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Effective action for stochastic partial differential equations

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Stochastic partial differential equations (SPDEs) are the basic tool for modeling systems where noise is important. SPDEs are used for models of turbulence, pattern formation, and the structural development of the universe itself. It is reasonably well known that certain SPDEs can be manipulated to be equivalent to (nonquantum) field theories that nevertheless exhibit deep and important relationships with quantum field theory. In this paper we systematically extend these ideas: We set up a functional integral formalism and demonstrate how to extract all the one-loop physics for an *arbitrary* SPDE subject to *arbitrary* Gaussian noise. It is extremely important to realize that Gaussian noise does *not* imply that the field variables undergo Gaussian fluctuations, and that these nonquantum field theories are fully interacting. The limitation to one loop is not as serious as might be supposed: Experience with quantum field theories (QFTs) has taught us that one-loop physics is often quite adequate to give a good description of the salient issues. The limitation to one loop does, however, offer marked technical advantages: Because at one loop almost any field theory can be rendered finite using zeta function technology, we can sidestep the complications inherent in the Martin-Siggia-Rose formalism (the SPDE analog of the Becchi-Rouet-Stora-Tyutin formalism used in QFT) and instead focus attention on a minimalist approach that uses only the physical fields (this ''direct approach'' is the SPDE analog of canonical quantization using physical fields). After setting up the general formalism for the characteristic functional (partition function), we show how to define the effective action to all loops, and then focus on the one-loop effective action and its specialization to constant fields: the effective potential. The physical interpretation of the effective action and effective potential for SPDEs is addressed and we show that key features carry over from QFT to the case of SPDEs. An important result is that the *amplitude* of the two-point function governing the noise acts as the loop-counting parameter and is the analog of Planck's constant $\hbar$ in this SPDE context. We derive a general expression for the one-loop effective potential of an arbitrary SPDE subject to translation-invariant Gaussian noise, and compare this with the one-loop potential for QFT.
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I. INTRODUCTION

Stochastic partial differential equations (SPDEs) are an essential tool in modeling systems where noise is relevant [1]. SPDEs are used for models of many macroscopic systems, from turbulence [2–4], to pattern-formation [5,6], to the structural development of the Universe itself [7–11]. It is known that certain SPDEs can be studied with tools that transform them into equivalent (stochastic) field theories which exhibit deep and important relationships with quantum field theory (QFT). See, for example, [1,2,5,6] and [12–15].

In this paper we set up the field-theoretical ''minimalist formalism'' for SPDEs, and demonstrate how to extract the one-loop physics for an *arbitrary* SPDE subject to additive Gaussian noise. It is important to realize that Gaussian noise does *not* imply that the field degrees of freedom undergo Gaussian fluctuations: the combined interplay of interactions and fluctuations will appear in the third (and higher) cumulants for the field $\phi(\vec{x},t)$. Also, the limitation to one-loop physics is not as serious as might be supposed: Experience with quantum field theories (QFTs) has taught us that one-loop physics is often quite adequate to give a good description of the salient issues [16–21]. In fact, in QFT, the calculation of one-loop quantities can be augmented by means of ''renormalization-group improved perturbation theory,'' which contains most of the relevant features of the physics to *all* orders in the expansion parameter [17,18]. (This was called ''magical perturbation theory'' by the authors [22] of Ref. [16].) Furthermore, at one loop (and higher), one can also introduce the effective action and effective potential [23–26], tools that allow one to determine the combined effects of interactions and fluctuations on the ground state of the system. Defining and calculating the one-loop effective action and effective potential is straightforward. Interpreting the physical significance of these quantities is more subtle. For arbitrary SPDEs it may not even be meaningful to define a notion of physical energy. Even when the physical energy makes sense, dissipative effects may vitiate energy conservation (even when noise is absent). We therefore spend some effort in establishing that certain key features of the effective action for QFTs carry over to SPDEs. In particular, we demonstrate that it is still meaningful to define and calculate the

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effective potential and look for its minima. The minima of the effective potential correspond to ground states of the system, and the locations of these minima are equal to stochastic expectation values of the fluctuating field in the presence of noise.

While it is possible to provide an abstract nonperturbative definition of the effective action [24], in order to proceed with explicit calculations (such as for the one-loop effective action) one needs a perturbative procedure based on an expansion in some small parameter. A well-known procedure of this type, the Martin-Siggia-Rose (MSR) formalism, already exists in the literature [1]. The MSR formalism invokes additional (unphysical) ''conjugate fields,'' which are generalizations of the fictitious fields sometimes introduced to deal with the dynamics of diffusion. These fictitious fields permit one to extend some of the procedures of conservative physical systems to diffusion. For instance, Morse and Feshbach state ''the dodge is to consider, . . . , a 'mirror-image' system with negative friction, into which the energy goes which is drained from the dissipative system.'' (See [27], p. 298.) In this paper we do not make use of the conjugate field formalism of MSR and, instead, proceed in a direct way in which we only have physical fields (plus possibly a nontrivial functional Jacobian that can be rewritten in terms of ghost fields). This approach simplifies the calculation since it halves the number of fields one has to deal with. These two *alternative* formalisms are very similar to the situation in spontaneously broken gauge field theories, where one can use two *equivalent* approaches to perturbation theory, such as ''unitary gauges'' versus ''renormalizable gauges:'' in one case the *particle content* is explicit and in the other *renormalizability* is explicit.

After setting up the path-integral formalism for the characteristic functional (partition function), $Z[J]$, we define both the perturbative [23] and nonperturbative effective action [24]. We then focus on the one-loop effective action and its restriction to constant (homogeneous and stationary) fields: the effective potential [20,21]. An important result is that the *amplitude* of the noise two-point correlation function acts as the loop-counting parameter and is the analog of Planck's constant $\hbar$ in this SPDE context.

We conclude by deriving the formula for the one-loop effective potential of a general SPDE subject to translation-invariant Gaussian noise. This formula has a strong resemblance to that obtained for ordinary QFTs and allows us to extend the use of QFT tools in the analysis of the SPDE's effective potential. We furthermore demonstrate that much of the physical intuition regarding the effective action in QFTs also carries over into SPDEs. Finally, we offer a discussion of our results. A number of more technical issues are relegated to the Appendixes.

II. STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

A. Elementary definitions

Consider the class of stochastic partial differential equations of the form

$$D\phi(\vec{x},t)=F[\phi(\vec{x},t)]+\eta(\vec{x},t), \tag{1}$$

where D is any linear differential operator, involving arbitrary time and space derivatives, which does *not* explicitly involve the field ϕ. Typical examples are

$$D=\frac{\partial}{\partial t}-\nu\vec{\nabla}^2 \qquad \text{diffusion equation,} \tag{2}$$

$$D=\frac{\partial^2}{\partial t^2}-\vec{\nabla}^2 \qquad \text{wave equation,} \tag{3}$$

$$D=\frac{\partial}{\partial t} \qquad \text{Langevin equation.} \tag{4}$$

The function $F[\phi]$ is any forcing term, generally nonlinear in the field ϕ. Typical examples are

$$F[\phi]=+\frac{\lambda}{2}(\vec{\nabla}\phi)^2 \tag{5}$$

in the Kardar-Parisi-Zhang (KPZ) equation,

$$F[\phi]=P[\phi] \tag{6}$$

in reaction-diffusion-decay systems (P is a polynomial),

$$F[\phi]=-\frac{\delta H[\phi]}{\delta\phi} \tag{7}$$

in ''purely dissipative'' SPDEs.

The forcing term will typically not contain any time derivatives, but this is not an essential part of the following analysis except insofar as time derivatives may complicate some of the Jacobian functional determinants that will be encountered below. Nonderivative terms linear in the field can be interpreted either as decay rates or (if a diffusion term is also present) as mass terms. They can be freely moved between the differential operator D and the forcing term $F[\phi]$. If they are considered part of the forcing term, then

$$F[\phi]=-\gamma\phi \qquad \text{describes a decay term,} \tag{8}$$

$$F[\phi]=-\nu m^2\phi \qquad \text{describes a mass term.} \tag{9}$$

The function $\eta(\vec{x},t)$ is a random function of its arguments and describes the noise that we assume is present in the system. For the remainder of this paper, we consider field-independent additive noise. At this stage the nature and probability distribution of the noise are completely arbitrary and do not need to be specified.

The noise represents our ignorance about precise details in the dynamics of the system. It could be due, for example, to fluctuations intrinsic to the dynamics (as in the case of quantum mechanics), or it could be thought of as representing the dynamics of short-scale degrees of freedom which have not completely decoupled from the macroscopic dynamics (e.g., thermal or turbulent noise), or it could be a way of implementing ignorance of the *exact* initial or boundary conditions in the system. Noise can also be a way of summarizing the necessary truncation of the deterministic dynamics of a many-body system when we try to describe it via a finite set of variables (e.g., a truncated BBGKY hierarchy).

If we think of turning off the noise, we do *not* require that the nonstochastic partial differential equation $D\phi = F[\phi]$ be derivable from an action principle (i.e., the nonstochastic partial differential equation need not arise from a Lagrangian formalism). Nevertheless, once we include noise, we demonstrate that the presence of noise automatically leads to a generalized action principle for the noisy system. It turns out that in the presence of Gaussian noise an equation of motion *proportional* to the factor $(D\phi - F[\phi])$ can always be derived by varying a well-defined ''classical'' action, and that the solutions to this equation of motion will coincide with those of the nonstochastic equation, provided a certain Jacobian determinant is nonsingular (i.e., invertible). This is explained in full detail in Appendix B.

B. Some typical examples

An example of considerable interest is the reaction-diffusion-decay system where the SPDE is taken to be [28–30]

$$\frac{\partial \phi}{\partial t} - \nu \vec{\nabla}^2 \phi = P[\phi] + \eta. \tag{10}$$

This equation is used, for example, to describe the density $\phi(\vec{x},t)$ of some chemical species as a function of space and time when the chemical is subject to both diffusion (via ν) and reaction or decay [via $P(\phi)$, a polynomial in the density field]. Expanding out the first few terms,

$$P(\phi) = P_0 + P_1\phi + P_2\phi^2 + P_3\phi^3 + \cdots, \tag{11}$$

we can identify P_0 with a constant (in space and time) source or sink, $-P_1$ with the decay rate, and P_2 with the reaction rate for the two-body reaction, etc. The noise accounts for random effects due to coupling to external sources, truncation of degrees of freedom, averaging over microscopic effects, etc. For n species of chemical reactant, the field ϕ is simply promoted to a vector in configuration space $\phi(\vec{x},t) \to \phi_i(\vec{x},t), \{i = 1, \ldots, n\}$. [The diffusion constant, ν, and decay rates then become matrices, the noise a vector, and the polynomial $P_i(\phi_j)$ a vector-valued polynomial with tensorial coefficients.]

