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Numerical evidence for the existence of a low-dimensional attractor and its implications in the rheology of dilute suspensions of periodically forced slender bodies

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We provide numerical evidence for the existence of a low-dimensional chaotic attractor in the rheology of dilute suspensions of slender bodies in a simple shear flow. The rheological parameters which characterize the stress deformation behavior of the suspension are calculated based on appropriate averages over the orientation vectors of the slender bodies. The system considered in this work, therefore, exhibits chaos in experimentally measurable averages over a large number of uncoupled chaotic oscillators. The numerical demonstration that these parameters may evolve chaotically may thus have important consequences for both chaos theory and suspension rheology. We also provide plausible explanations for the existence of a low-dimensional chaotic attractor in the rheological parameters in terms of the expressions for the rheological parameters and the coupling between individually chaotically evolving orientations and the expressions for the rheological parameters. [S1063-651X(99)05712-8]

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INTRODUCTION

The importance of the field of suspension rheology, both as a scientific discipline and for its various technological applications, has been widely recognized. The properties of fluid suspensions of small particles depend generally on the nature of the fluid, the properties of the suspension particles, and the degree of isotropy of the suspension. For example, the physical properties of heterogeneous media like metal alloys, composites, polymer solutions, electrorheological fluids, etc. are influenced mostly by the properties of the constituent materials and the manner in which they are distributed. In order to obtain a wide variety of properties with the same constituent materials, it is thus necessary to obtain a wide variety of orientation distributions of the constituent materials. For a complete description of the orientation distribution of the particles in a suspension, one must consider many influencing factors, such as the nature and type of the underlying fluid or flow, particle-particle interactions which are a result of the disturbance that the presence of each particle in the suspension produces on all the surrounding particles in the medium; and the effect of Brownian diffusion resulting from the bombardment of the suspension particles by the surrounding fluid molecules. A number of models of suspensions incorporating one or more of these factors have been proposed to describe the orientations of the particles in a suspension, and various applications of these models have been reported in the literature [1–5]. Szeri and co-workers discussed the possibility of periodic and quasiperiodic attractors for the orientation vectors in two-dimensional time-dependent flows and in three-dimensional recirculating flow fields in a series of papers [6–10]. The type of flow fields they considered in their work do not allow for the possibility of chaotic attractors for the evolution of the orientation vectors. For the case of two-dimensional time-dependent flows, their equations for the evolution of the orientation vectors decouple, and hence chaos is not possible. For the three-dimensional flows they studied, the equations of the evolution of the orientation vectors can be written in terms of a fundamental matrix governed by a linear set of equations with periodic coefficients, and hence these equations do not allow for the possibility of chaos.

Demonstration of chaotic dynamics in suspension rheology is a recent development, and remains an area as yet not fully explored. Ramamohan *et al.* [11] were the first to show that the orientation of slender rods in a simple shear flow under the action of an external periodic force varies chaotically for a certain range of values of the parameter corresponding to the external force. They also showed that viscometric material functions, which are indicators of the collective behavior of all the particles in a suspension, also exhibit chaotic dynamics when the orientations of individual particles evolve chaotically.

In this work we explore in more detail certain aspects of the chaotic behavior of the rheological parameters that seem to have broader implications for the collective behavior of spatially extended dynamical systems. The system considered, namely, the system of a periodically forced suspension of orientable particles in a simple shear flow, is a physically realizable example of a spatially extended system in which individual oscillators may evolve chaotically. The most popular model systems for studying the time evolution mechanisms of spatially extended systems are coupled map lattices, cellular automata, and lattices of coupled ordinary differential equations. However, even for these simple systems little is known about the behavior of spatial averages when the local dynamics is chaotic. Two types of behavior have been noted; in one class of oscillators these averages settle down to stationary behavior, and in the other they exhibit robust fluctuations. The system we consider in this work represents an example of a spatially extended system of individually chaotically varying oscillators, for which suit-

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able spatial averages (rheological parameters) also evolve chaotically.

Spatially extended dynamical systems have recently become a subject of extensive research. There are only a few results in the literature, even for these simple systems, concerning the behavior of the spatial averages of such systems when the local dynamics is chaotic. The possibility of the spatial averages of individually chaotically varying oscillators also varying chaotically has only recently been discussed [12]. In this paper, we present numerical evidence for the existence of a low-dimensional chaotic attractor for the rheological parameters of the suspension, with the number of orientation vectors varying from 16 to 484. The variation of the number of the orientation vectors from 16 to 484 implies that the dimension of the system considered varies from 33 to 969. This is because the evolution of each orientation vector is governed by a set of two coupled nonautonomous nonlinear ordinary differential equations. We provide numerical evidence that the invariant properties of the attractor remain unchanged even when we increase the number of orientation vectors. We present plausible explanations for these results in terms of the forms of the averages representing the rheological parameters and the nonlinear coupling of the averages and individual orientations.

