right], \tag{16}$$

where $\phi(z)=\int_0^z f_0(z')dz', C_1$ is an arbitrary constant.

Further, by calculation of $f_b(z)$, we can use the ''cold'' solution $\Psi(z)\approx C_1e^{i\phi(z)}$ that allows us to express the unknown constants $C_\pm$ in terms of C_1. Requiring a periodicity of $\Psi(z)$: $\Psi(z+d)=\Psi(z)$, yields the dispersion relation

$$\exp\left(-i\int_0^d f_0(z)dz\right)-1=\int_0^d f_{b0}(z)e^{-i\phi(z)}dz, \tag{17}$$

where

$$f_{b0}(z)=\frac{\nu d}{\beta^2|\varepsilon|^{1/2}}\left(\frac{\omega}{k_0\nu}\frac{\pi^2}{\sin^2\pi\Delta}-\pi\cot\pi\Delta\right)I(\Delta)e^{i\Delta k_0z-i\phi(z)}$$
$$\times\frac{\sin\dfrac{\omega}{v}X(z)-iX'(z)\cos\dfrac{\omega}{v}X(z)}{X(z)(\sin k_zX(z)-iX'(z)\cos k_zX(z))},$$

$$I(\Delta)=\frac{1}{d}\int_{-d/2}^{d/2}e^{-i\Delta k_0z}\Psi(z)dz.$$

V. ''COLD'' SOLUTION AND ITS PROPERTIES

The first-order ''cold'' dispersion relation can be obtained from Eq. (17) assuming that $f_{b0}(z)=0$, which yields the set of relations

$$\int_0^d f_0(z)dz=-2\pi m,\quad m=0,\pm1,\pm2,\ldots. \tag{18}$$

Thus, instead of the infinite determinant (4), which specifies radial and axially shifted modes simultaneously, we now have the infinite set of dispersion relations. Every dispersion relation is connected with a certain axially shifted branch. As a matter of fact, let $k_z(\omega)$ be the solution of Eq. (18) at $m=0$. Then it can easily be shown that $k_n(\omega)=k_z(\omega)+nk_0$ is the solution of Eq. (18) for $m=n$, with an accuracy of the order of $|\varepsilon|\ll 1$ for not very large $|n|$. The deviation from exact periodicity is due to the approximate equation (15) being used instead the exact one (14). It can easily be seen that if $k_z(\omega)$ and $\Psi(z)$ are the eigenvalue and eigenfunction of Eq. (14), respectively (without the beam) then $k_{zm}(\omega)=k_z(\omega)+mk_0$ is also an eigenvalue of (14) but with the other eigenfunction $\Psi_m(z)=\Psi(z)e^{-imk_0z}$. Since $k_z(\omega)$ contains, in turn, an infinite set of curves corresponding to the different radial modes $k_z(\omega)=\{k_{zq}(\omega)\}$, $q=1,2,\ldots$, we have a so-called ''dense'' spectrum [17] when each point within the range $\omega<\Omega_e$, $-\infty<k_z<\infty$ in the (ω,k_z) plane lies either on the curves given by the functions $k_{zq}(\omega)$, $q=0,1,2,\ldots$ or on the curves given by the functions $k_{zqm}(\omega)=k_{zq}(\omega)+mk_0$, $m=0,1,2,\ldots$, or approach infinitely closely to them. However, it should be especially emphasized that for a full definition of the total field on the waveguide axis $E_z(0,z)=\exp(ik_z z)\Psi(z)$ (and consequently in the whole waveguide as well), it is sufficient to exploit any single axially shifted set of eigenvalues (containing only radial modes) and relative eigenfunctions.

Turning to the approximate solution, we note that we can consider only branch for $m=0$ in Eq. (18), for example, with the corresponding expressions for the periodical function $\Psi(z)=C_1\exp(i\phi(z))$ which contain all spatial harmonics. The dispersion relation $\int_0^d f_0(z)dz=0$ defines only radial modes enabling us to avoid the problem of the ''dense'' spectrum [17]. In the limit when the height of ripples tends to zero $(X(z)\rightarrow|\varepsilon|^{1/2}x_0)$, this dispersion relation passes to a dispersion relation for a smooth planar waveguide filled with homogeneous plasma, $\cos(k_z|\varepsilon|^{1/2}x_0)=0$, having solutions like ordinary electrostatic Trivelpiece-Gould waves. Meanwhile $\Psi(z)$ tends to C_1 and $E_z(0,z)$ tends to $C_1e^{ik_z z}$, which is also inherent for a smooth wall waveguide. It should be noted that such a limit for the branch with $m\neq 0$ in Eq. (18) gives $\Psi_m(z)\rightarrow C_1e^{-imk_0 z}$, but the corresponding value for $E_z(0,z)=\Psi_m(z)e^{ik_{zm}z}$ again tends to $C_1e^{ik_z z}$.

