Escape from intermittent repellers: Periodic orbit theory for crossover from exponential to algebraic decay

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We apply periodic orbit theory to study the asymptotic distribution of escape times from an intermittent map. The dynamical zeta function exhibits a branch point which is associated with an asymptotic power law escape. By an analytic continuation technique we compute a pair of complex conjugate zeroes beyond the branch point, associated with a preasymptotic exponential decay. The crossover time from an exponential to a power law is also predicted. The theoretical predictions are confirmed by numerical simulation. Applications to conductance fluctuations in quantum dots are discussed. [S1063-651X(99)13812-1]

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I. INTRODUCTION

Exponential distribution of escape times from chaotic scattering systems should be expected only if the associated repeller is hyperbolic. For intermittent repellers one expects asymptotic power law decay [1,2]. Nevertheless, in numerical simulations one often observes what appears to be a perfect exponential [3,4]. As we will show here, the crossover to a power law may be hard to detect, because the crossover time may be so long to preclude any descent statistics.

The importance of intermittency cannot be overemphasized. A generic Hamiltonian system exhibits a mixed phase space structure. A typical trajectory is intermittently trapped close to the stable islands [5]. But even ergodic billiards may exhibit intermittency, typically if they have neutrally stable orbits. Popular billiards such as the Stadium and the Sinai billiards are of this type.

A quantum dot is an open scattering system in two dimensions, obtained by connecting leads to a cavity. Inspired by quantum chaos research, one likes to contrast shapes of the cavity corresponding to chaotic motion, such as the stadium, with shapes corresponding to integrable motion, such as to the rectangle or the square. Both extreme cases are sensitive to naturally occurring imperfections and one naturally ends up with mixed phase space systems where one component hopefully dominates. Consequently, the signals of underlying chaos or integrability do not show up in as clear cut manner as one would have hoped for.

So far, most of the published analysis of these problems has been numerical and heuristic. In this paper, we obtain from the periodic orbit theory quantitative predictions concerning the asymptotic distribution of escape times from an intermittent map. We will demonstrate that a *preasymptotic* exponential escape law is associated with a pair of complex conjugate zeroes of the zeta function. These zeros will be computed with a resummation technique introduced in Ref. [6]. The truly asymptotic escape distribution will be a power law, and is associated with a branch point of the zeta function. The strength of this power law will also be provided by the resummation scheme, with the exponent determined from the analytic form of the marginal fixpoint. The relative magnitudes of the pre-exponential and the power law will yield a good estimate of the crossover time, which will be surprisingly high.

II. ESCAPE AND PERIODIC ORBITS

Much of the early work on cycle expansion [7] was concerned with escape from (hyperbolic) repellers, so we can follow Ref. [7] rather closely when deriving the basic formulas relating escape to the periodic orbits of the repeller. Consider a one-dimensional (1D) map, on some interval I, with L monotone branches $f_i(x)$ where $0 \leqslant i \leqslant L-1$. Each branch $f_i(x)$ is defined on an interval I_i. A generating partition is then given by $\mathcal{C}^{(1)} = \{I_0, I_1 \dots I_{L-1}\}$. We want the map to admit an unrestricted symbolic dynamics. We therefore require all branches to map their domain $f_i(I_i) = I$ onto some interval $I \supset \mathcal{C}^{(1)}$ covering $\mathcal{C}^{(1)}$. A trajectory escapes whenever some iterate of the map $x \notin \mathcal{C}^{(1)}$.