CHAOS IN SUSPENSION RHEOLOGY

We use the results of Batchelor [13], as modified by Berry and Russel [14], for the equations governing the dynamics of a slender body in an infinite expanse of fluid. These equations have physical meaning in the case of dilute suspensions in which particle-particle interactions are neglected. Physically this corresponds to the limit $nl^3 \ll 1$, where ''n'' is the number density of the particles in the suspension and ''l'' is half the length of the slender body. Since we neglect particle-particle interactions, the rheological parameters represent averages over the instantaneous orientations of the particles alone. The rheological parameters show no dependence upon the spatial distribution of the particles in the dilute limit, since particle-particle interactions are not taken into account. Hence the spatial distribution of the particles is immaterial as long as the distribution is uniform with n particles per unit volume. The particles are assumed to be sufficiently small that the boundaries of the physical apparatus containing the suspension do not significantly affect the rheology of the bulk of the suspension. The experimental feasibility of studying the dynamics and rheology of small particles under the effect of constant external force fields has been demonstrated by a number of authors [15–20]. The extension to periodic external forces, as studied in this paper, should not pose any additional difficulties.

Under these conditions in a simple shear flow, the equations governing the dynamics of a periodically forced slender body in spherical coordinates are [11]

$$\dot{\theta} = \sqrt{2}\sin\theta\cos\theta\sin\phi\cos\phi + (k_1\cos\theta\cos\phi + k_2\cos\theta\sin\phi - k_3\sin\theta)\cos(\omega t),$$

$$\dot{\phi} = -\sqrt{2}\sin^2\phi + \left(-k_1\frac{\sin\phi}{\sin\theta} + k_2\frac{\cos\phi}{\sin\theta}\right)\cos(\omega t). \quad (1)$$

In the above equations θ and ϕ are the polar and azimuthal angles made by the unit vector which represents the orientation of the particle at the instant t. A typical slender rod has length $2l$ and radius a. k_1, k_2, and k_3 are the components of the orientation-independent part of the external torque, and ω is the frequency of the external driver. The undisturbed velocity profile of the flow is chosen as $\mathbf{v}_0 = \dot{\gamma} y \mathbf{i}$ where $\dot{\gamma}$ is the shear rate, y is the y coordinate, and $\mathbf{i}$ is the unit vector in the x direction. Since the rheological parameters of suspensions of orientable particles are generally determined by the orientations of the particles alone, and are unaffected by their translatory motion, we choose a coordinate system which moves along with the particle and thus neglect any translatory motion of the particle. The singularity in Eq. (1) for ϕ may be removed for computational purposes by taking $\phi_1 = \phi\sin\theta$. The above equations are dimensionless and scaled as follows:

Length	l
Velocity	$\frac{l\dot{\gamma}}{\sqrt{2}}$
Force	$\frac{8}{3\sqrt{2}}\pi\eta_s l^2\dot{\gamma}\frac{1}{\ln(2r)}$
Torque	$\frac{8}{3\sqrt{2}}\pi\eta_s l^3\dot{\gamma}\frac{1}{\ln(2r)}$
Time	$\frac{\sqrt{2}}{\dot{\gamma}}$

where η_s is the solvent viscosity, l is half the length of the rod, and r is the aspect ratio of the particle. In this system the orientation of the particles evolves chaotically for certain ranges of values of the parameters k_1, k_2, and k_3 and initial conditions of θ and ϕ [11]. The system shows a quasiperiodic transition to chaos in this range. In this system, transient chaos also exists, originating from the collision of a stable nonchaotic attractor with a chaotic attractor. The existence of such complex behavior in this system, which is the simplest of a class of systems, is indicative of the possibility for the class of systems considered to exhibit a wide variety of chaotic properties. Further, this system falls into that class of systems for which fluctuations in averages may exist, as discussed by Bunimovich and Jiang [12]. Chaotic dynamics exhibited by similar systems shows potential for practical applications like particle separation based on shape [21] and the possibility of intelligent rheology [22].

Since the results of computations of the evolution of the above system of equations are not easily accessible to experimental verification, the effects of these chaotic orientation distributions on rheological parameters like the viscometric material functions were also studied [23]. The apparent viscosities and normal stress differences are important material functions which characterize the rheology of a suspension in shear flows, and which can be measured by a viscometer. For periodically forced dilute suspensions of non-Brownian slender rods in a simple shear flow, these material functions have expressions in terms of certain orientation averages as given below [notations as in Eq. (1)] [23]:

$$[\eta_1]=75\langle\sin^4\theta\sin^2 2\phi\rangle+300\sqrt{2}k_1\cos(\omega t)\times[\langle\sin^3\theta\sin\phi\cos^2\phi\rangle-\langle\sin\theta\sin\phi\rangle]+75\sqrt{2}k_2\cos(\psi t)\langle\sin^4\theta\sin^2 2\phi\rangle+75\sqrt{2}k_3\cos(\omega t)\times\langle\sin\theta\sin 2\theta\sin 2\phi\rangle, \quad (2)$$

$$[\eta_2]=75\langle\sin^4\theta\sin^2 2\phi\rangle+150\sqrt{2}k_1\cos(\omega t)\times\langle\sin^3\theta\sin 2\phi\cos\phi\rangle+300\sqrt{2}k_2\cos(\omega t)\times[\langle\sin^3\theta\sin^2\phi\cos\phi\rangle-\langle\sin\theta\cos\phi\rangle]+75\sqrt{2}k_3\cos(\omega t)\langle\sin\theta\sin 2\theta\sin 2\phi\rangle, \quad (3)$$