A second well-known example is the massive KPZ equation (equivalent to the massive noisy Burgers equation) [5,6,13,14,31]

$$\frac{\partial \phi}{\partial t} - \nu \vec{\nabla}^2 \phi = -\nu m^2 \phi + \frac{\lambda}{2}(\vec{\nabla}\phi)^2 + \eta. \tag{12}$$

In the fluid dynamical interpretation of the KPZ equation, the fluid velocity is taken to be $\vec{v} = -\vec{\nabla}\phi$. This model problem leads to a form of ''turbulence'' which is known in the literature as Burgulence [32,33].

A third example is the enormous class of SPDEs known as ''purely dissipative'' systems [15]. Purely dissipative systems have SPDEs of the form

$$\frac{\partial \phi}{\partial t} = -\frac{\delta H[\phi]}{\delta \phi(\vec{x})} + \eta. \tag{13}$$

These equations fall into our classification of general SPDEs as particular types of Langevin equations with $D = \partial_t$ and with a driving term that is a (functional) gradient $F[\phi] = -\delta H[\phi]/\delta\phi(\vec{x})$. The nomenclature ''purely dissipative'' is justified by the fact that in the *absence* of noise these systems satisfy

$$\frac{\partial H[\phi]}{\partial t} = -\int \left(\frac{\delta H[\phi]}{\delta \phi(\vec{x})}\right)^2 d^d\vec{x} \leq 0. \tag{14}$$

Note that the reaction-diffusion-decay (RDD) system can be interpreted as an example of a purely dissipative system if we take $D = \partial_t$ and

$$H_{\mathrm{RDD}}[\phi] = \int \left[\frac{\nu}{2}(\vec{\nabla}\phi)^2 + \int_0^{\phi(x)} P(\tilde{\phi})\, d\tilde{\phi}\right] d\vec{x}. \tag{15}$$

On the other hand, the KPZ system is *not* a purely dissipative system,

$$\begin{aligned}\frac{\delta F_{\mathrm{KPZ}}[\phi(x)]}{\delta \phi(y)} &= \nu m^2 \delta(x-y) + \lambda \vec{\nabla}_x \phi(x) \cdot \vec{\nabla}_x \delta(x-y) \\ &= \nu m^2 \delta(x-y) - \lambda \vec{\nabla}_y \phi(y) \cdot \vec{\nabla}_y \delta(x-y) \\ &\neq \frac{\delta F_{\mathrm{KPZ}}[\phi(y)]}{\delta \phi(x)}.\end{aligned} \tag{16}$$

The class of purely dissipative SPDEs is a very wide one, but there are many SPDEs that are not of purely dissipative type. We do *not* want to restrict attention to purely dissipative systems in this paper, rather we want to keep the discussion as general as possible.

III. STOCHASTIC AVERAGES, CHARACTERISTIC FUNCTIONAL, FEYNMAN RULES

We will focus on the stochastic partial differential equation

$$D\phi(\vec{x},t) = F[\phi(\vec{x},t)] + \eta(\vec{x},t) \tag{17}$$

and analyze it using functional integral techniques: Feynman diagrams, the effective action, and the effective potential. We develop the field theory via the most direct route, with no conjugate fields present.

We postpone to subsequent papers more technically involved approaches such as the Martin-Siggia-Rose Lagrangian (with its extra unphysical conjugate fields used for bookkeeping purposes) and the hidden BRST supersymmetry implicit in these stochastic differential equations [1,12,15,34].

In this section we develop the necessary tools to construct the basic field theory and nonequilibrium statistical mechanics associated with Eq. (17). We will assume *uniqueness* of the solution to Eq. (17), and in order to calculate the characteristic functional, we will introduce an *ensemble average* over noise realizations, and the notion of δ *functionals*. Once the *characteristic functional* is available, we find it useful to introduce *ghosts in the manner of Faddeev-Popov* before deriving the *Feynman rules*. We will only need to make one assumption about the noise: that it be Gaussian, i.e., that all its cumulants are vanishing except the first, $\langle \eta(\vec{x},t)\rangle$, and second, $G_\eta = \langle \eta(\vec{x},t)\,\eta(\vec{x}',t')\rangle$. (See, e.g., [35].)

A. Step 1: Uniqueness

Let us assume that the partial differential equation (17), plus initial conditions, is a well-posed problem. Thus, given a particular realization of the noise, η, the differential equation is assumed to have a unique solution which we designate as

$$\phi_{\rm soln}(\vec{x},t|\eta). \tag{18}$$

This assumption is relatively mild but does imply that the nonlinearity is sufficiently weak so as not to drive us past a bifurcation point. On the other hand, it is known that noise in concert with nonlinearities can lead to the phenomenon of delayed bifurcation in nonlinear parabolic SPDEs [36]. If the partial differential equation is ill-posed, in the sense that the solutions are not unique, additional analysis must be developed on a case-by-case basis. A specific example of this behavior is spontaneous symmetry breaking in QFT, which causes the naive loop expansion to violate the convexity properties of the effective potential. This situation must be dealt with by an improved loop expansion [37–39].

B. Step 2: Ensemble average

For any function $Q(\phi)$ of the field ϕ we introduce the ensemble average (over the noise), defined by

$$\langle Q(\phi)\rangle \equiv \int (\mathcal{D}\eta)\mathcal{P}[\eta]Q(\phi_{\rm soln}(\vec{x},t|\eta)), \tag{19}$$

where $\mathcal{P}[\eta]$ is the probability density functional of the noise. It is normalized to 1, but is otherwise completely arbitrary, that is,

$$\int (\mathcal{D}\eta)\mathcal{P}[\eta]=1. \tag{20}$$

The symbol $\mathcal{D}\eta$ indicates a functional integral over all instances (or realizations) of the noise.

C. Step 3: δ functionals

We next use a functional δ function to write the following identity:

$$\phi_{\rm soln}(\vec{x},t|\eta) \equiv \int (\mathcal{D}\phi)\,\phi\,\delta[\phi-\phi_{\rm soln}(\vec{x},t|\eta)]$$
$$= \int (\mathcal{D}\phi)\,\phi\,\delta[D\phi - F[\phi]-\eta]\sqrt{\mathcal{J}\mathcal{J}^\dagger}, \tag{21}$$

where we have performed a change of variables and introduced the Jacobian functional determinant, defined by

$$\mathcal{J}\equiv \det\left(D-\frac{\delta F}{\delta\phi}\right) \tag{22}$$

and its adjoint

$$\mathcal{J}^\dagger \equiv \det\left(D^\dagger - \frac{\delta F^\dagger}{\delta\phi}\right). \tag{23}$$

The above is just the functional analog of a standard δ-function result: If $f(x)=0$ has a unique solution at $x=x_0$, then

$$x_0 = \int dx\, x\,\delta(x-x_0) = \int dx\, x\,\delta(f(x))|f'(x)|$$
$$= \int dx\, x\,\delta[(f(x))\sqrt{f'(x)[f'(x)]^*}. \tag{24}$$

The δ function forces one to pick up only one contribution from the solution of the equation $f(x)=0$, and the derivative is there to provide the correct measure to the integral. In the functional case the derivative becomes a determinant. It is in fact the Jacobian determinant associated with the change of variables from ϕ to $D\phi - F[\phi]$. It is now easy to see that one also has the identity

$$Q(\phi_{\rm soln}(\vec{x},t|\eta)) \equiv \int (\mathcal{D}\phi)Q(\phi)\,\delta(D\phi - F[\phi]-\eta)\sqrt{\mathcal{J}\mathcal{J}^\dagger}. \tag{25}$$

Furthermore, the ensemble average over the noise, Eq. (19), becomes

$$\langle Q(\phi)\rangle = \int (\mathcal{D}\eta)(\mathcal{D}\phi)\mathcal{P}[\eta]Q(\phi)$$
$$\times\,\delta(D\phi - F[\phi]-\eta)\sqrt{\mathcal{J}\mathcal{J}^\dagger}. \tag{26}$$

The noise integral is easy to perform, with the result that for arbitrary stochastic averages one has

$$\langle Q(\phi)\rangle = \int (\mathcal{D}\phi)\mathcal{P}[D\phi - F[\phi]]Q(\phi)\sqrt{\mathcal{J}\mathcal{J}^\dagger}. \tag{27}$$

We see from this equation that the effect of the noise only appears in the stochastic average through its probability distribution $\mathcal{P}[D\phi - F[\phi]]$. It is worthwhile to point out that the main difference, at this stage of the formalism, between the present ‘‘minimal’’ approach and that of MSR lies in the way the δ functional is handled. In MSR, instead of integrating directly over the noise, as is done here, the δ functional is replaced by its functional Fourier integral representation. This is the step wherein the conjugate field enters. If this latter route is taken, the noise integration can be performed exactly only for Gaussian noise. In the minimal formalism, by contrast, the integration over the noise can be done exactly for arbitrary noise. It thus lends itself immediately for handling non-Gaussian systems: For general noise distributions we can explicitly write down the probability distribution for the fields as

$$\mathbf{P}[\phi] = \mathcal{P}[D\phi - F[\phi]]\sqrt{\mathcal{J}\mathcal{J}^\dagger}. \tag{28}$$

We will not explore further the possibility of arbitrary noise in this paper, since Gaussian noise (which manifestly does *not* imply Gaussian fluctuations of the fields) is already sufficiently general to be of great practical interest.

The presence of the functional determinant is essential: it must be kept to ensure proper counting of the solutions to the original stochastic differential equation. In QFT this functional determinant is known as the Faddeev-Popov determinant and is essential in maintaining unitarity [19,21], i.e., conservation of probability. In some particular cases the functional determinant is field-independent, and it is safe to neglect it. We discuss this more fully in Appendix A and in the companion papers [30,31], but for the sake of generality we will carry these determinants along (with little extra cost) for the rest of this paper.

D. Step 4: Characteristic functional (partition function)

A particularly useful quantity is the generating functional, or characteristic functional (partition function), defined by taking

$$Q(\phi)=\exp\left(\int d^d\vec{x}\,dt\,J(\vec{x},t)\,\phi(\vec{x},t)\right) \tag{29}$$

in Eq. (19). We define it as follows:

$$Z[J]\overset{\text{def}}{=}\left\langle \exp\left(\int dx\,J(x)\,\phi(x)\right)\right\rangle \tag{30}$$

$$=\int(\mathcal{D}\phi)\mathbf{P}[\phi]\exp\left(\int dx\,J\phi\right) \tag{31}$$

$$=\int(\mathcal{D}\phi)\mathcal{P}[D\phi-F[\phi]]\exp\left(\int dx\,J\phi\right)\sqrt{\mathcal{J}\mathcal{J}^{\dagger}}, \tag{32}$$

with an obvious condensation of notation, $dx=d^d\vec{x}\,dt$. When there is no risk of confusion, we will suppress the dx completely. This key result will enable us to calculate the effective action and the effective potential in a direct way.