Thus, the solutions associated with different m's in Eq. (18) are equivalent, and we can restrict ourselves to a consideration of just one of them (for example with $m=0$), avoiding the problem of the ''dense'' spectrum since it contains only radial modes. Finally, note that since $\int_0^d \mathrm{Im}\{f_0(z)\}dz\equiv 0$, Eq. (18) is equivalent to

$$\int_0^d \mathrm{Re}\{f_0(z)\}dz=2\pi m,\quad m=0,\pm 1,\pm 2,\ldots .$$

VI. CHERENKOV INSTABILITIES OF PLASMA SPATIAL HARMONICS

Turning to the dispersion relation with the beam, note that since $f_{b0}(z)\sim\nu\ll 1$, from Eq. (17) we can drive the more simple relation

$$\int_0^d f_0(z)dz=i\int_0^d f_{b0}(z)e^{-i\phi(z)}dz, \tag{19}$$

which coincides with Eq. (18) in leading order with respect to ν, and corresponds to the branch which tends to that described by Eq. (18) with $m=0$ at $f_{b0}(z)\rightarrow 0$.

In the general case, the integrals in Eq. (18) can be calculated by numerical methods. In order to obtain analytical expressions for the increments of the instabilities, consider the sinusoidally rippled waveguide $X_0(z)=x_0(1+\alpha\cos k_0 z)$, and expand $f_0(z)$ and $f_{b0}(z)$ in α powers up to the first order. The resulting dispersion relation can be written in the form

$$\cos(k_z x_0|\varepsilon|^{1/2})=\frac{\nu d}{\beta^2|\varepsilon|^{1/2}}\left(\frac{\omega}{k_0 v}\,\frac{\pi^2}{\sin^2\pi\Delta}-\pi\cot\pi\Delta\right)\times|I(\Delta)|^2\sin\left(\frac{\omega}{v}x_0|\varepsilon|^{1/2}\right), \tag{20}$$

where

$$I(\Delta)=\frac{1}{d}\int_{-d/2}^{d/2}e^{-i\Delta k_0 z}\exp\left(-i\alpha\frac{k_z}{k_0}\sin k_0 z+\alpha\cos k_0 z\right)dz.$$

One of the main distinctions of Eq. (20) from that obtained earlier is that the right-hand side of Eq. (20) contains the resonant denominators $\sin^{-2}\pi\Delta$ and $\sin^{-1}\pi\Delta$ which take into account the interaction of the beam with all plasma harmonics simultaneously, and we can analyze the contribution of each of them. They can lead to a widening at the region of unstable frequencies and wave numbers. The term proportional to $\sin^{-1}\pi\Delta$ causes an asymmetry between fast and slow space charge waves with respect to beam lines $\omega=k_z v+mk_0$, which will be especially remarkable at low frequencies. It should be noted that in the limit of $\alpha\rightarrow 0$ and $\Delta\ll 1$, the obtained equation (20) coincides exactly with that for the smooth planar waveguide.

At frequencies when $\Delta\approx m$, we certainly have unstable solutions. In this case the integral $I(\Delta)$ has a clear physical meaning. It is not difficult to recognize that Eq. (3) is proportional to the mth Fourier coefficient of the longitudinal electric field on the waveguide axis:

$$I(\Delta)|_{\Delta=m}=\frac{1}{d}\int_{-d/2}^{d/2}e^{-imk_0 z+i\phi(z)}dz\sim\frac{1}{d}\int_{-d/2}^{d/2}e^{-imk_0 z}\Psi(z)dz\equiv a_m. \tag{21}$$