$$[\tau_1]=150[\langle\sin^4\theta\cos^2\phi\sin 2\phi\rangle-\langle\sin^2\theta\cos^2\theta\sin 2\phi\rangle]+300\sqrt{2}k_1\cos(\omega t)[\langle\sin^3\theta\cos^3\phi\rangle-\langle\sin\theta\cos\phi\rangle-\langle\sin\theta\cos\phi\cos^2\theta\rangle]+300\sqrt{2}k_2\cos(\omega t)\times[\langle\sin^3\theta\sin\phi\cos^2\phi\rangle-\langle\sin\theta\sin\phi\cos^2\theta\rangle]+300\sqrt{2}k_3\cos(\omega t)[\langle\sin^2\theta\cos^2\phi\cos\theta\rangle+\langle\cos\theta\rangle-\langle\cos^3\theta\rangle], \quad (4)$$

$$[\tau_2]=150[\langle\sin^4\theta\sin^2\phi\sin 2\phi\rangle-\langle\sin^2\theta\cos^3\theta\sin 2\phi\rangle]+300\sqrt{2}k_1\cos(\omega t)[\langle\sin^3\theta\sin^2\phi\cos\phi\rangle-\langle\sin\theta\cos^2\theta\cos\phi\rangle]+300\sqrt{2}k_2\cos(\omega t)\times[\langle\sin^3\theta\sin^3\phi\rangle-\langle\cos\theta\rangle-\langle\sin\theta\sin\phi\cos^2\theta\rangle]+300\sqrt{2}k_3\cos(\omega t)[\langle\sin^2\theta\sin^2\phi\cos\theta\rangle+\langle\cos\theta\rangle-\langle\cos^3\theta\rangle], \quad (5)$$

The angular brackets represent orientation averages of quantities over various possible orientations of the particles. Kumar [24] studied the case when the particles are aligned in 81 various directions, and evaluated the averages by discretizing the $\theta\phi$ space into 81 grids. This has been done by dividing the range $[0, \pi]$ of θ by $[a_0, a_1, \ldots, a_9]$, where

$$a_i=\cos^{-1} b_i, \quad \text{for} \quad i=0,1,\ldots,9$$

and

$$b_j=-1+j(-\cos\pi+\cos 0)/9, \quad \text{for} \quad j=0,1,\ldots,9,$$

and the range $[0,2\pi]$ of ϕ into nine equal intervals of width $2\pi/9$. He started off from a set of 81 particles uniformly distributed in phase space initially with one particle in each bin. Each particle, in principle, can represent an infinite number of particles aligned in a common direction. If (θ_i, ϕ_j) describes the particle orientation at time t for i, $j=1,2,\ldots,81$, then the orientation average of $B(\theta,\phi)$ at time t is given by

$$\langle B(\theta,\phi)\rangle=\frac{\int\int B(\theta,\phi)\,\delta(\theta-\theta_i)\,\delta(\phi-\phi_j)\,d(\cos\theta)\,d\phi}{\int\int \delta(\theta-\theta_i)\,\delta(\phi-\phi_j)\,d(\cos\theta)\,d\phi}=\frac{1}{81}\sum_{i=1}^{9}\sum_{j=1}^{9} B(\theta_i,\phi_j). \quad (6)$$

Here, the discretization of the $\theta\phi$ space is used to distribute the particles initially, and this distribution becomes closer to the continuous uniform distribution as we increase the number of orientation vectors. The evolution of each of the distributed particles was then calculated using the evolution equations (1) with a standard Runge-Kutta method with an adaptive step size [25]. Each one of the orientation averages at time t is calculated using the orientation description of the 81 particles at time t according to Eq. (6). The numerical integration of Eq. (1) in single precision was shown to be equivalent to treating Eqs. (1) as Langevin equations with weak Brownian motion, as Kumar was able to reproduce the results of Leal and Hinch [3] (also see Ref. [23]) for weak Brownian motion. Note the nonlinear coupling between the expressions for the rheological parameters, namely Eqs. (2)–(5), and the equations for the evolution of the individual orientation vectors, namely, Eqs. (1). We note that both of these sets of equations contain the parameters k_1, k_2, and k_3, and hence the nonlinear interaction between these parameters can result in a low-dimensional chaotic attractor for the rheological parameters.