E. Step 5: Gaussian noise

We will now make some assumptions about the noise: We assume it to be Gaussian. Without loss of generality we can take the noise to have zero mean, since if the mean is nonzero we can always redefine the forcing term $F[\phi]$ to make the noise have zero mean. We therefore take the noise to be Gaussian of zero mean, so that the only nonzero cumulant is the second-order one. We do not need to make any more specific assumptions about the functional realization of the noise: the noise might (for instance) be white, power-law, colored, pink, $1/f$ noise, or shot noise, and our considerations below apply to all of these cases. As long as the noise is Gaussian, its probability distribution can be written as

$$\mathcal{P}[\eta]=\frac{1}{\sqrt{\det(2\pi G_\eta)}}\times\exp\left(-\tfrac{1}{2}\int\int dx\,dy\;\eta(x)G_\eta^{-1}(x,y)\,\eta(y)\right). \tag{33}$$

The characteristic functional (partition function) is thus seen from Eq. (32) to be

$$Z[J]=\frac{1}{\sqrt{\det(2\pi G_\eta)}}\int(\mathcal{D}\phi)\sqrt{\mathcal{J}\mathcal{J}^{\dagger}}\exp\left(\int J\phi\right)\times\exp\left(-\tfrac{1}{2}\int\int(D\phi-F[\phi])G_\eta^{-1}(D\phi-F[\phi])\right). \tag{34}$$

This characteristic functional (partition function) contains all the physics of the model since it allows for the calculation of averages, correlation functions, thermodynamic variables, etc. Note that the noise has been completely eliminated and survives only through the explicit appearance of its two-point correlation function in the above. Since the characteristic functional is now given as a path integral over the physical field, all the standard machinery of statistical field theory (and quantum field theory) can be brought to bear [40]. See, for example, [15,19–21] and [41–51].

This formula for the partition function demonstrates that (modulo Jacobian determinants) all of the physics of any stochastic differential equation can be extracted from a functional integral based on the "classical action"

$$\mathcal{S}_{\text{classical}}=\tfrac{1}{2}\int\int(D\phi-F[\phi])G_\eta^{-1}(D\phi-F[\phi]). \tag{35}$$

This "classical action" is a generalization of the Onsager-Machlup action [52]. The Onsager-Machlup paper dealt with stochastic differential equations rather than partial differential equations (mechanics rather than field theory) and was limited to noise that was temporally white. As their formalism was developed with the notions of linear-response theory in mind, Onsager and Machlup assumed the "forcing term" $F[\phi]$ to be linear, so that *both* the noise and the field fluctuations were Gaussian. In our formalism all these assumptions can be relaxed: the forcing term can be nonlinear and in general the field fluctuations will not be Gaussian even if the noise is Gaussian.

F. Step 6: Faddeev-Popov ghosts

We mentioned previously that the Jacobian determinant is often (not always) field-independent. This is a consequence of the causal structure of the theory as embodied in the fact that we are only interested in *retarded* Green functions. The situation here is in marked contrast with that in QFT where the relativistic nature of the theory forces the use of *Feynman* Green functions ($+i\epsilon$ prescription). As we explain in Appendix A, this change radically alters the behavior of the functional determinant.

In order to avoid too many special cases, and to have a formalism that can handle both constant and field-dependent Jacobian factors, we exponentiate the determinant via the introduction of a pair of Faddeev-Popov ghost fields [15,19],

$$\mathcal{J}\equiv \det\left(D-\frac{\delta F}{\delta\phi}\right)$$
$$=\frac{1}{\det(2\pi I)}\int(\mathcal{D}[g^\dagger,g])\exp\left(-\frac{1}{2}\int g^\dagger\left[D-\frac{\delta F}{\delta\phi}\right]g\right), \tag{36}$$

where I is the identity operator on spacetime. The g field is a so-called (complex) scalar ghost field. It is a field of anti-commuting complex variables and behaves in a manner similar to an ordinary scalar field except that there is an extra minus sign for each ghost loop. We also need to use the *conjugate* ghost field $g^\dagger$ to handle the determinant.

We should point out that if the operator $D-(\delta F/\delta\phi)$ is self-adjoint, then $\mathcal{J}=\mathcal{J}^\dagger$. In this case $\sqrt{\mathcal{J}\mathcal{J}^\dagger}$ reduces to $|\mathcal{J}|=|\mathcal{J}^\dagger|$. In QFTs the relevant operators occurring in the Jacobian determinants are obtained from second functional derivatives of the action and are automatically self-adjoint. In contrast, for SPDEs there is no guarantee that $D-(\delta F/\delta\phi)$ be self-adjoint, and in fact for the examples previously discussed (KPZ, reaction-diffusion-decay, and purely dissipative) this operator is not self-adjoint. Instead we rely on the much weaker property that the operator $D-(\delta F/\delta\phi)$ is *real* in order to write $\sqrt{\mathcal{J}\mathcal{J}^\dagger}=\sqrt{\mathcal{J}\mathcal{J}^*}=|\mathcal{J}|$. In all cases we are interested in, the relevant operators are not only real, but positive, so that the absolute value symbol can be ignored and the characteristic functional equation (34) is given by

$$Z[J]=\frac{1}{\sqrt{\det[(2\pi)^3G_\eta]}}\int(\mathcal{D}\phi)(\mathcal{D}g)(\mathcal{D}g^\dagger)$$
$$\times\exp\left(-\tfrac{1}{2}\int\int(D\phi-F[\phi])G_\eta^{-1}(D\phi-F[\phi])\right)$$
$$\times\exp\left(-\tfrac{1}{2}\int g^\dagger\left[D-\frac{\delta F}{\delta\phi}\right]g\right)\exp\left(\int J\phi\right). \tag{37}$$

This procedure trades off the functional determinants for two extra functional integrals. The advantage of this procedure becomes clear when one develops the perturbation theory. [This Faddeev-Popov trick for exponentiating the Jacobian determinant is also essential in finding the hidden Becchi-Rouet-Stora-Tyutin (BRST) supersymmetry.]

It must be noted that neither pair of ghost field variables couples to an external source. This means they can only appear in internal lines in Feynman diagrams, a fact that will be used later on when we discuss loops and loop counting.

In Appendix A, we take a closer look at the Jacobian functional determinant, its causal structure, and its specific form for local driving forces. In the latter part of this Appendix, we make use of the perturbation theory based on Feynman diagrams to evaluate this functional determinant from another perspective.

G. Step 7: Feynman rules

With the partition function in the form given above (with two independent ghosts), it is now easy to develop a formal Feynman diagram expansion. We wish to treat the driving term $F[\phi]$ as the perturbation and expand around the free-field theory defined by setting $F=0$. With this convention the free action is, explicitly,

$$\mathcal{S}_{\text{free}}\equiv\int\int\left\{\frac{1}{2}[D\phi]G_\eta^{-1}[D\phi]\right\}dx\,dy+\int\left\{\frac{1}{2}g^\dagger Dg\right\}dx. \tag{38}$$

There are two particle propagators in this free action, one for the ϕ field, and two for the ghost fields. Formally,

$$G_{\phi\phi}=[D^\dagger G_\eta^{-1}D]^{-1}=[D^{-1}G_\eta(D^\dagger)^{-1}], \tag{39}$$

$$G_{g^\dagger g}=[D]^{-1}. \tag{40}$$

Here $D^\dagger$ is the adjoint operator of D, defined by partial integration. (For instance, if $D=\partial_t-\nu\vec{\nabla}^2$, then $D^\dagger=-\partial_t-\nu\vec{\nabla}^2$.) In the interests of generality, we reiterate the fact that we have not assumed translation invariance for the noise (although the noise is now Gaussian). The momentum-frequency representation for the propagators is

$$G_{\phi\phi}(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2)=\frac{G_\eta(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2)}{D^\dagger(\vec{k}_1,\omega_1)D(\vec{k}_2,\omega_2)}, \tag{41}$$

$$G_{g^\dagger g}(\vec{k},\omega)=\frac{1}{D(\vec{k},\omega)}. \tag{42}$$

The Feynman vertices come from the interaction piece of the action, which in this convention is

$$\mathcal{S}_{\text{interaction}}=\int\int\left\{-[D\phi]G_\eta^{-1}F[\phi]+\frac{1}{2}F[\phi]G_\eta^{-1}F[\phi]\right\}dx\,dy-\int\left\{\frac{1}{2}g^\dagger\frac{\delta F}{\delta\phi}g\right\}dx. \tag{43}$$

The nature of the vertices (obtained by functional differentiation with respect to the fields present in the theory) depends on the structure of the forcing term $F[\phi]$. In the mean time we formally assert

$$\phi-F[\phi]\text{ vertex:}\qquad -\frac{D(\vec{k}_1,\omega_1)\phi F[\phi]}{G_\eta(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2)}, \tag{44}$$

$$F[\phi]-F[\phi]\text{ vertex:}\qquad +\frac{1}{2}\frac{F[\phi]F[\phi]}{G_\eta(\vec{k}_1,\omega_1;\vec{k}_2,\omega_2)}, \tag{45}$$

$$\text{ghost vertex:}\qquad -\frac{1}{2}g^\dagger\frac{\delta F[\phi]}{\delta\phi}g. \tag{46}$$

Note that to turn these schematic Feynman rules into practical computational tools, we will need to assume that $F[\phi]$ is some specific local functional of the field ϕ (typically a polynomial or polynomial with derivatives). When we define the effective action, we will again see that the formalism can be successfully developed even for non-translation-invariant noise, and this derivation of the Feynman rules matches the generality of the definition of the effective action. This concludes, for now, the most general aspects of the discussion.