Consequently the instability near $\Delta\approx m$ can be conditionally interpreted as the Cherenkov instability of the mth spatial plasma harmonic. The spatial growth rates for them can be estimated as

$$\delta_{mn}=\sqrt{\frac{3}{2}}\left[\frac{\nu x_0}{\beta^2}\left(\frac{\omega_{mn}}{k_0 v}\right)^2\left(\frac{d}{x_0|\varepsilon(\omega_{mn})|^{1/2}}\right)J_m^2\left(\frac{\alpha k_z(\omega_{mn})}{k_0}\right)\times\left(1+\frac{mk_0}{k_z(\omega_{mn})}\right)^2\right]^{1/3}k_0, \tag{22}$$

where $J_m(x)$ is the Bessel function of the mth order, and $k_z(\omega_{mn})=(\omega_{mn}/v)+mk_0$, ω_{mn} are the synchronous frequencies at which the shifted beam lines $\omega=k_z v+mk_0$ cross with the curve for the nth radial mode (see Fig. 2). They can be approximately determined from the relation

$$\left(\frac{\omega}{v}+mk_0\right)|\varepsilon(\omega)|^{1/2}x_0=\pi(n+1/2),$$

$$m=0,\pm 1,\pm 2,\ldots,\quad n=0,1,2,\ldots$$

It is interesting to compare the increment for the ''-1'' spatial harmonic (lowest radial mode) given by Eq. (22) for $m=-1$ and $n=0$ with that following from the traditional approach [18]:

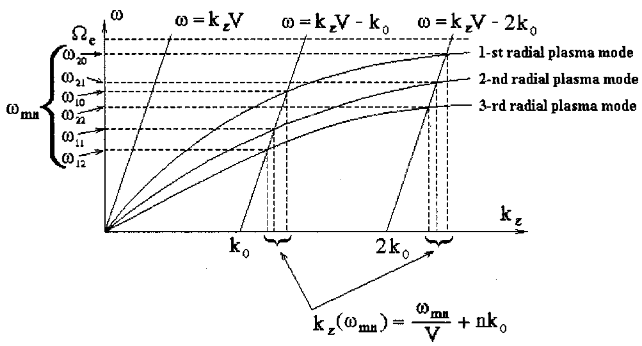

FIG. 2. Qualitative picture of the Cherenkov interaction between the spatial plasma and beam harmonics (for clarity, only three curves for radial plasma modes and three curves for spatial beam harmonics are shown).

$$\delta'_{-10} = \sqrt{\frac{3}{2}}\left[\frac{\pi \nu x_0}{\beta^2}\left(\frac{\omega_{-10}}{k_0 v}\right)^2 \frac{|k_z(\omega_{-10})|}{k_0}\alpha^2\right]^{1/3} k_0. \quad (23)$$

It can be easily seen that

$$\delta_{-10} \simeq \left(\frac{2d}{\pi x_0 |\varepsilon(\omega_{-10})|^{1/2}}\right)^{1/3} \delta'_{-10} > \delta'_{-10},$$

since $(d/x_0|\varepsilon|^{1/2}) \gg 1$ is provided in our approximation. Thus, the obtained value for the spatial growth rate turns out to be remarkably larger than that following from the previous considerations based on the analysis of the truncated determinant [Eq. (4)].

The obtained enhancement in the increment can be physically interpreted by the following way. Within our approach we are able to find a periodic function $\Psi(z)$ which contains all spatial harmonics of the plasma wave field. We also take into account all beam harmonics which form the beam part of the IE kernel [see Eq. (11)] also being periodical. Thus the frequency providing the synchronization of the lowest beam harmonic with, for example, the -1 spatial plasma harmonic, simultaneously provides the synchronization between the 1 spatial beam harmonic and the lowest spatial plasma harmonic, the -1 beam spatial harmonic and the -2 spatial plasma harmonic, and so on, i.e., the nth spatial beam harmonic with the $(n-1)$th spatial plasma harmonic. Each spatial beam harmonic amplifies the corresponding synchronous spatial harmonic of the plasma wave field. Hence the resulting value for increment (22) takes into account all these elementary interactions and characterizes the growth of the total field at this frequency.

Meanwhile the value for increment (23) takes into account only the interaction between the lowest spatial beam harmonic and the -1 spatial harmonic of the plasma wave field. The contribution from resonant interactions between the highest spatial beam and plasma harmonics tends to be lost; hence the behavior of the total field is changed.

The obtained estimations also predict a more efficient beam-plasma wave energy transfer in the nonlinear regime. Considering trapping as a basic mechanism of the Cherenkov instability saturation, we can write the estimation for the trapping amplitude of plasma oscillations [4]:

$$m(v_{\mathrm{ph}} - v)^2 = \frac{eE_z v}{\omega} \quad (24)$$

$v_{\mathrm{ph}} = \omega/(\omega/v + \delta) \simeq v(1 - \delta v/\omega)$, and δ is a linear increment. Thus Eq. (24) shows that, in the saturation regime, E_z scales with a linear increment like δ^2.