As a first step in analyzing the effects of the chaotic dynamics on the rheology, in this work we have studied a suspension of slender rods with the number of directions of alignment varying from 16 to 484. Kumar and Ramamohan [23] provided numerical evidence for the existence of chaos in the rheological parameters when $k_1=k_3=0$ and k_2, varying in the range $0\leqslant k_2\leqslant 0.30$ in steps of 0.01, when the particles are aligned in 81 directions. However, Kumar and Ramamohan [23] did not provide numerical evidence for the existence of a low-dimensional attractor in their work. Hence, in this paper, we consider the typical cases $k_1=k_3=0$ and $k_2=0.21$, with $\omega=1$. We studied the time series of $[\tau_1]$, $[\tau_2]$, $[\eta_1]$, and $[\eta_2]$ by taking a minimum of $\tau_s(10^{2+0.4d})$ number of data points with a sampling time τ_s, which is chosen such that the autocorrelation time lies between 1 and 10, where d is the approximate dimension of the attractor of the time series of the appropriate rheological parameter. In order to eliminate the possibility of any transients in the chaotic case, we eliminated the first 15 000 points of the Poincarè section from the time series. All the tests were performed using the software ''Chaos Data Analyzer Professional Version 2.1'' of the Academic Software Library of the American Physical Society.

RESULTS AND DISCUSSION

We estimated the correlation dimension of the attractor using the Grassberger-Procaccia algorithm with a minimum of $10^{2.0+0.4d}$ points [26,27] (Fig. 1). We also performed a false nearest neighbors test, and found that after an embedding dimension of 3–4, the number of false nearest neighbors reduced to a plateau near zero (Fig. 2). We also performed a nonlinear prediction test, and found that with an embedding dimension of around 5 or 6, the nonlinear prediction error decreased (Fig. 3). We also performed a principal component analysis of the time series for each of the four rheological parameters, and found that there were about three principal eigenvalues of the correlation matrix (Fig. 4). Upon increasing the number of initial orientation vectors, we found that the fractal dimension of all the rheological parameters

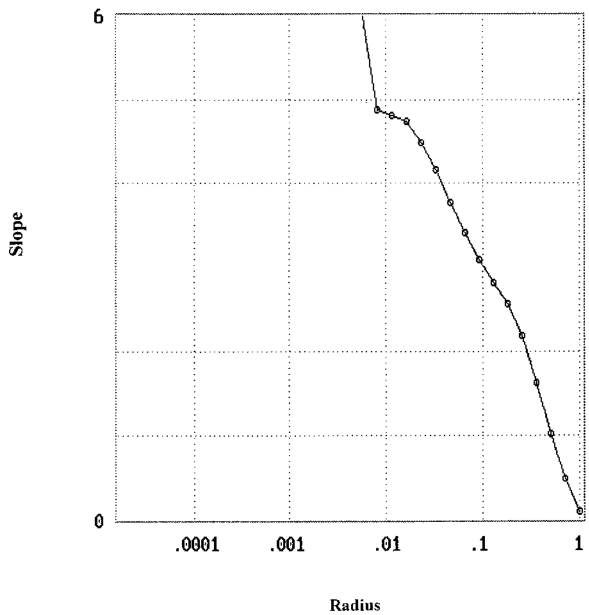

FIG. 1. Plot showing the determination of the correlation dimension of the rheological parameter [η_1] when the particles are aligned in 400 directions for $k_1=k_3=0$, $k_2=0.21$, and $\omega=1$.

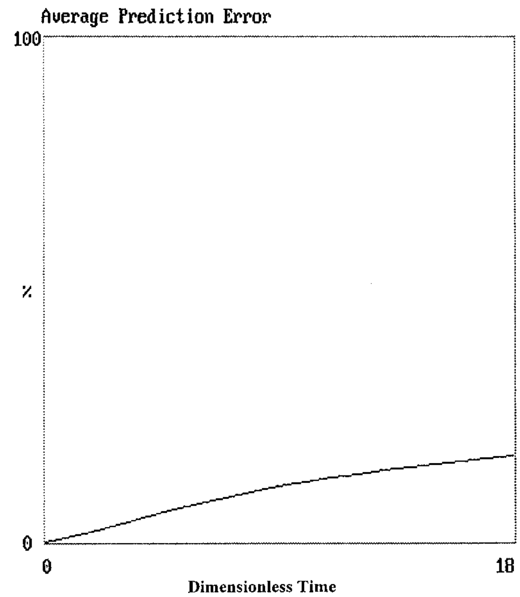

FIG. 3. Plot of the average prediction error of the time series of the parameter [η_1] when the particles are aligned in 400 directions for $k_1=k_3=0$, $k_2=0.21$, and $\omega=1$.

settled down to values between 3.5 and 4.0 (Fig. 5). All the tests were performed on each of the four time series, and the results were consistent.

We have treated the case of the number of directions of alignment equal to 400 separately, and have studied this case in detail for each parameter. In each case, we observed that there were no sharp peaks in the probability distribution, and that the data could not be fitted by lower order polynomials. The power spectra of the data, which show broadband noise, also confirm the existence of the chaos in the rheological parameters. We also observed that the Lyapunov exponent, which is a measure of the sensitivity of the system to its initial conditions of [τ_1], [τ_2], [η_1], and [η_2], were more or less similar and positive. The Lyapunov exponents and correlation dimension of these parameters for $k_2=0.21$ when the particles are aligned in 400 directions are given in Table I.