When it comes to actual calculations in specific models, the majority of these models have noise that is not only Gaussian but is also translation invariant. In the interests of simplicity, we now (finally, and only for the rest of this particular section) indicate the effects of assuming translation invariance for the noise. This lets us take simple Fourier transforms in the difference variable $x-y$ (more precisely, $\vec{x}-\vec{y}$ and t_x-t_y) to see that in momentum-frequency space

$$G_{\phi\phi}(\vec{k},\omega)=\frac{G_\eta(\vec{k},\omega)}{D^\dagger(\vec{k},\omega)D(\vec{k},\omega)}=\frac{G_\eta(\vec{k},\omega)}{D(-\vec{k},-\omega)D(\vec{k},\omega)}, \tag{47}$$

$$G_{g^\dagger g}(\vec{k},\omega)=\frac{1}{D(\vec{k},\omega)}. \tag{48}$$

The Feynman diagram vertices are now (ϕ and g are here understood to be Fourier transformed)

$$\phi-F[\phi]\ \text{vertex:}\qquad -\frac{D(\vec{k},\omega)\phi F[\phi]}{G_\eta(\vec{k},\omega)}, \tag{49}$$

$$F[\phi]-F[\phi]\ \text{vertex:}\qquad +\frac{1}{2}\frac{F[\phi]F[\phi]}{G_\eta(\vec{k},\omega)}, \tag{50}$$

$$\text{ghost vertex:}\qquad -\frac{1}{2}g^\dagger\frac{\delta F[\phi]}{\delta\phi}g. \tag{51}$$

As always, there is a certain amount of freedom in writing down the Feynman rules. It is always possible to take part of the quadratic piece in the total action and move it from the free action to the interaction term or vice versa. We have already seen that a linear term (e.g., $\nu m^2\phi$) in the forcing function $F[\phi]$ can with equal facility be reassigned to the differential operator D via the scheme $D\to D-\nu m^2, F[\phi]\to F[\phi]-\nu m^2\phi$. This procedure can always be used to completely eliminate any linear term in $F[\phi]$. Similar but more complicated behavior occurs if the forcing function contains both constant and quadratic pieces (e.g., $a+b\phi^2$). With the conventions given above, the interaction term contains (at least) a ''cosmological constant,'' $\frac{1}{2}a^2\int\int G_\eta^{-1}(x,y)dx\,dy$, a quadratic piece, $ab\int\int G_\eta^{-1}(x,y)\phi^2(y)dx\,dy$, and a ϕ^4 interaction. The quadratic piece could be moved into the free action at the cost of making the expression for the free propagator a little more complicated. This freedom in writing down the Feynman rules does not imply any ambiguity in the physical results: Moving quadratic pieces around from the interaction term to the free action will modify the Feynman rules but will not affect any physical quantities.

IV. EFFECTIVE ACTION: LOOP EXPANSION

In order to set up the formalism for the effective action, and its loop expansion, it is useful to first separate the two-point function for the noise into a *shape*, $g_2(x,y)$, and a constant *amplitude*, $\mathcal{A}$, via the correspondence

$$G_\eta(x,y)\overset{\text{def}}{=}\mathcal{A}g_2(x,y). \tag{52}$$

For the case of Gaussian white noise, this is automatically satisfied by the definition $G_\eta(x,y)\to\mathcal{A}\delta(x-y)$. For more general Gaussian noises (which describe, for instance, the effects of small scale degrees of freedom not fully decoupled from the physics of ϕ, such as in the case of a heat bath into which ϕ has been immersed), this form of the two-point function allows the interpretation of the noise intensity $\mathcal{A}$ as a characterization of the bath-system coupling. This will become more evident when we compare the ''effective potential'' in noisy environments, Eq. (72) below, with the same object for zero-temperature quantum field theory, Eq. (73). The normalization of the shape function $g_2(x,y)$ is essentially arbitrary, and any convenient normalization will suffice.

Another advantage of singling out the intensity parameter $\mathcal{A}$ is that it is the loop-counting parameter for this formulation of SPDEs. To see this, one starts by writing the characteristic functional (with external sources rescaled for convenience) as

$$\begin{aligned}Z[J]=&\frac{1}{\sqrt{\det[(2\pi)^3G_\eta]}}\int(\mathcal{D}\phi)(\mathcal{D}g)(\mathcal{D}g^\dagger)\\&\times\exp\left(-\tfrac{1}{2}\int\int\frac{(D\phi-F[\phi])g_2^{-1}(D\phi-F[\phi])}{\mathcal{A}}\right)\\&\times\exp\left(-\tfrac{1}{2}\int g^\dagger\left[D-\frac{\delta F}{\delta\phi}\right]g\right)\exp\left(\frac{\int J\phi}{\mathcal{A}}\right).\end{aligned} \tag{53}$$

The generating function (Helmholtz free energy in statistical field theory) for connected correlation functions is defined by

$$W[J]=+\mathcal{A}\{\ln Z[J]-\ln Z[0]\}. \tag{54}$$

The effective action (Gibbs free energy in statistical field theory) is then defined *nonperturbatively* in terms of $W[J]$ by taking its Legendre transform [20,24],

$$\Gamma[\phi;\phi_0]=-W[J]+\int\phi J,\qquad\frac{\delta W[J]}{\delta J}=\phi,$$

$$\frac{\delta\Gamma[\phi;\phi_0]}{\delta\phi}=J. \tag{55}$$

Here ϕ_0 is some suitable background (mean) field, which is taken to be the stochastic expectation value of ϕ in the absence of external sources, $J=0$. It is often but not always zero, and we retain it for generality.

The previous equation defines the nonperturbative effective action. For any specific example the previous equation is not very useful, and we often have to restrict ourselves to a perturbative calculation of the effective action, after singling out an expansion parameter. One can always develop a Feynman diagram expansion provided that the classical action can be separated into a ''quadratic piece'' and an ''interacting term,'' as we have already done.

In the loop expansion the sum of all connected diagrams coupled to external sources $J(x)$ is exactly $W[J]$ as defined above, and the effective action $\Gamma[\phi;\phi_0]$ corresponds to all (amputated) one-particle irreducible graphs (1PI), that is, Feynman diagrams that cannot be made disconnected by cutting only one propagator.

In the following argument, we will be considering diagrams contributing to the effective action. Recall that ghosts can only appear as internal lines, since they are not coupled to external sources.

To see the role of the amplitude $\mathcal{A}$ as a loop-counting parameter, note that each field propagator is proportional to $\mathcal{A}$ while each ghost propagator is independent of $\mathcal{A}$. The vertices that do not include ghosts are proportional to $\mathcal{A}^{-1}$, while ghost vertices are independent of $\mathcal{A}$. Thus each Feynman diagram contributing to the effective action is proportional to $A^{I_\phi - V_\phi}$, where I_ϕ is the number of nonghost propagators and V_ϕ is the number of nonghost vertices. But each ghost vertex is attached to exactly two ghost propagators (except for tadpole ghost loops), and each ghost propagator is attached to exactly two ghost vertices (except for tadpole ghost loops). In the case of tadpole ghost loops, exactly one propagator is attached to exactly one ghost vertex. This implies that if one assigns a factor $\mathcal{A}$ to each ghost propagator and a factor of $\mathcal{A}^{-1}$ to each ghost loop, then one will not change the total number of factors of $\mathcal{A}$ assigned to the Feynman diagram. Thus the Feynman diagrams are proportional to $\mathcal{A}^{I-V}$, where I is the total number of (internal) propagators in the Feynman diagram and V is the total number of vertices, now including ghosts.

It is the result of a standard topological theorem that for any graph (not just any Feynman diagram) $I-V=L-1$, where L is the number of loops [19,21,43]. It is then easy to see that field theories based on SPDEs exhibit exactly the same loop-counting properties as QFTs except that the loop-counting parameter is now the *amplitude* of the noise two-point function (instead of Planck's constant $\hbar$). The only subtle part of the argument has been in dealing with the Faddeev-Popov ghosts, and it is important to realize that this argument is completely independent of the details of the differential operator D and the forcing term $F[\phi]$. When it comes to calculating the diagrams contributing to the effective action, the extra explicit factor of $\mathcal{A}$ inserted in the definition of $W[J]$ above guarantees that the 1PI graphs contribute to $\Gamma[\phi;\phi_0]$ with a weight that is exactly $\mathcal{A}^L$. This demonstrates that $\mathcal{A}$ is a *bona fide* expansion parameter.

At this point, it becomes natural to make a comparison with the MSR (Martin-Siggia-Rose) formalism for the calculation of the effective action in stochastic field theories, where one introduces a field *conjugate* to ϕ. Historically, this conjugate field first arose in setting up a variational approach to the diffusion equation (cf. Morse and Feshbach [27]). The following remarks will help one to understand the differences and the complementarity of our approach to the MSR approach; the bottom line is related to technical issues associated with proving all-orders renormalizability [1,12,15]. The direct approach developed in this paper is akin to the ghost-free axial gauge of QCD or the so-called unitary gauge in the standard model of particle physics: This is a formalism well-adapted to isolating the physical degrees of freedom, at least perturbatively, but is not well-adapted to proving the all-orders renormalizability of the theory. (Proving one-loop renormalizability for specific theories is not too difficult, and we will address this issue in a pair of companion papers [30,31].)

In analogy with the situation in QFT, one has three possible responses to this state of affairs.

(i) Use the MSR formalism for all calculations. This is comparable to using BRST-invariant versions of the standard model of particle physics to calculate scattering cross sections and decay rates (That is, overkill).

(ii) One could appeal to the fact that the SPDEs considered in this paper are hardly likely to be thought of as fundamental theories in the particle physics sense; these SPDEs are much more like ''effective field theories,'' in that the noise and fluctuations in real physical systems are manifestations of our lack of knowledge of the short-distance physics. Viewed as effective theories, renormalizability is no longer the main guiding light it was once thought to be [20].

(iii) At a very *practical* level one can choose to be guided by experience with quantum field theories. It is well known that one-loop physics is often sufficient for extracting most of the physical information from a system. Calculations beyond one-loop, while certainly important at a fundamental level, are often more than is really needed. One of the great technical simplifications of one-loop physics is that, via zeta function technology, essentially any field theory can be regularized at one loop without excessive complications [53–55].

For these reasons we will now restrict our attention to a one-loop calculation (apart from the discussion of Feynman diagrams and the loop expansion, everything up to this point has been valid nonperturbatively, while those discussions were still valid to all orders in perturbation theory). In the next section we calculate the one-loop effective action.