The energy density of plasma oscillations in the saturation being proportional, $|E_z|^2$ scales with the increment like δ^4. Hence the enhancement in the energy transfer predicted by our estimations is rather considerable. The energy density of the plasma oscillations in the saturation regime is higher than that following from conventional estimations by a factor of $(d/x_0|\varepsilon|^{1/2})^{4/3} \gg 1$.

VII. CONCLUSION

Introducing the plasma into periodical structures leads to crucial changes in their electrodynamic properties, regardless of the type of structure and plasma configuration. In the low frequency region $\omega < \Omega_e$, an interesting sort of spectral behavior such as a ''dense'' spectrum certainly appears [19]. Conventional techniques can hardly be used to analyze such spectral behavior, therefore, its effect on the operation of various experimentally realized plasma-filled devices such as BWO's [5–7], TWT's [8], pasotron's, and some others has not yet been studied.

In this paper a constructive approach allowing a consideration of beam interaction with low frequency plasma waves in a periodical plasma-filled waveguide has been suggested. It provides a possibility to obtain a greater understanding of the numerical difficulties associated with the analysis of such systems. It is shown that the relative boundary value problem is equivalent to a singular IE. The singularity in the kernel of the IE seems to be the main reason why the conventional analysis in this case is hardly possible in principle, yielding the nonconvergence of the numerical results [18]. The simplest method of regularization is proposed, which provides a passage from the initial singular IE (6) to the second type of IE with the regular kernel [Eq. (10)]. As we can easily see from Eq. (10), the singularity in the kernel of the initial IE contributes to shifts of the unknown function argument, leading to significant distinctions in the final results. In turn, the IE (10), with a regular kernel, also hardly permits direct numerical analysis, since its eigenvalues $k_{zqm} = k_{zq}(\omega) + mk_0 (q = 0,1,2 \ldots$ and $m = 0, \pm 1, \pm 2, \ldots)$, with corresponding eigenfunctions $\Psi_{qm}(z) = \Psi_q(z)e^{-imk_0 z}$, where integer q numerates the radial modes, cause the ''dense'' spectrum. However, it should be pointed out that any single axially shifted branch containing only radial modes with corresponding eigenfunctions is enough to fully specify the total field distribution within the waveguide, and others do not give us any new information concerning this matter. Thus to avoid the ''dense'' spectrum we must select only one branch from all axially shifted ones.

The obtained approximate solution demonstrates the analytical method of such a separation, thereby providing passage from a ''dense'' spectrum to a normal type of spectrum, with a further analysis of the latter. The obtained concrete expressions for the eigenvalues and eigenfunctions can be used in a numerical analysis of more general cases which can be carried out on the basis of some iterative procedure. The second type of IE [Eq. (10)] is certainly very suitable for this aim.

Turning to the approximate solution, we note that it can also have an independent meaning, providing a possibility to understand in detail the peculiarities of beam-plasma interaction in periodical systems, and visually demonstrating a method of overcoming the problem of the ''dense'' spectrum. Concerning the concrete results, it should be stressed that the static part of the field formed by the highest spatial plasma and beam harmonics can be of principal importance. The obtained values for the spatial growth rates turn out to be remarkably larger than that obtained by the conventional approach, providing a truncation of the infinite matrix [18].

It should also be noted that the first-order approximation considered cannot provide the full information about the features of the plasma wave spectrum. In particular, it does not split the dispersion curves near the points $k_z=(m/2)k_0$, where a solution like Eq. (1) loses the property of linear independence. Thus the obtained values for the spatial increments (22) are valid if the synchronous points $(\omega_{mn},\ k_{zmn})$ are rather far from these points. It can be shown that the effects of splitting of the dispersion curves and formation of the forbidden bands can already be treated in the second approximation, when we will have a second-order differential equation instead of a first-order one [Eq. (15)]. Analysis of this issue can comprise the subject of the separate paper.