Some typical phase-space plots, the plots of time derivative X' versus X at each data point; Return maps, the plot of the values of the time series versus the previous value of the time series when its derivative is equal to a constant for the rheological parameters when the particles are aligned in 400 directions are given in Figs. 6 and 7. The range of the values explored by [η_1], [η_2], [τ_1] and [τ_2] remains more or less

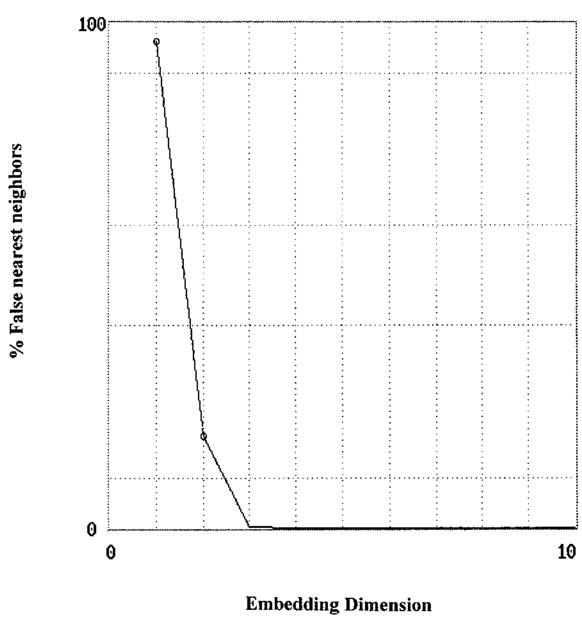

FIG. 2. Plot of percentage false nearest neighbors of the parameter [η_1] with embedding dimension when the particles are aligned in 400 directions for $k_1=k_3=0$, $k_2=0.21$, and $\omega=1$.

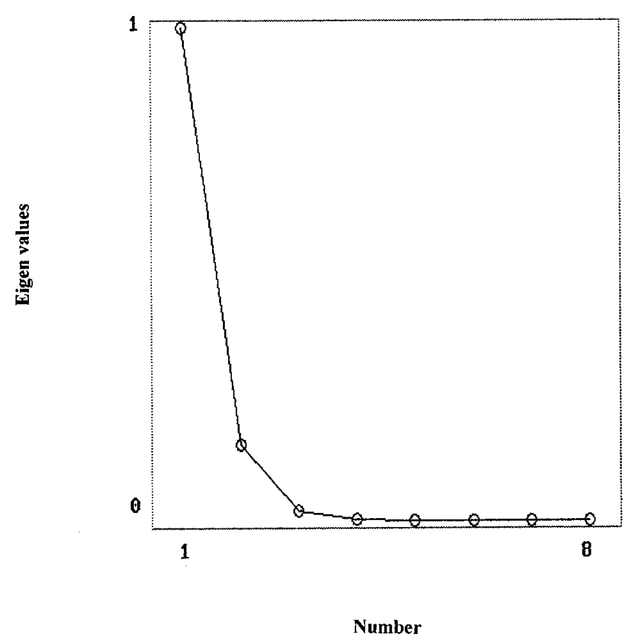

FIG. 4. Plot of the eigenvalues of the correlation matrix of the parameter [η_1] when the particles are aligned in 400 directions for $k_1=k_3=0$, $k_2=0.21$, and $\omega=1$.

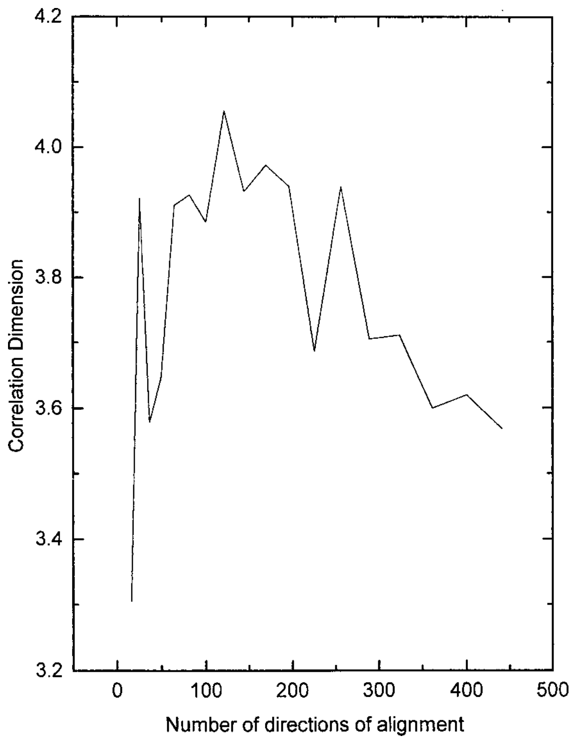

FIG. 5. Plot of the correlation dimension for $[\eta_1]$ as a function of the number of directions of alignment of the particles for $k_1 = k_3 = 0$, $k_2 = 0.21$, and $\omega = 1$.

same with an increasing number of initial orientation vectors. Figure 8 shows a plot of $[\eta_1]$ versus $[\eta_2]$ for different numbers of initial orientations, indicating that the range of values explored by $[\eta_1]$ and $[\eta_2]$ remain approximately the same.