V. EFFECTIVE ACTION: ONE LOOP

It is well known that the effective action for a field theory can be obtained by performing a Legendre transform on the logarithm of the characteristic functional (partition function). Writing

$$Z[J]=\int \mathcal{D}\phi \exp\left(\frac{-\mathcal{S}[\phi]+\int J\phi}{a}\right), \tag{56}$$

where a is the parameter characterizing the fluctuations, one gets for the one-loop effective action (first order in a)

$$\Gamma[\phi;\phi_0]=\mathcal{S}[\phi]-\mathcal{S}[\phi_0]+\tfrac{1}{2}a\{\ln\det(\mathcal{S}_2[\phi]) - \ln\det(\mathcal{S}_2[\phi_0])\}+O(a^2). \tag{57}$$

Here $\mathcal{S}_2=\delta^2\mathcal{S}/\delta\phi(x)\delta\phi(y)$ is the matrix of second-order functional derivatives of the action $\mathcal{S}[\phi]$ (often called the Jacobi field operator). For QFT the loop-counting parameter a is Planck's constant $\hbar$, and $\mathcal{S}_2$ is a second-order partial differential operator that depends on the field ϕ via some potential-like term. The determinants of partial differential operators can be defined and calculated by a variety of techniques. The notation $\mathcal{S}[\phi_0]$ is actually shorthand for $\mathcal{S}[\langle\phi[J=0]\rangle]$, and for a symmetric ground state

($\langle \phi[J=0]\rangle = 0$) one often has $\mathcal{S}[0]=0$. These terms contribute a constant offset to the effective action. In QFT these terms are interpreted as a field-independent contribution to the vacuum energy and are traditionally ignored, although in the context of cosmology they contribute (sometimes catastrophically) to the cosmological constant. In the interest of generality we will make them explicit. When we consider field theories based on SPDEs, the loop-counting parameter a becomes $\mathcal{A}$, which we singled out as the amplitude for the noise, and the bare action in Eq. (53) is replaced by Eqs. (34) and (37),

$$\mathcal{S}[\phi] \to \tfrac{1}{2}\int\int \{(D\phi - F[\phi]) g_2^{-1} (D\phi - F[\phi])\} \times d^d\vec{x}\, dt\, d^d\vec{y}\, dt' - \frac{1}{2}\mathcal{A}(\ln \mathcal{J} + \ln \mathcal{J}^\dagger) \tag{58}$$

$$= \mathcal{S}_{\text{classical}}[\phi] - \frac{1}{2}\mathcal{A}(\ln \mathcal{J} + \ln \mathcal{J}^\dagger), \tag{59}$$

where on the second line we have denoted by $\mathcal{S}_{\text{classical}}[\phi]$ the double integral in the previous line. This is the quantity that we have previously defined as the nonlinear generalization of the Onsager-Machlup action to arbitrary Gaussian noise [52].

The noise at this stage is Gaussian, and does not need to be translation invariant. We have explicitly kept the Jacobian functional determinant. Inserting Eq. (59) into the formula for the one-loop effective action [Eq. (57)], we obtain the following general result (applicable to any SPDE):

$$\begin{aligned}\Gamma[\phi;\phi_0] = {} & \mathcal{S}_{\text{classical}}[\phi] - \mathcal{S}_{\text{classical}}[\phi_0] \\ & + \mathcal{A}\left\{\frac{1}{2}\ln\det(\mathcal{S}_2[\phi]) - \frac{1}{2}\ln\det(\mathcal{S}_2[\phi_0]) \right. \\ & - \frac{1}{2}\ln \mathcal{J}[\phi] - \frac{1}{2}\ln \mathcal{J}^\dagger[\phi] + \frac{1}{2}\ln \mathcal{J}[\phi_0] \\ & \left. + \frac{1}{2}\ln \mathcal{J}^\dagger[\phi_0]\right\} + O(\mathcal{A}^2).\end{aligned} \tag{60}$$

To make this more explicit, the fluctuation operator $\mathcal{S}_2(\phi)$ (also known as the Jacobi field operator) is

$$\begin{aligned}\mathcal{S}_2[\phi] = {} & \left(D^{\leftarrow} - \frac{\delta F^{\leftarrow}}{\delta\phi}\right) g_2^{-1} \left(D - \frac{\delta F}{\delta\phi}\right) \\ & - (D\phi - F[\phi]) g_2^{-1} \frac{\delta^2 F}{\delta\phi\,\delta\phi}.\end{aligned} \tag{61}$$

Here the $\leftarrow$ indicates that these operators should be thought of as acting to the left. Also $g_2^{-1}(x,y)$ is to be understood as a "matrix" with implicit sums over the indices x,y (i.e., integrations over the variables.) Note that if $F[\phi]$, contains derivatives of ϕ, then $\delta F/\delta\phi$ will be a differential operator. Performing an integration by parts, this can be converted to a statement about the adjoint operator acting to the right, i.e., we can rewrite $\mathcal{S}_2[\phi]$ as

$$\begin{aligned}\mathcal{S}_2[\phi] = {} & \left(D^{\dagger} - \frac{\delta F^{\dagger}}{\delta\phi}\right) g_2^{-1} \left(D - \frac{\delta F}{\delta\phi}\right) \\ & - (D\phi - F[\phi]) g_2^{-1} \frac{\delta^2 F}{\delta\phi\,\delta\phi}.\end{aligned} \tag{62}$$

Putting all this together gives the following one-loop result for the effective action:

$$\begin{aligned}\Gamma[\phi;\phi_0] = {} & \tfrac{1}{2}\int\int d^d\vec{x}\, dt\, d^d\vec{y}\, dt' \{(D\phi - F[\phi]) g_2^{-1} \\ & \times (D\phi - F[\phi])\} - \frac{1}{2}\mathcal{A}(\ln \mathcal{J} + \ln \mathcal{J}^\dagger) \\ & + \frac{1}{2}\mathcal{A}\ln\det\left[\left(D^{\dagger} - \frac{\delta F}{\delta\phi}^{\dagger}\right) g_2^{-1} \left(D - \frac{\delta F}{\delta\phi}\right) \right. \\ & \left. - (D\phi - F[\phi]) g_2^{-1} \frac{\delta^2 F}{\delta\phi\,\delta\phi}\right] \\ & - (\phi \to \phi_0) + O(\mathcal{A}^2).\end{aligned} \tag{63}$$

Grouping together the terms proportional to $\mathcal{A}$, and using the representation of the functional determinant, enables us to rewrite the above in the alternative form

$$\begin{aligned}\Gamma[\phi;\phi_0] = {} & \tfrac{1}{2}\int\int d^d\vec{x}\, dt\, d^d\vec{y}\, dt' \{(D\phi - F[\phi]) g_2^{-1} \\ & \times (D\phi - F[\phi])\} \\ & + \frac{1}{2}\mathcal{A}\ln\det\left[I - \left\{\left(D^{\dagger} - \frac{\delta F}{\delta\phi}^{\dagger}\right)^{-1} g_2 \left(D - \frac{\delta F}{\delta\phi}\right)^{-1} \right.\right. \\ & \left.\left. \times\left((D\phi - F[\phi]) g_2^{-1} \frac{\delta^2 F}{\delta\phi\,\delta\phi}\right)\right\}\right] \\ & - (\phi \to \phi_0) + O(\mathcal{A}^2).\end{aligned} \tag{64}$$

This expression for the one-loop effective action is instructive. It is made up of two contributions whose origin and physics are quite different. On the one hand, the first term (the generalized Onsager-Machlup term) gives a contribution whose form is directly related to both the noise *shape factor* and the non-noisy part of the equation of motion, including nonlinearities. On the other hand, the log-determinant term is proportional to the noise *amplitude* (which we have seen is the loop-expansion parameter) and its specific form depends *also* on the structures of D and $F[\phi]$, as well as on properties of the noise shape function. Therefore, noise plays a central role in the physics of the SPDE and, as will be discussed below, particularly in the nature of the ground state of the stochastic system described by Eq. (1).

VI. EFFECTIVE POTENTIAL: ONE LOOP

We now concentrate on field configurations that are homogeneous and static. For such field configurations the effective action reduces to a quantity known as the "effective potential." In this section we will *calculate* the effective potential, deferring the discussion of its physical *interpretation* (in the context of SPDEs) to the next section.

The effective potential is defined as

$$\mathcal{V}[\phi;\phi_0]=\frac{\Gamma[\phi;\phi_0]}{\Omega}, \tag{65}$$

with ϕ a homogeneous and static field configuration, and Ω the volume of spacetime. The effective potential at one loop is given by

$$\begin{aligned}\mathcal{V}[\phi;\phi_0]=&\frac{1}{2}F^2[\phi]\left\{\int d^d\vec{x}\,dt\,g_2^{-1}\right\}-\frac{1}{2}\frac{\mathcal{A}}{\Omega}\ln\det\left(D-\frac{\delta F}{\delta\phi}\right)\\&-\frac{1}{2}\frac{\mathcal{A}}{\Omega}\ln\det\left(D^\dagger-\frac{\delta F^\dagger}{\delta\phi}\right)\\&+\frac{1}{2}\frac{\mathcal{A}}{\Omega}\ln\det\left[\left(D^\dagger-\frac{\delta F^\dagger}{\delta\phi}\right)g_2^{-1}\left(D-\frac{\delta F}{\delta\phi}\right)\right.\\&\left.+F[\phi]\left\{\int d^d\vec{x}\,dt\,g_2^{-1}\right\}\frac{\delta^2F}{\delta\phi\delta\phi}\right]\\&-(\phi\rightarrow\phi_0)+O(\mathcal{A}^2).\end{aligned} \tag{66}$$

In order to turn this into a more tractable expression, it is useful to introduce a frequency-momentum representation. First notice that

$$\begin{aligned}\int d^d\vec{x}\,dt\,g_2^{-1}(\vec{x},t)=&\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}d^d\vec{x}\,dt\,\tilde{g}_2^{-1}(\vec{k},\omega)\\&\times\exp[-i(\omega t-\vec{k}\cdot\vec{x})]\\=&\tilde{g}_2^{-1}(\vec{k}=\vec{0},\omega=0).\end{aligned} \tag{67}$$

[It is clear from the formula for the one-loop effective potential equation (66) that the above integral has to be finite, or rendered finite by appropriate renormalizations of the noise correlation function and the parameters it contains.]