The nth level partition $\mathcal{C}^{(n)} = \{I_q ; n_q = n\}$ can be constructed iteratively. Here q are words of length n built from the alphabet $\mathcal{A} = \{i; 0 \leqslant i \leqslant L-1\}$. An interval is thus defined recursively according to

$$I_{iq} = f_i^{-1}(I_q), \tag{1}$$

where iq is the concatenation of letter i with word q. A concrete example will be given in Eq. (16) and Fig. 1. Next define the characteristic function for the nth level partition

$$\chi^{(n)}(x) = \sum_q^{(n)} \chi_q(x), \tag{2}$$

where

$$\chi_q(x) = \begin{cases} 1 & x \in I_q, \\ 0 & x \notin I_q. \end{cases} \tag{3}$$

An initial point surviving n iterations must be contained in $\mathcal{C}^{(n)}$. Starting from an initial (normalized) distribution $\rho_0(x)$ we can express the fraction that survives n iterations as

$$\Gamma_n = \int \rho_0(x) \chi^{(n)}(x) dx. \tag{4}$$

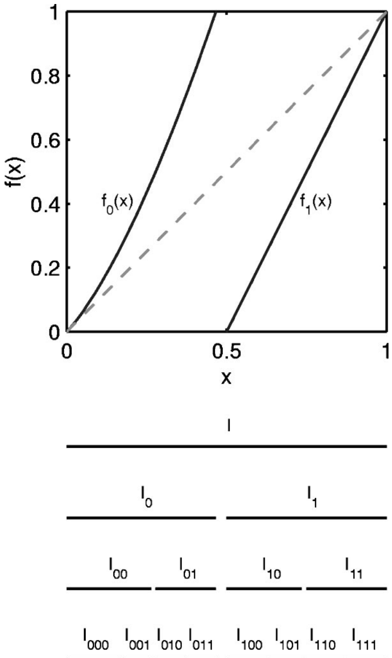

FIG. 1. The intermittent map (16) for the parameter values $s=0.7$ and $p=1.2$. The map is defined on the interval I. Below the map is also shown the partitions $\mathcal{C}^{(1)}=\{I_0,I_1\}$, $\mathcal{C}^{(2)}$, and $\mathcal{C}^{(3)}$.

We choose the distribution $\rho_0(x)$ to be uniform on the interval I. The survival probability is then given by

$$\Gamma_n=a\sum_q^{(n)} |I_q|, \tag{5}$$

where

$$a^{-1}=\int_I dx=|I|. \tag{6}$$

Assuming hyperbolicity the size of I_q can be related to the stability $\Lambda_q=(d/dx)f^n(x)|_{x\in q}$ of periodic orbit $\overline{q}$ according to

$$|I_q|=b_q\frac{1}{|\Lambda_q|}, \tag{7}$$

where $b_q=O(|I|)$, can be bounded close to the size of I. This results from the fact that (i) $f^n(I_q)=I$, (ii) the smallness of $|I_q|$, and (iii) the fact that derivative is bounded by the assumption of hyperbolicity. We will eventually relax this assumption, but for the moment we will stick to it.

The survival fraction can now be bounded by a sum over periodic orbit according to

$$C_1(N)\sum_q^{(n)} \frac{1}{|\Lambda_q|}<\Gamma_n<C_2(N)\sum_q^{(n)} \frac{1}{|\Lambda_q|} \tag{8}$$

for all $n>N$.

The periodic orbit sum in Eq. (8) will be denoted Z_n

$$\sum_q^{(n)} \frac{1}{|\Lambda_q|}\equiv Z_n \tag{9}$$

and can be rewritten as a sum over primitive periodic orbits (period n_p) and their repetitions

$$Z_n=\sum_p n_p\sum_{r=1}^{\infty}\frac{\delta_{n,rn_p}}{|\Lambda_p|^r}. \tag{10}$$

It is closely related to the trace of the Perron-Frobenius operator

$$Z_n\approx \mathrm{tr}\,\mathcal{L}^n=\int dx\,\delta[x-f^n(x)]=\sum_p n_p\sum_{r=1}^{\infty}\frac{\delta_{n,rn_p}}{|\Lambda_p^r-1|}. \tag{11}$$

By introducing the zeta function

$$\zeta^{-1}(z)=\prod_p\left(1-\frac{z^{n_p}}{|\Lambda_p|}\right), \tag{12}$$

Z_n can be expressed as a contour integral

$$Z_n=\frac{1}{2\pi i}\int_\gamma z^{-n}\left(\frac{d}{dz}\log\zeta^{-1}(z)\right)dz, \tag{13}$$

where the small contour γ encircles the origin in negative direction.