We generated six sets of surrogate data for the parameter $[\eta_1]$. This was done by randomizing the phase of the time series such that the resulting data had the same power spectrum and autocorrelation, but a different probability distribution. We observed that the mean of the dimension of the set of surrogate data was 3σ away from the dimension of the original data, and the dimension of the data was in all cases lower than the dimension of the surrogate data sets, and thus the existence of nonlinear structure is statistically significant.

To provide some reasoning for the reduction of a system of 949 dimensions to an attractor of about 3.6 dimensions, we plotted $[\eta_1]$, $[\eta_2]$, $[\tau_1]$, and $[\tau_2]$ as functions of θ and ϕ (Fig. 9). We observe from these figures that there are a number of peaks and valleys in each of the plots, so that only a fraction of the orientations at any instant contribute significantly to values of $[\eta_1]$, $[\eta_2]$, $[\tau_1]$, and $[\tau_2]$ away from the mean. The remaining set of orientation vectors do not contribute significantly to the values of $[\eta_1]$, $[\eta_2]$, $[\tau_1]$, and $[\tau_2]$ and can be considered as contributing a noise term to the rheological parameter. This coupled with the structure of the attractors observed for aligned distributions reported in Ref. [28], led probably to only a fraction of the orientation vectors playing significant roles in the final attractor. This fraction of significant orientation vectors thus may be governed by a system of the dimension observed in this work, while the remaining directions contribute only a noise term. Note the nonlinear coupling between the expressions for the rheological parameters, namely, Eqs. (2)–(5), and the equations for the evolution of the individual orientation vectors, namely, Eqs. (1). We note that both these sets of equations contain the parameters k_1, k_2, and k_3, and hence the non-

TABLE I. Lyapunov exponents and correlation dimension for the parameters $[\eta_1]$, $[\eta_2]$, $[\tau_1]$, and $[\tau_2]$ for $k_1 = k_3 = 0$, $k_2 = 0.21$, and $\omega = 1$ when the particles are aligned in 400 directions.

	$[\eta_1]$	$[\eta_2]$	$[\tau_1]$	$[\tau_2]$
Correlation dimension	3.620	3.557	4.167	3.234
Lyapunov exponent	0.276	0.150	0.250	0.208
	±0.007	±0.006	±0.006	±0.007

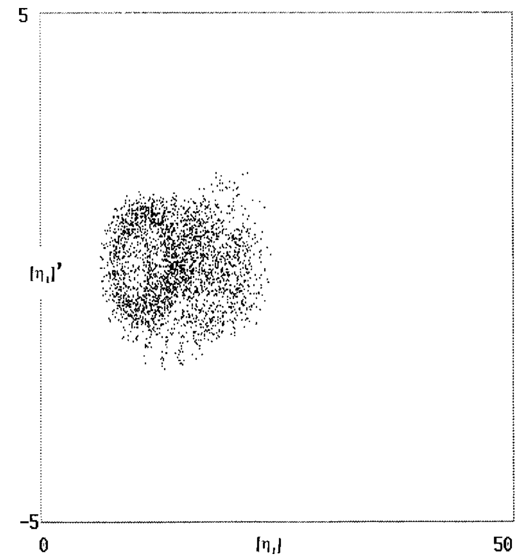

FIG. 6. Phase-space plot of $[\eta_1]$ when particles are aligned in 400 directions for $k_1 = k_3 = 0$, $k_2 = 0.21$, and $\omega = 1$. Plot of $[\eta_1]$ vs derivative of $[\eta_1]$.

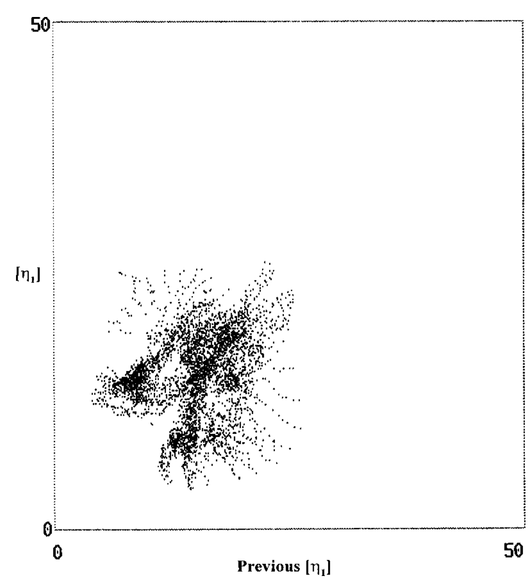

FIG. 7. Return map of $[\eta_1]$ when particles are aligned in 400 directions for $k_1 = k_3 = 0$, $k_2 = 0.21$, and $\omega = 1$. Plot of $[\eta_1]$ vs the previous value of $[\eta_1]$ when its derivative is a constant.