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APPENDIX

For a calculation of the function

$$F(x)=\sum_{n=-\infty}^{\infty}\frac{h_n e^{ink_0x}}{(\omega-h_nv)^2},\tag{A1}$$

we use the representation

$$\frac{h_n}{(\omega-h_nv)^2}=\frac{1}{k_0v^2}\left[1+\frac{\omega}{k_0v}\frac{\partial}{\partial\Delta}\right]\left(\frac{1}{n-\Delta}\right),$$

where $\Delta=(\omega-k_zv)/k_0v$. After that it is enough to calculate the sum

$$f(\Delta,x)=\sum_{n=-\infty}^{\infty}\frac{e^{ink_0x}}{n-\Delta}.\tag{A2}$$

Using the equalities

$$\frac{1}{n-\Delta}=\frac{n}{n^2-\Delta^2}+\frac{\Delta}{n^2-\Delta^2},\tag{A3}$$

$$\sum_{n=-\infty}^{\infty}\frac{ne^{ink_0x}}{n^2-\Delta^2}=2i\sum_{n=1}^{\infty}\frac{n\sin nk_0x}{n^2-\Delta^2},\tag{A4}$$

$$\sum_{n=-\infty}^{\infty}\frac{e^{ink_0x}}{n^2-\Delta^2}=2\sum_{n=1}^{\infty}\frac{\cos nk_0x}{n^2-\Delta^2}-\frac{1}{\Delta^2},\tag{A5}$$

and expressions for table sums on the right-hand side of Eqs. (A4) and (A5) [28], $f(\Delta,x)$ can be represented in the form

$$f(\Delta,x)=-\pi\frac{\exp[-i\Delta((2m+1)\pi-k_0x)]}{\sin\pi\Delta},$$

$$md\leqslant x\leqslant(m+1/2)d.\tag{A6}$$

However, for practical calculations it is more convenient to use another representation which is valid within the interval $x\in(-d,d)$:

$$f(\Delta,x)=-\pi\frac{\exp[i\Delta(k_0x-\pi\,\mathrm{sgn}(x))]}{\sin\pi\Delta}.\tag{A7}$$

One can easily show that expressions (A6) and (A7) are equivalent over the pointed interval.

Substituting Eq. (12) into Eq. (11), we can transform the term with the static part of the beam kernel in the following way:

$$\begin{aligned}\int_{-d/2}^{d/2}&G_b^{st}(z,z')\Psi(z')dz\\&=-\frac{\pi i\nu}{|\varepsilon|^{1/2}\beta^2}\left(1+\frac{\omega}{k_0v}\frac{\partial}{\partial\Delta}\right)\int_{-d/2}^{d/2}[(1+X'(z))\\&\quad\times f(\Delta,z+X(z)-z')e^{ik_zX(z)}-(1-X'(z))\\&\quad\times f(\Delta,z-X(z)-z')e^{-ik_zX(z)}]\Psi(z')dz'.\end{aligned}\tag{A8}$$

Note that when $z\in(-d/2,d/2)$, the argument that function $f(\Delta,y)$ belongs to the first term under integration on the right-hand side of Eq. (A8) changes in the range $y=y_+(z,z')=z+X(z)-z'\in(-d+s,d+s)$, where $s=X(d/2)$. However, we can use Eq. (12b) for $f(\Delta,y)$ when its argument is over the interval $y\in(-d,d)$. To achieve this we merely shift the limits of integration on the equal value. Due to the periodicity of the integrand with period d, the value of the integral does not change. Thus,

$$\int_{-d/2}^{d/2} f(\Delta,y_+(z,z'))\Psi(z')dz' = \int_{-d/2+s}^{d/2+s} f(\Delta,y_+(z,z'))\Psi(z')dz. \tag{A9}$$

Now, when $z\in(-d/2,d/2)$, the argument of $f(\Delta,y)$ on the right-hand side of Eq. (A9) changes in the range $y = y_+(z,z') = z+X(z)-z' \in (-d,d)$; consequently we can use the representation (12b) for $f(\Delta,y)$. Manipulations with the second term on the right-hand side of Eq. (A8) are identical. As a result we have

$$\int_{-d/2}^{d/2} f(\Delta,y_-(z,z'))\Psi(z')dz = \int_{-d/2-s}^{d/2-s} f(\Delta,y_-(z,z'))\Psi(z')dz, \tag{A10}$$

where $y_-(z,z') = z-X(z)-z'$. Substituting Eq. (12b) into Eqs. (A9) and (A10), and then transforming the right-hand side of Eq. (A8) with the help of Eqs. (A9) and (A10), after simple algebraic manipulations we obtain Eq. (13).

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