The expansion of the zeta function to a power series is usually referred to as a *cycle expansion*:

$$\zeta^{-1}(z)=\sum c_n z^n. \tag{14}$$

This representation converges up to the leading singularity. Its domain of convergence is therefore usually larger than that of the product representation (12), which diverges at (nontrivial) zeroes. If the zeta function $\zeta^{-1}(z)$ is analytic in a disk extending beyond the leading zero z_0, then the periodic orbit sum Z_n, and hence the survival probability Γ_n, will decay asymptotically as

$$Z_n\sim z_0^{-n}\equiv e^{-\kappa n}, \tag{15}$$

where $\kappa=\ln z_0$ is the escape rate.

We will introduce intermittency in connection with a specific model. We then consider an intermittent map $x\mapsto f(x)$ with two branches $(L=2)$, where I is chosen as the unit interval

$$f(x)=\begin{cases} f_0(x)=x(1+p(2x)^s) & x\in I_0=\{x;0\leq x<\xi\},\\ f_1(x)=2x-1, & x\in I_1=\{x;1/2\leq x\leq 1\}.\end{cases} \tag{16}$$

The map is intermittent if $s>0$ and allows escape if $p>1$. The map is shown in Fig. 1, together with some of its partitions. The right edge of the left branch I_0, here denoted ξ, is defined implicitly by $f_0(\xi)=1$. The trajectory escapes when $\xi(s)<x<\frac{1}{2}$.

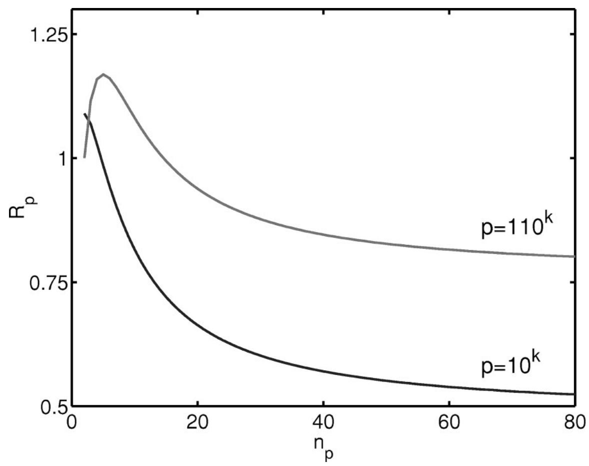

FIG. 2. The quantity R_p plotted for the sequences $p=10^k$ and $p=110^k$ versus length n_p.

The intermittent property is related to the fact that the cycle $\overline{0}$ is neutrally stable $f'(0)=1$. Consequently, cycle stabilities can no longer be exponentially bounded with length. This loss of hyperbolicity makes it difficult to relate the survival probability Γ_n to the periodic orbit sum Z_n. Indeed, Eq. (7) is brutally violated in some cases, as can be realized from the following example. The problem of intermittency is best represented by the family of periodic orbits 10^k. It follows from Eqs. (1) and (16) that $|I_{10^k}|=\frac{1}{2}|I_{0^k}|$. It can be shown [6] that $|I_{0^k}|\sim 1/k^{1/s}$ and thus

$$|I_{10^k}|\sim\frac{1}{k^{1/s}} \tag{17}$$

which should be compared with the asymptotic behavior of the stabilities

$$1/\Lambda_{10^k}\sim\frac{1}{k^{1+1/s}}. \tag{18}$$

The difference in power laws seems to spoil every possibility of a bound such as Eq. (8). However, Eq. (7) is not necessary to establish a bound such as Eq. (8). It suffices if the ratio