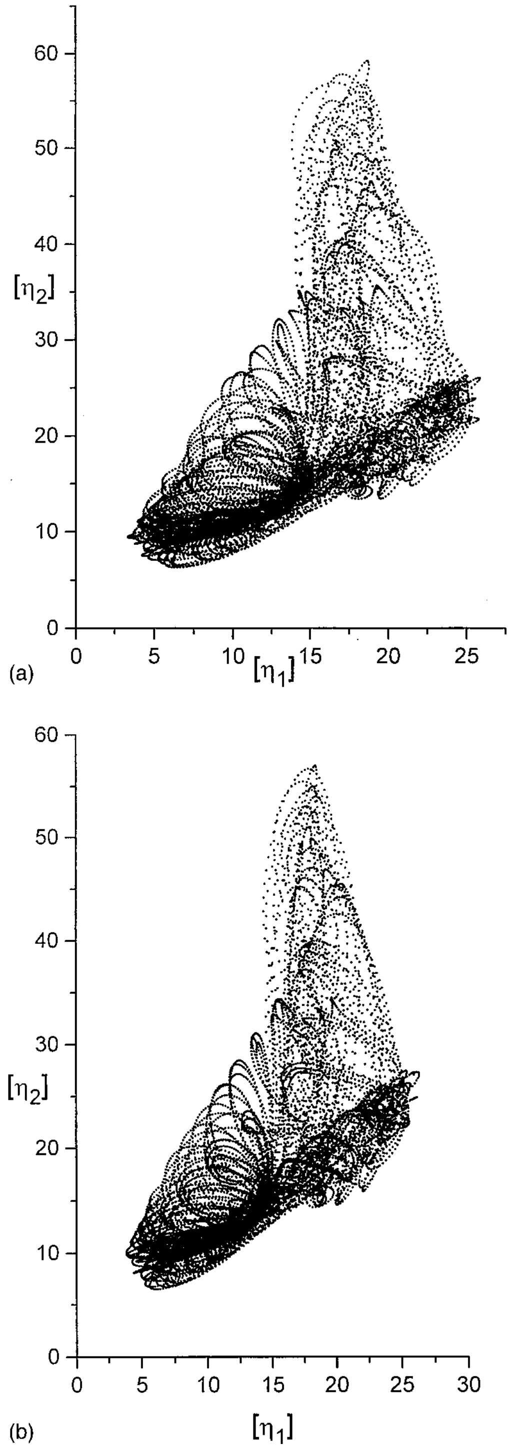

FIG. 8. (a) Plot of [η_1] vs [η_2] when particles aligned in 225 directions for $k_1=k_3=0$, $k_2=0.21$, and $\omega=1$. (b) Plot of [η_1] vs [η_2] when particles aligned in 400 directions for $k_1=k_3=0$, $k_2=0.21$, and $\omega=1$.

linear interaction between these parameters can result in a low-dimensional chaotic attractor for the rheological parameters [29,30].

IMPLICATIONS FOR CHAOS THEORY

Suspension rheology provides a standard physically realizable system of a spatially extended system, and suspensions under conditions in which individual particles show chaotic dynamics provide an excellent model system to investigate the various possibilities of spatiotemporal chaos and nontrivial collective behavior. The system of a dilute suspension ($nl^3 \ll 1$) of periodically forced weak Brownian slender rods in a simple shear flow is a model system in which individual particles in the chaotic parametric regimes of their orientations constitute individually chaotically varying uncoupled oscillators. For nonspherical particles, the orientation averages discussed above, over a finite number of orientations, represent averages over an infinity of oscillators, since in a distribution with particles aligned in only a few common directions each particle can in principle represent an infinite number of particles aligned along the same direction. Our results presented above show that a system of uncoupled oscillators can settle down to a low-dimensional attractor of appropriate averages, and may be the first example of a physically realizable system, showing evidence of low-dimensional chaos in appropriate experimentally measurable averages.

The study of the rheology of a dilute suspension of spheroids in a simple shear flow as a model system to study aspects of spatiotemporal chaos has several advantages over other model systems. First, unlike the various systems considered in the literature, the present system is a physically realizable one. Second, it allows the consideration of orientation averages over finitely many orientations, in place of spatial averages over an infinite number of spatial positions, since, in an aligned suspension, each particle can represent a large number of particles aligned along the same direction. Third, *the time scale of the fluctuations of the oscillators in real time can be adjusted to any desired value by suitably adjusting the shear rate*. Hence if chaos can be shown to be possible in dilute suspensions of periodically forced spheroids, this may be the first system to demonstrate the possibility of nontrivial chaotic collective behavior. Fourth, in the case of periodically forced suspensions of spheroids, individual particles can, in principle, be controlled to oscillate chaotically as desired [23,28]; also, for orientation averages, ''the law of large numbers'' is not, in principle, applicable. Fifth, in suspension models of spatially extended systems, the effect of local correlations can be incorporated by considering particle-particle interactions, and nearest neighbor coupling can be approximated by weak interactions in sufficiently dilute suspensions. Sixth, the conditions under which appropriate spatial averages of a spatially extended system vary chaotically can be studied by considering the conditions under which the orientation averages of individually chaotically varying orientations vary chaotically; numerical simulations for the latter are likely to be simpler than those for other models considered so far in the literature. Seventh, the dependence of the rheological parameters on the orientations is such that only a fraction of the orientations contribute significantly to the rheological parameters. This, coupled with the nonlinear interaction between the expressions for the rheological parameters and the expressions for the individual spheroids in terms of the parameters k_1, k_2, and k_3, provides a plausible explanation for the evolution of the rheological parameters being governed by a low-dimensional attractor.