We next make use of the following identity valid for a translation invariant operator X:

$$\begin{aligned}\ln\det X=&\int d^d\vec{x}\,dt\int d^d\vec{k}_1d\omega_1\int d^d\vec{k}_2d\omega_2\langle\vec{x},t|\vec{k}_1,\omega_1\rangle\\&\times\ln X(\vec{k}_1,\omega_1)\,\delta^d(\vec{k}_1,\vec{k}_2)\,\delta(\omega_1,\omega_2)\langle\vec{k}_2,\omega_2|\vec{x},t\rangle\\=&\Omega\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\ln X(\vec{k},\omega).\end{aligned} \tag{68}$$

Applying this to the one-loop effective potential yields

$$\begin{aligned}\mathcal{V}[\phi;\phi_0]=&\frac{1}{2}F^2[\phi]\tilde{g}_2^{-1}(\vec{k}=\vec{0},\omega=0)\\&-\frac{1}{2}\mathcal{A}\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\ln\left[D(\vec{k},\omega)-\frac{\delta F}{\delta\phi}\right]\\&-\frac{1}{2}\mathcal{A}\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\ln\left[D^\dagger(\vec{k},\omega)-\frac{\delta F^\dagger}{\delta\phi}\right]\\&+\frac{1}{2}\mathcal{A}\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\ln\left[\left(D^\dagger(\vec{k},\omega)-\frac{\delta F^\dagger}{\delta\phi}\right)\right.\\&\times\tilde{g}_2^{-1}(\vec{k},\omega)\left(D(\vec{k},\omega)-\frac{\delta F}{\delta\phi}\right)\\&\left.+F[\phi]\frac{\delta^2F}{\delta\phi\delta\phi}\tilde{g}_2^{-1}(\vec{k}=\vec{0},\omega=0)\right]\\&-(\phi\rightarrow\phi_0)+O(\mathcal{A}^2).\end{aligned} \tag{69}$$

(Note that g_2 plays two rather different roles above.) We now adopt the simplifying *convention* that

$$\int d^d\vec{x}\,dt\,g_2^{-1}(\vec{x},t)=1=\tilde{g}_2^{-1}(\vec{k}=0,\omega=0). \tag{70}$$

This is only a *convention*, not an additional restriction on the noise, since it only serves to give an absolute meaning to the normalization of the amplitude $\mathcal{A}$.

With these conventions, the one-loop effective potential can be written as

$$\begin{aligned}\mathcal{V}[\phi;\phi_0]=&\frac{1}{2}F^2[\phi]-\frac{1}{2}\mathcal{A}\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\ln\left[D(\vec{k},\omega)-\frac{\delta F}{\delta\phi}\right]\\&-\frac{1}{2}\mathcal{A}\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\ln\left[D^\dagger(\vec{k},\omega)-\frac{\delta F^\dagger}{\delta\phi}\right]\\&+\frac{1}{2}\mathcal{A}\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\ln\left[\left(D^\dagger(\vec{k},\omega)-\frac{\delta F^\dagger}{\delta\phi}\right)\right.\\&\left.\times\tilde{g}_2^{-1}(\vec{k},\omega)\left(D(\vec{k},\omega)-\frac{\delta F}{\delta\phi}\right)+F[\phi]\frac{\delta^2F}{\delta\phi\delta\phi}\right]\\&-(\phi\rightarrow\phi_0)+O(\mathcal{A}^2),\end{aligned} \tag{71}$$

which can be recast into

$$\begin{aligned}\mathcal{V}[\phi;\phi_0]=&\frac{1}{2}F^2[\phi]+\frac{1}{2}\mathcal{A}\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\\&\times\ln\left[1+\frac{\tilde{g}_2(\vec{k},\omega)F[\phi]\dfrac{\delta^2F}{\delta\phi\delta\phi}}{\left(D^\dagger(\vec{k},\omega)-\dfrac{\delta F^\dagger}{\delta\phi}\right)\left(D(\vec{k},\omega)-\dfrac{\delta F}{\delta\phi}\right)}\right]\\&-(\phi\rightarrow\phi_0)+O(\mathcal{A}^2).\end{aligned} \tag{72}$$

This formula is one of the central results of this paper. It shows that noise-induced fluctuations modify the zero-loop piece of the potential in a way which is reminiscent of the situation in both statistical and quantum field theory. For example, in QFT one has [20,21]

$$\mathcal{V}_{\rm QFT}[\phi;\phi_0]=V(\phi)+\frac{1}{2}\hbar\int\frac{d^d\vec{k}\,d\omega}{(2\pi)^{d+1}}\times\ln\left[1+\frac{\dfrac{\delta^2 V}{\delta\phi\,\delta\phi}}{\omega^2+\vec{k}^2+m^2}\right]-(\phi\to\phi_0)+O(\hbar^2). \tag{73}$$

We see in Eq. (72) that the ground-state structure of the SPDE (which we will soon see is obtained by minimizing $\mathcal{V}[\phi;\phi_0]$) depends on both the noise correlations and the nonlinearities induced by the forcing term. We also see explicitly how the noise amplitude is essential in the competition between deterministic and stochastic effects.

The major difference between the effective potential for SPDEs and QFT lies in the fact that for SPDEs the scalar propagator of QFT is replaced with a propagator which has a more complex structure for the equivalent of the ''mass'' term. This difference is due to the causal structure of SPDEs.

Notice also that for SPDEs one can naturally adapt the noise to be both the source of fluctuations *and* the regulator to keep the Feynman diagram expansion finite. This follows immediately by inspection of Eq. (72), which shows that the (momentum- and frequency-dependent) noise shape function $\tilde{g}_2$ will affect the momentum and frequency behavior of the one-loop integral. The finiteness, divergence structure, and renormalizability of this integral will depend very much on the functional form of $\tilde{g}_2$. It is thus clear that we can use the noise shape function to regulate the integral, if we wish.

VII. INTERPRETATION

The *physical interpretation* of the effective action and the effective potential for SPDEs is considerably more subtle than that for the more usual QFTs. The situation is complicated by the fact that for a completely general SPDE it may not be meaningful to define a physical energy. Even when the SPDE is sufficiently special so that some physical notion of energy may be defined, the system may be subject to dissipation: The *physical* energy need not be conserved, even in the absence of noise. Thus the effective action and effective potential for SPDEs are not related to the physical energy. This means that *some* of the physical intuition built up from QFTs may be misleading and it becomes important to reassess the notion of effective action and effective potential to see how much survives in the SPDE context.

The great virtue of the effective action and effective potential in QFT is that they contain all the information regarding the ground state of the system and its fluctuations: From a knowledge of the effective potential, one can ascertain under what conditions the system will display one degree of symmetry or another. It is essential that most of these properties carry over to the case of SPDEs, otherwise the effective action and effective potential would be mere mathematical constructs without physical relevance. Fortunately the key features do in fact carry over: (i) the stationary points of the effective action still correspond to stochastic expectation values of the fields in the absence of an external current; (ii) the effective potential governs the probability that the *spacetime average* of the field takes on specific values; (iii) even when the notion of physical energy is lacking, we will see that there is a notion of quasienergy for SPDEs, with the quasienergy being a measure of the extent to which the system has been driven away from its nonstochastic (zero-noise) configuration; and (iv) the one-loop effective action will be demonstrated to describe the (approximate) probability for an initial field configuration to evolve into some final (in the asymptotic sense) field configuration under the influence of the stochastic noise.

A. Equations of motion in the presence of fluctuations

If one makes use of the definition of the effective action as a Legendre transform, it is easy to see that

$$\frac{\delta\Gamma[\phi;\phi_0]}{\delta\phi}=J[\phi], \tag{74}$$

where $J[\phi]$ is that external current required in order that

$$\langle\phi[J]\rangle=\phi. \tag{75}$$

In particular, by taking $J=0$,

$$\frac{\delta\Gamma[\phi;\phi_0]}{\delta\phi}=0\Leftrightarrow\phi=\langle\phi[J=0]\rangle. \tag{76}$$

Stationary points of the effective action occur at those (mean)-field configurations which are zero-external-current stochastic expectation values of the fluctuating field. (Proof of this may be found, for instance, on p. 65 of Weinberg [20].) It is important to realize that one never needs to invoke the notion of energy to obtain this result. The QFT interpretation of this result, which we now see extends to SPDEs, is that the effective action gives the equations of motion once fluctuations (noise) are taken into account. (This is a nonperturbative result, not limited to the one-loop approximation.)

B. Probability distribution for the spacetime average field

We have previously seen that the probability distribution for the fluctuating field, considered as a function over spacetime, to take on the value $\phi(\vec{x},t)$ is given by the functional

$$\mathbf{P}[\phi]=\mathcal{P}[D\phi-F[\phi]]\sqrt{\mathcal{J}\mathcal{J}^\dagger}. \tag{77}$$

Now suppose we coarse-grain, by looking at the spacetime average of the field ϕ as defined by

$$\frac{\int_{\Omega_s\times T}\phi(\vec{x},t)\,d^d\vec{x}\,dt}{\Omega_s T}, \tag{78}$$

and ask what is the probability that this spacetime average take on a specific numerical value $\bar{\phi}$? (For definiteness we impose periodic boundary conditions in space Ω_s and time T and interpret $\Omega_s T$ as the volume of the spacetime box. This has the technical advantage that the partition function $Z[J]$ is then needed only for sources J that are strictly independent of space and time.)

The probability we are interested in is easily calculated to be

$$\mathrm{Prob}\left(\int_{\Omega_s\times T} d^d\vec{x}\,dt\,\phi(\vec{x},t)=\bar{\phi}\Omega_s T\right)$$
$$=\int(\mathcal{D}\phi)\mathbf{P}[\phi]\delta\left(\int_{\Omega_s\times T} d^d\vec{x}\,dt\,\phi(\vec{x},t)-\bar{\phi}\Omega_s T\right) \quad (79)$$
$$=\int(\mathcal{D}\phi)\int d\lambda\,\mathbf{P}[\phi]$$
$$\times\exp\left(i\lambda\left[\int_{\Omega_s\times T} d^d\vec{x}\,dt\,\phi(\vec{x},t)-\bar{\phi}\Omega_s T\right]\right) \quad (80)$$
$$=\int d\lambda\,Z[J(x)=i\lambda]\exp(-i\lambda\Omega_s T\bar{\phi}). \quad (81)$$

We now take the limit as $\Omega_s T$ becomes very large, and apply the method of stationary phase. By definition we have

$$Z[J(x)=i\lambda]=\exp[\Omega_s T\{i\lambda\phi(\lambda)-\mathcal{V}[\phi(\lambda);\phi_0]/\mathcal{A}\}] \quad (82)$$

with the subsidiary condition

$$\left.\frac{\delta\mathcal{V}[\phi;\phi_0]}{\delta\phi}\right|_{\phi(\lambda)}=i\lambda\mathcal{A}. \quad (83)$$

It is easy to demonstrate that

$$\mathrm{Prob}\left(\int_{\Omega_s\times T} d^d\vec{x}\,dt\,\phi(\vec{x},t)=\bar{\phi}\Omega_s T\right)$$
$$\propto\exp\left(-\Omega_s T\left[\frac{\mathcal{V}[\bar{\phi};\phi_0]}{\mathcal{A}}+O\left(\frac{1}{\Omega_s T}\right)\right]\right). \quad (84)$$

Thus the effective potential governs the probability distribution of the spacetime average of the fluctuating field. Minima of the effective potential correspond to maxima of the probability density of the spacetime average field. The way we have set up the argument applies equally well to QFTs and SPDEs and makes no reference to the notion of physical energy. (This result is nonperturbative but approximate—it is not limited to one loop. If we take either the infinite volume or infinite time limits, then with probability 1, the spacetime average field must equal one of the minima of the effective potential.)