$$R_p=\frac{|\Lambda_p|}{n_p}\sum_{k=1}^{n_p}|I_{\mathcal{S}^k p}| \tag{19}$$

stays bounded. That it to say that it suffices if the *average size* of the intervals along a cycle can be related to the stability, rather than each interval separately. Here $\mathcal{S}$ denotes the cyclic shift operator $\mathcal{S}(p=s_1,s_2s_n)=s_2s_n,s_1$. We check this numerically on two sequences 10^k and 110^k, the former being most prone to intermittency. The result is plotted in Fig. 2. We note that for both sequences, R_p appear to tend to well defined limits, where 10^k exhibit the largest deviation from unity. Indeed, it is reasonable to assume that the sequence 10^k provides a lower bound

$$R_p>\lim_{k\to\infty}R_{10^k},\quad \forall p. \tag{20}$$

In view of this, the numerical results strongly suggest that R_p stays bounded, and that, for this particular system, C_1 in Eq. (8) can be chosen as $C_1=0.5$ and C_2 presumably close to unity. This is a surprisingly low price to pay for the complication of intermittency.

The sizes of the intervals I_{0^n} has no relation whatsoever to the stability of the cycle $\overline{0}$ (being unity). We exclude the intervals I_{0^n} from our considerations by *pruning* the fixed point from the zeta function

$$\zeta^{-1}(z)=\prod_{p\neq 0}\left(1-\frac{z^{n_p}}{|\Lambda_p|}\right). \tag{21}$$

The contribution from I_{0^n} to Γ_n can be added separately if required. Since the results rely on summation along periodic orbits, it might break down for some choices of the initial distribution $\rho_0(x)$.

III. RESUMMATION AND SIMULATION

After having argued that the survival probabilities Γ_n still can be bounded close to periodic orbit sums Z_n we turn to the problem of computing the asymptotics of these quantities. The coefficients of the cycle expansion (14) for the map (16) decay asymptotically as

$$c_n\sim\frac{1}{n^{1+1/s}}, \tag{22}$$

which induces a singularity of the type $(1-z)^{1/s}$ in the zeta function [6]. If $1/s$ is an integer, the singularity is $(1-z)^{1/s}\log(1-z)$.

To evaluate the periodic orbit sum it is convenient to consider a resummation of the zeta function around the branch point $z=1$:

$$\zeta^{-1}(z)=\sum_{i=0}^{\infty}c_iz^i=\sum_{i=0}^{\infty}a_i(1-z)^i+(1-z)^{1/s}\sum_{i=0}^{\infty}b_i(1-z)^i. \tag{23}$$

In practical calculations one has only a finite number of coefficients c_i, $0\leq i\leq n_c$ of the cycle expansion at disposal. Here n_c is the cutoff in (topological) length. In Ref. [6] we proposed a simple resummation scheme for the computation of the coefficients a_i and b_i in Eq. (23). We replace the infinite in Eq. (23) sums by finite sums of increasing degrees, n_a and n_b, and require that

$$\sum_{i=0}^{n_a}a_i(1-z)^i+(1-z)^{1/s}\sum_{i=0}^{n_b}b_i(1-z)^i =\sum_{i=0}^{n_c}c_iz^i+O(z^{n_c+1}). \tag{24}$$

One proceeds by expanding $(1-z)^i$ and $(1-z)^{i+1/s}$ around $z=0$, skipping all powers z^{n_c+1} and higher. If $n_a+n_b+2=n_c+1$ one is then left with a solvable linear system of equations yielding the coefficients a_i and b_i. It is natural to require that $|n_b+1/s-n_a|<1$ so that the maximal powers of the two sums in Eq. (24) are adjacent. Then, for each cutoff