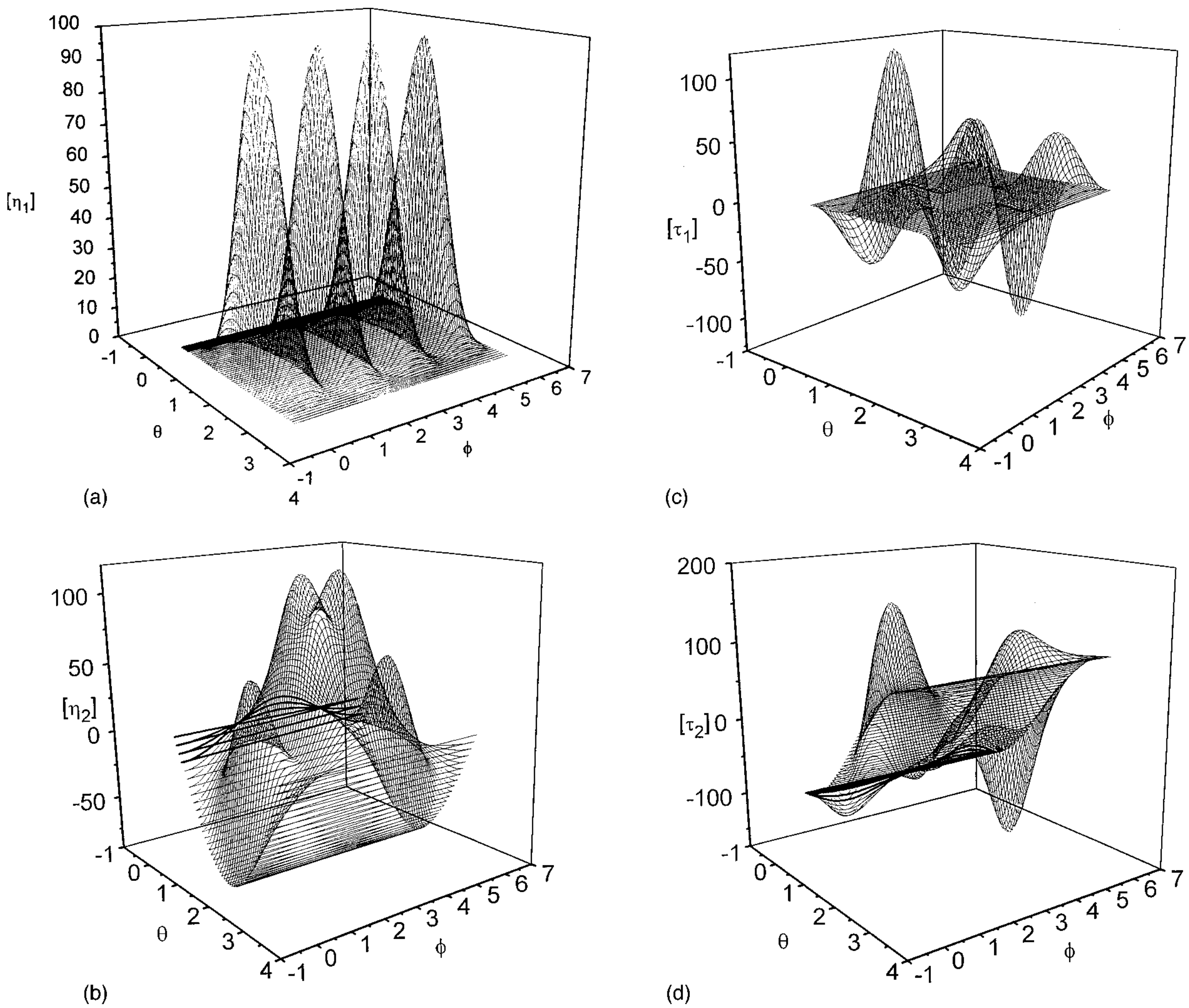

FIG. 9. Plot of $[\eta_1]$, $[\eta_2]$, $[\tau_1]$, and $[\tau_2]$ as functions of θ and ϕ for $k_1=k_3=0$, $k_2=0.21$, $\omega=1$, and $t=0$.

CONCLUSION

We have reviewed our results regarding the existence of chaos in the rheological parameters of periodically forced dilute suspensions of slender rods or spheroids in simple shear flow. An example of a physically realizable system has been presented to show that a spatially extended system of uncoupled chaotic oscillators can exhibit a chaotic nontrivial collective behavior. Numerical evidence for the existence of a low-dimensional attractor in the rheological parameters has also been presented. This system may be the first example of a physically realizable system which may show low-dimensional chaos in suitable experimentally measurable parameters which represent averages over a large number of uncoupled chaotic oscillators. The advantages of this example system over existing ones as an ideal model system to study aspects of spatiotemporal chaos and nontrivial collective behavior have also been discussed.

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