C. Action and quasienergy for SPDEs

Even though the physical energy may not be defined for arbitrary SPDEs, we nevertheless can demonstrate that there always exists a positive-semidefinite functional of field configurations, the tree-level action, and a related ''quasienergy,'' whose minima correspond to maxima of the probability distribution of field configurations.

From the way the functional formalism has been set up, we can always define and calculate an effective action and an effective potential even if the underlying nonstochastic version of the partial differential equation does not arise from a Lagrangian formulation. We have already seen that the effective action has a natural interpretation in terms of the equations of motion once fluctuations are taken into account, and that the effective potential governs fluctuations in the spacetime average of the field. We now go one step further: We distinguish two concepts of ''energy,'' the ''true physical energy'' and the ''quasienergy,'' and show that even if the physical energy is undefined (or possibly not useful due to dissipative effects), the quasienergy is still a useful measure of the extent to which fluctuations modify the nonstochastic equations of motion. We start from our general SPDE (1),

$$D\phi=F[\phi]+\eta, \quad (85)$$

and its nonstochastic version,

$$D\phi=F[\phi]. \quad (86)$$

Sometimes this nonstochastic partial differential equation will arise from some Lagrangian, often it will not.

Even if the PDE ($D\phi=F[\phi]$) does not arise from a Lagrangian, the results of this paper demonstrate that it is always possible to assign a tree-level action to the stochastic system:

$$\mathcal{S}_{\mathrm{classical}}=\frac{1}{2}\int\int dx\,dy(D\phi-F[\phi])g_2^{-1}(D\phi-F[\phi])\geqslant 0. \quad (87)$$

This classical action is positive semidefinite, and has minima (which are equal to zero) at field configurations that satisfy the zero-noise equations of motion. This is most obvious for white noise, when the action is a perfect square, but the result is general. The noise two-point correlation function [being an (infinite-dimensional) covariance matrix] is by definition positive definite. Therefore, its inverse is also positive definite and similarly the (infinite-dimensional) matrix g_2^{-1} is a positive definite operator. Thus this classical action (the generalized Onsager-Machlup action [52]) is always greater than or equal to zero.

The classical action thus measures the extent to which a given field configuration fails to satisfy the zero-noise equations of motion, the measure of the deviation being weighted by the *shape* of the noise correlations. (In fact, if the amplitude $\mathcal{A}$ of the noise is set to zero, the action is identically equal to zero.)

We now define the quasienergy by

$$\mathcal{S}[\phi]=\int E_{\mathrm{quasi}}[\phi]dt. \quad (88)$$

We justify calling this object the quasienergy by the fact that if we treat it as a Hamiltonian functional, and put the resulting object into the partition function of an equilibrium statistical field theory, we get the generating functional for all the correlation functions (ignoring ghost Jacobians for the moment). Explicitly, we can write

$$E_{\mathrm{quasi}}[\phi]=\frac{1}{2}\int\int d^d\vec{x}\,d^d\vec{y}\,dt'(D\phi-F[\phi])g_2^{-1}$$
$$\times(D\phi-F[\phi]). \quad (89)$$

Note that the quasienergy depends both on the PDE and on the shape of the noise correlation function. If the amplitude $\mathcal{A}$ of the noise is set to zero, this quasienergy is conserved and is exactly equal to zero. This quasienergy can be thought of as a nonlinear generalization of the Onsager-Machlup ''energy'' to arbitrary Gaussian noise. The particular label one chooses to apply to this quantity is not important as long as one bears carefully in mind that this ''energy'' need not be the physical energy.

If we now restrict ourselves to homogeneous and static fields, and consider the effective potential as defined above, then by the procedures used in quantum and stochastic field theories, the effective potential (multiplied by the volume of space) is the stochastic expectation value of the quasienergy $\langle E_{\rm quasi}[\phi]\rangle$ in the *presence* of the noise-induced fluctuations, and subject to the constraint $\langle\phi\rangle=\phi$. A proof of this result is provided on pp. 72 and 73 of Weinberg [20]. Though that proof is phrased in a Lorentzian-signature QFT language, it readily carries over to Euclidean-signature equilibrium statistical field theory. Once the physical energy is replaced by the quasienergy, the proof can be extended to SPDEs as well.

We have been able to show that minima of the effective potential also minimize the quasienergy, and therefore the noise-induced deviations from the zero-noise equations of motion.

D. Transition probabilities

What is the probability that a certain initial field configuration $\phi_i(\vec{x})$ at time t_i evolves into a final field configuration $\phi_f(\vec{x})$ at time t_f? We have already developed the appropriate machinery to address this question. Indeed

$$
\begin{aligned}
\text{Prob}(\phi_f(\vec{x}),t_f;\phi_i(\vec{x}),t_i) &\propto \int (\mathcal{D}\eta)\mathcal{P}[\eta]\,\delta[\phi_{\rm soln}(\vec{x},t_i;\eta)-\phi_i(\vec{x})]\,\delta[\phi_{\rm soln}(\vec{x},t_f;\eta)-\phi_f(\vec{x})]\\
&\propto \int (\mathcal{D}\eta)(\mathcal{D}\phi)\mathcal{P}[\eta]\,\delta[\phi_{\rm soln}(\vec{x},t;\eta)-\phi(\vec{x},t)]\,\delta[\phi(\vec{x},t_i)-\phi_i(\vec{x})]\,\delta[\phi(\vec{x},t_f)-\phi_f(\vec{x})]\\
&\propto \int (\mathcal{D}\eta)(\mathcal{D}\phi)\mathcal{P}[\eta]\,\delta[D\phi-F[\phi]-\eta]\sqrt{\mathcal{J}\mathcal{J}^\dagger}\,\delta[\phi(\vec{x},t_i)-\phi_i(\vec{x})]\,\delta[\phi(\vec{x},t_f)-\phi_f(\vec{x})]\\
&\propto \int (\mathcal{D}\phi)\mathcal{P}[D\phi-F[\phi]]\sqrt{\mathcal{J}\mathcal{J}^\dagger}\,\delta[\phi(\vec{x},t_i)-\phi_i(\vec{x})]\,\delta[\phi(\vec{x},t_f)-\phi_f(\vec{x})]\\
&\propto \int (\mathcal{D}\phi)\mathbf{P}[\phi]\,\delta[\phi(\vec{x},t_i)-\phi_i(\vec{x})]\,\delta[\phi(\vec{x},t_f)-\phi_f(\vec{x})]\\
&\propto \int (\mathcal{D}\phi)\exp(-\mathcal{S}[\phi]/\mathcal{A})\sqrt{\mathcal{J}\mathcal{J}^\dagger}\,\delta[\phi(\vec{x},t_i)-\phi_i(\vec{x})]\,\delta[\phi(\vec{x},t_f)-\phi_f(\vec{x})]\\
&\propto \int_{\phi(\vec{x},t_i)=\phi_i(\vec{x})}^{\phi(\vec{x},t_f)=\phi_f(\vec{x})} (\mathcal{D}\phi)\exp(-\mathcal{S}[\phi]/\mathcal{A})\sqrt{\mathcal{J}\mathcal{J}^\dagger}.
\end{aligned}
\tag{90}
$$

This is formally identical to the formula usually encountered in equilibrium statistical field theory, and everything so far is nonperturbatively correct.

Now take a saddle-point approximation: Find an interpolating field $\phi_{\rm int}(\vec{x},t)$ that minimizes $\mathcal{S}[\phi]$ and interpolates from $\phi_i(\vec{x})$ to $\phi_f(\vec{x})$. Perform the Gaussian integral about the saddle point. Then by definition of the one-loop effective action

$$
\text{Prob}(\phi_f(\vec{x}),t_f;\phi_i(\vec{x}),t_i)\approx\exp\left[-\frac{\Gamma[\phi_{\rm int}]}{\mathcal{A}}\right]. \tag{91}
$$

This is only a one-loop result, but it demonstrates that the effective action for SPDEs inherits many of the important features of the effective action for QFTs.

E. Summary

From the above, we see that the effective action and effective potential for SPDEs exhibit many of the key features of the effective action and effective potential of QFTs. This is important because it guarantees that not only is it relatively easy to calculate the one-loop effective potential, but also it is useful to do so: As is the case for QFTs, minima of the effective potential for SPDEs provide information about expectation values of the fields. The effective action also provides information about fluctuations in spacetime averaged fields, it gives information about the noise-induced deviations from the nonstochastic equations of motion, and it governs the transition probabilities whereby initial field configurations evolve to final field configurations. Thus, both the effective potential and the effective action are as useful for SPDEs as they are for QFTs.

Furthermore, as demonstrated in recent work by Alexander and Eyink [56–58], the effective potential is also a useful tool in a strong noise regime far from equilibrium. The major difference between those papers and our own formalism is that they work within the MSR approach. They also focus on strong noise regimes, while we emphasize that for many purposes a one-loop calculation is both computationally efficient and quite sufficient to extract many key features of the physics of the system. The two approaches are complementary, and where they overlap, they are in complete agreement.

VIII. DISCUSSION

In this paper we have developed a general and powerful formalism applicable to arbitrary SPDEs. We have shown how to convert *arbitrary* correlation functions associated with *arbitrary* SPDEs into functional integrals. (And for this first step the noise does not have to be Gaussian.) For Gaussian noise (not necessarily translation invariant) we have carried the formalism further, setting up the basic ingredients needed for Feynman diagram expansions with the noise amplitude serving as the loop-counting parameter, and defining a nonperturbative effective action in analogy with QFT.

We hope to have convinced the reader that the ''direct approach'' developed in this paper is both useful and complementary to the more traditional MSR formalism [1,12,15]. Some questions can more profitably be asked and answered in this ''direct'' formalism. For instance, the fact that the noise amplitude is the loop-counting parameter is easy to establish in this ''direct'' formalism, but appears to have no analog result in the MSR formalism. The effective action gives rise naturally to the concept of an effective potential, a powerful construct well known and studied within the QFT context, where it serves to classify and compute ground states and allows one to investigate symmetry properties and patterns of symmetry breaking (both spontaneous and dynamic). An analogous construct can also be defined and calculated for stochastic field theories based on SPDEs, and we have done so in this work. However, for arbitrary SPDEs, such as those contemplated here, the notion of ground state and effective potential must be approached with extra care and their physical interpretation clarified. We have taken pains to do so, establishing that the minimum of the construct we call the effective potential corresponds to solving the full equations of motion (for homogeneous and static field configurations) in the presence of noise.