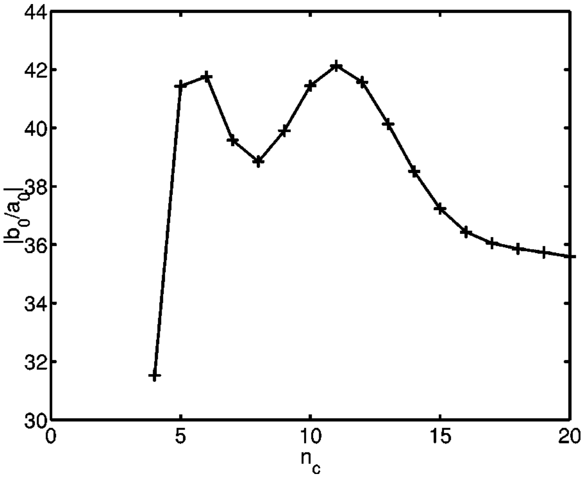

FIG. 3. The ratio b_0/a_0 versus cutoff length n_c.

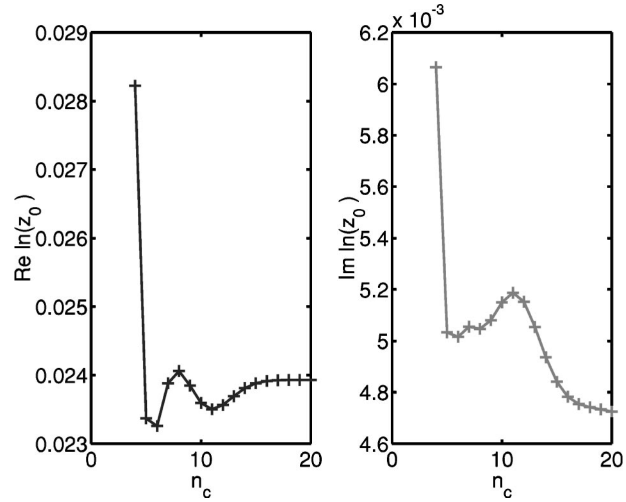

FIG. 4. The real and imaginary part of $\ln z_0$ versus cutoff length n_c.

length n_c the integers n_a and n_b are uniquely determined, and one can study convergence of the coefficients a_i and b_i, or various quantities derived from them (see below), with respect to increasing values of n_c.

If the zeta function is entire (except for the branch cut) the periodic orbit sum Z_n can be written

$$Z_n=\sum_{\alpha} z_{\alpha}^{-n}+\frac{1}{2\pi i}\int_{\gamma_{\rm cut}} z^{-n}\left(\frac{d}{dz}\log \zeta^{-1}(z)\right)dz. \quad (25)$$

The sum is over all zeroes z_α of the zeta function (assuming they are not degenerate) and the contour $\gamma_{\rm cut}$ goes round the branch cut in anti clockwise direction. If poles and/or natural boundaries are present, expression (25) must be modified accordingly.

The leading asymptotic behavior is provided by the vicinity of the branch point $z=1$, and is found to be [8]

$$Z_n \sim \frac{b_0}{a_0}\frac{1}{s}\frac{1}{\Gamma(1-1/s)}\frac{1}{n^{1/s}}, \quad n\to\infty. \quad (26)$$

The relevant ratio b_0/a_0, obtained from the resummation scheme, versus cutoff length n_c is plotted in Fig. 3. In all numerical work we have used the parameters $s=0.7$ and $p=1.2$ and computed all periodic orbits up to length 20. The convergence in Fig. 3 is perhaps not overwhelming but we should bear in mind that we study a quantity which would diverge in a conventional cycle expansion; we are not merely accelerating convergence as in Ref. [6], we are actually attempting an analytic continuation.

There is also a pair of complex conjugate zeroes, $z_0=x_0 \pm i y_0$ beyond the branch cut. They contribute both to the residue sum in Eq. (25) and to the integral around the cut in Eq. (25). But since their imaginary parts $\pm y_0$ are small, they will, in effect, contribute a factor x_0^{-n} to the periodic orbit sum Z_n. This zero will dominate Z_n in some range $0 \ll n_c \ll n_{\rm cross}$ before the asymptotic power law sets in.

In Fig. 4 we study the convergence of the real and imaginary part of z_0 obtained from the resummation scheme above, for different cutoffs n_c. The zero is computed by Newton-Raphson iteration of the left hand side of Eq. (24), with coefficients a_i and b_i provided by the resummation scheme, for various value of the cutoff length n_c. Again we note that the analytic continuation technique works quite satisfactorily.

The probability of escaping at iteration n is

$$p_n=\Gamma_{n-1}-\Gamma_n. \quad (27)$$

We get for this distribution

$$p_n \approx \begin{cases} x_0^{-n}, & 0 \ll n \ll n_{\rm cross}, \\ \dfrac{b_0}{a_0}\dfrac{1}{s^2}\dfrac{1}{\Gamma(1-1/s)}\dfrac{1}{n^{1+1/s}}, & n_{\rm cross} \ll n. \end{cases} \quad (28)$$

Here we have neglected the interval I_{0^n}, having the same asymptotic decay law as the periodic orbit sum Z_n. Due to the uncertainty in the bounds (8) it can be neglected.

The crossover $n=n_{\rm cross}$ takes place when the two terms in Eq. (28) are of comparable magnitude. For our standard set of parameters ($p=1.2$, $s=0.7$) it is found to be $n_{\rm cross} \approx 300$.

To check our predictions we run a simulation of the system. The result can be seen in Fig. 5. We note that the slope of the exponential, the power and magnitude of the power law, as well as the crossover time agrees very well with our predictions.

A reader still in any doubt on the effectiveness of a resummed cycle expansion should consider the following. The simulation in Fig. 5 averaged over 10^8 initial points, yet, in itself the result would not be very conclusive. A direct evaluation of Z_n up to say $n=600$ would require roughly 10^{170} periodic orbits. We have not bothered to perform such a cross check. But a resummed cycle expansion provides reliable answers with a length cutoff as low as $n_c=15$, corresponding to 4719 prime cycles. Admittedly, we benefited from knowing the asymptotic power law of the cycle expansion. However, if this is not the situation, this power law is easily extracted if one uses stability ordering [8].

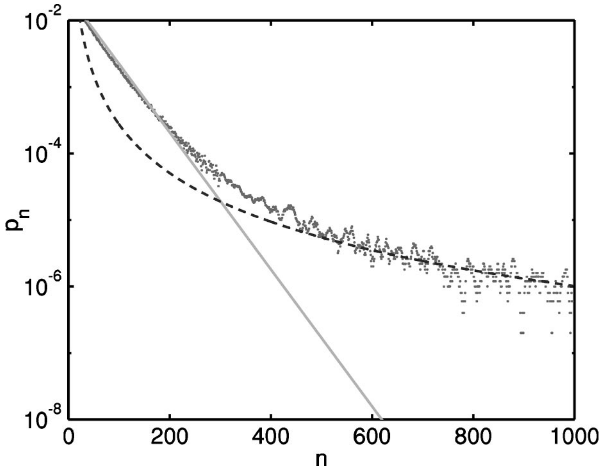

FIG. 5. Distribution of escape times obtained from simulation (shaky curve) and the pre-exponential (full line) and the asymptotic power law (dashed line) obtained from resummation.

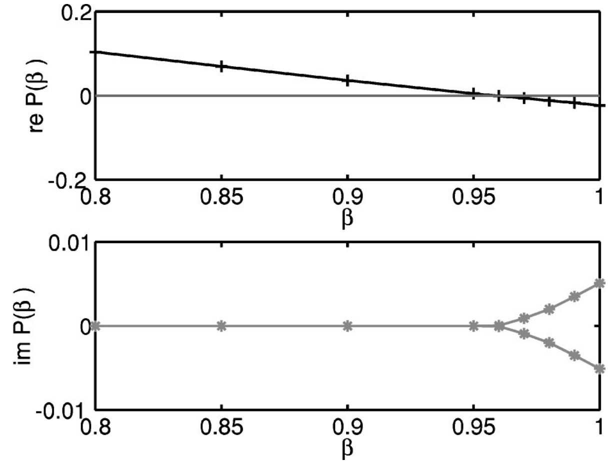

FIG. 6. The real and imaginary part of the leading zero(s) versus the thermodynamic parameter β.

IV. A ZERO IS CUT IN TWO

The occurrence of a dominating zero beyond the branch point is, in fact, very natural and probably generic for a wide class of open systems. Consider the one-parameter family of zeta functions

$$\zeta^{-1}(z;\beta)=\prod_p\left(1-\frac{z^{n_p}}{|\Lambda_p|^\beta}\right). \tag{29}$$

For small enough β there is a leading zero $z_0(\beta)$ within the domain of convergence $|z_0(\beta)|<1$. This is related to the topological pressure [9,10] according to $\mathcal{P}(\beta)=-\log z_0(\beta)$. For instance, $\mathcal{P}(0)$ is the topological entropy. For a certain β (actually the fractal dimension of the repeller) the zero collides with the branch point $z=1$, splits into two, and the complex conjugate pair continues to move out beyond the branch point. This is an example of a phase transition [10,11].

In Fig. 6 we plot the logarithm of the leading zero(s) $[-\ln z_0(\beta)]$ versus β. It is obtained from a resummation analogous to the one discussed above, see Ref. [6]. It can be interpreted as the topological pressure only as long it is real.

V. MESOSCOPIC DISCUSSION

The particular form of the distribution of escape times does depend on the initial distribution $\rho_0(x)$. In this paper we have restricted ourselves to a uniform initial distribution. To model chaotic scattering one must imagine that particles can be injected according to any distribution. For example, one can construct a chaotic scatterer from a bounded billiard by drilling holes anywhere on the boundary and injecting particles with any conceivable distribution of angles. This may even effect the asymptotic power law [12]. Periodic orbit theories can also account for other initial distributions than uniform. However, the preceding discussions about relating periodic orbit sums to survival probabilities warns us to be cautious when doing so for intermittent systems. As it appears, the general rule of thumb, first an exponential, then a cross over to some power law, can be extended to open Hamiltonian systems with a mixed phase space structure [2,12].

An immediate application concerns conductance fluctuations in quantum dots [13]. The Fourier transform $\hat{C}(x)$ of the correlation function $C(\Delta k)=\langle T(k)T(k+\Delta k)\rangle_k$, where $T(k)$ is the transmission as function of the Fermi wave number, can, after several approximations, be related to the escape distribution $p(L)$ [13]

$$\hat{C}(x)\sim\int_0^\infty dL\,p(L+x)p(L). \tag{30}$$

If there is a crossover to a power law in $p(L)$ there will be an associated crossover in $\hat{C}(x)$. For an intermittent chaotic system, the crossover time may be very long—the quasiregular region component of phase space will not make itself noticed until very long times. If the elastic mean free path of electrons is much shorter than the length corresponding to the crossover time, the quasiregular component will never ve detected in this type of experiment. Or the other way around, a small deviation from an integrable structure induces chaotic layers in phase space. This chaotic layer may lead to exponential escape for small times, and the experimental outcome may very well resemble predictions for fully chaotic systems.

In experiments a (weak) magnetic field is a more natural control parameter than the Fermi energy. Instead of the distribution of dwelling times one has to consider the distribution of enclosed areas, a related but more subtle concept which we plan to address in future work. One has observed Lorentzian shape (predicted for chaotic systems) of the so called weak localization peak even in near integrable structures [14]. This has been attributed to naturally occurring imperfections [15,16] and rhymes well with the classical considerations above. Admittedly, we have now moved far from our original intermittent map and entered the realm of speculation. What we do want to point out in this paper is that these kind of problems are well suited for periodic orbit computations—zeta functions are powerful tools for making long time predictions, even for intermittent chaos, once the problems of analytical continuation can be overcome.

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