We feel that the most interesting result of this analysis is a general formula for the one-loop effective potential for *any* SPDEs subject to translation-invariant Gaussian noise. This is still an extremely broad class of problems, and in a pair of companion papers we will specialize this analysis to two particular cases. First, we discuss the noisy Burgers equation (KPZ equation), where the effective potential approach immediately leads us to such interesting observations as the existence of dynamical symmetry breaking (DSB) and the Coleman-Weinberg mechanism [31]. Second, we discuss the reaction-diffusion-decay system, and explicitly calculate the renormalized effective potential for one, two, and three spatial dimensions [30]. These are issues that are extremely difficult to address using the MSR approach.

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APPENDIX A: JACOBIAN FUNCTIONAL DETERMINANT

The Jacobian functional determinant is often (but not always) field independent, and can often (but not always) be discarded. In this appendix we explore this issue in more detail.

1. Causality: Retarded Green functions

This discussion is a generalization of Rivers [21], pp. 155 and 156. There are also relevant comments in De Dominicis and Peliti [12], Appendix B, part C (pp. 370 and 371). See also the footnote on p. 214 of Frisch [3], and the discussion in Zinn-Justin (pp. 372 and 373 [15]). We are interested in evaluating

$$\mathcal{J} \equiv \det\left(D - \frac{\delta F}{\delta \phi} \right). \tag{A1}$$

To proceed, we make some specific assumptions about the form of D. Let us confine attentions to the class of differential operators

$$D_n \equiv \frac{\partial^n}{\partial t^n} - D_0(\vec{\nabla}). \tag{A2}$$

(If we take $D_0 = \vec{\nabla}^2$, then D_1 is the diffusion operator while D_2 is the wave operator, so this class of differential operators is still broad enough to cover almost everything of physical interest.) Now write

$$\mathcal{J}_n \equiv \det\left(\partial_t^n - D_0 - \frac{\delta F}{\delta \phi} \right) \tag{A3}$$

$$= \det(\partial_t^n) \det\left[I - G_n \left(D_0 + \frac{\delta F}{\delta \phi} \right) \right] \tag{A4}$$

$$= \det(\partial_t^n) \exp \operatorname{Tr} \ln\left[I - G_n \left(D_0 + \frac{\delta F}{\delta \phi} \right) \right] \tag{A5}$$

$$= \det(\partial_t^n) \exp\left\{ - \sum_{m=1}^{\infty} \operatorname{Tr}\left(\frac{1}{m} \left[G_n \left(D_0 + \frac{\delta F}{\delta \phi} \right) \right]^m \right) \right\}. \tag{A6}$$

Here G_n is the retarded Green function corresponding to ∂_t^n. Explicitly

$$G_n(t,t') = \frac{(t-t')^{n-1}}{(n-1)!}\Theta(t-t'). \tag{A7}$$

One can easily check that this is a Green function by computing, for $n>1$,

$$\partial_t G_n(t,t') = \frac{(t-t')^{n-2}}{(n-2)!}\Theta(t-t') = G_{n-1}(t,t'), \tag{A8}$$

and noting that

$$\partial_t G_1(t,t') = \delta(t-t'). \tag{A9}$$

Finally, the retarded nature of the Green function is due to the presence of the Heaviside step function.

The traces Tr in the formula for the Jacobian $\mathcal{J}_n$ are spacetime traces. We may write this as $\mathrm{Tr} = \mathrm{tr}_{\mathrm{time}}\,\mathrm{tr}_{\mathrm{space}}$, and concentrate (for now) only on the trace over time $\mathrm{tr}_{\mathrm{time}}$. For $n>1$ it is easy to see that $\mathrm{tr}_{\mathrm{time}}(G_n)=0$, and in fact that for all $m>0$, $\mathrm{tr}_{\mathrm{time}}([G_n]^m)=0$. To generalize this argument to the spacetime trace, we need to make the assumption that $F[\phi(\vec{x},t)]$ does not explicitly contain any time derivatives. If this is the case, we can write

$$\frac{\delta F[\phi(\vec{x},t)]}{\delta\phi(\vec{y},t')} = \delta(t-t')\frac{\delta F[\phi(\vec{x},t)]}{\delta\phi(\vec{y},t)}. \tag{A10}$$

This now implies that for the spacetime trace ($n>1; m>0$),

$$\mathrm{Tr}\left(\left[G_n\left\{D_0 + \frac{\delta F[\phi]}{\delta\phi}\right\}\right]^m\right) = 0. \tag{A11}$$

Thus the retarded nature of the Green function causes all the trace terms to vanish and we have the exact result that for $n>1$ and $F[\phi]$ not containing time derivatives

$$\mathcal{J}_n \equiv \det\left(\partial_t^n - D_0 - \frac{\delta F}{\delta\phi}\right) \tag{A12}$$

$$= \det(\partial_t^n). \tag{A13}$$

This means that the functional determinant is simply a field-independent constant. It is therefore irrelevant and may be discarded. (In particular, for $n=2$, the stochastic wave equation, one never has to evaluate the functional determinant.) [Note: This argument also works provided $n>1+$ (the highest order of time derivatives occurring in $F[\phi]$). Proving this is an easy exercise.] The partition function (characteristic functional) is now, for $n>1$,

$$Z[J] \propto \int (\mathcal{D}\phi)\exp\left[-\frac{1}{2}\int\int (D_n\phi - F[\phi])G_\eta^{-1} \times (D_n\phi - F[\phi])\right]\exp\left(\int J\phi\right). \tag{A14}$$

For $n=1$ the situation is almost as good. First note that

$$(G_1)^2(t,t') = \int d\bar{t}\, G_1(t,\bar{t})G_1(\bar{t},t') \tag{A15}$$

$$= \int d\bar{t}\,\Theta(t-\bar{t})\Theta(\bar{t}-t') \tag{A16}$$

$$= (t-t')\Theta(t-t'). \tag{A17}$$

Thus $\mathrm{tr}([G_1]^2)=0$, and it is easy to show that for $m>1$, $(G_1)^m = G_m$ so that $\mathrm{tr}([G_1]^m)=0$. The only term that survives is $\mathrm{tr}(G_1)=\Theta(0)$. But $\Theta(0)$ is ill-defined and must be specified by some particular prescription. The prescription which is most useful in this context is the symmetric one wherein $\Theta(0)$ is nonzero and equals $\frac{1}{2}$. This may be justified by a limiting procedure as described, for example, in the text by Zinn-Justin [15] (Chap. 4, pp. 69 and 70). This symmetric prescription is equivalent to adopting the *Stratonovich calculus* for stochastic equations. Choosing $\Theta(0)=0$ is equivalent to the *Ito calculus*. The Ito calculus simplifies the Jacobian determinant (to unity) at the cost of destroying equivariance under field redefinitions (the Ito calculus explicitly breaks coordinate invariance in field space). See, for instance, Eyink [56] or Zinn-Justin [15]. We will stick with the symmetric prescription (Stratonovich calculus) for this paper, though suitable modifications for the Ito calculus are straightforward if at times tricky (the loss of reparametrization invariance under field redefinitions implies that all arguments involving a change of variables must be carefully reassessed).

Now for $n=1$ only one of the trace terms in the functional determinant survives and we have (with the assumption that $F[\phi]$ contains no time derivatives)

$$\mathrm{Tr}\left[G_1\frac{\delta F[\phi(x)]}{\delta\phi(y)}\right] = \mathrm{Tr}\left[G_1\frac{\delta F[\phi(\vec{x},t)]}{\delta\phi(\vec{y},t)}\right]$$

$$= \Theta(0)\int dt\,\mathrm{tr}_{\mathrm{space}}\left[\frac{\delta F[\phi(\vec{x},t)]}{\delta\phi(\vec{y},t)}\right]$$

$$= \Theta(0)\mathrm{Tr}\left[\frac{\delta F[\phi(\vec{x},t)]}{\delta\phi(\vec{y},t)}\right]. \tag{A18}$$

This implies

$$\mathcal{J}_1 \equiv \det\left(\partial_t - D_0 - \frac{\delta F}{\delta\phi}\right) \tag{A19}$$

$$= \det(\partial_t)\exp\left\{-\Theta(0)\mathrm{Tr}\left[D_0 + \frac{\delta F[\phi(\vec{x})]}{\delta\phi(\vec{y})}\right]\right\} \tag{A20}$$

$$= \det(\partial_t)\exp\{-\tfrac{1}{2}\mathrm{Tr}[D_0]\}\exp\left\{-\frac{1}{2}\mathrm{Tr}\left[\frac{\delta F[\phi(\vec{x})]}{\delta\phi(\vec{y})}\right]\right\}. \tag{A21}$$

The first two factors in the last line are field independent and so may be discarded with the result that

$$\mathcal{J}_1 \propto \exp\left\{ -\frac{1}{2}\mathrm{Tr}\left[\frac{\delta F[\phi(\vec{x})]}{\delta \phi(\vec{y})} \right] \right\}. \tag{A22}$$

The partition function (characteristic functional) is now

$$Z[J] \propto \int (\mathcal{D}\phi)\exp\left[-\frac{1}{2}\int \int (D_1\phi - F[\phi]) \times G_\eta^{-1}(D_1\phi - F[\phi]) \right] \tag{A23}$$

$$\times \exp\left(-\frac{1}{2}\mathrm{Tr}\left[\frac{\delta F[\phi(\vec{x})]}{\delta \phi(\vec{y})} \right] \right) \exp\left(\int J\phi \right). \tag{A24}$$

This means that for stochastic differential equations that are first order in time, the functional determinant must be kept. There are specific choices of the nonlinear driving term $F[\phi]$ that lead to even further simplifications.

2. Jacobian functional determinant for local driving forces

Suppose that $F[\phi(x)]$ is a *local* functional of the field ϕ. This implies that there exists a local function $\mathcal{F}(\phi,\vec{\nabla})$ such that

$$\frac{\delta F[\phi(\vec{x},t)]}{\delta \phi(\vec{y},t')} = \mathcal{F}(\phi(\vec{x},t),\vec{\nabla})\,\delta(t-t')\,\delta(\vec{x}-\vec{y}). \tag{A25}$$

Evaluating the functional determinant now gives

$$\mathcal{J} = \exp\left(-\frac{1}{2}\delta^d(\vec{0})\int dt\, d^d\vec{x}\, \mathcal{F} \right). \tag{A26}$$

Insofar as we trust the formal result $\delta^d(\vec{0})=0$ (see, for example, [15]) we can discard the functional determinant as an irrelevant constant. This formal result is a somewhat contentious issue, and we have found that it is often more useful and safer to either prove that the Jacobian is a field-independent constant, [31] or to carry the Jacobian along for the whole calculation [30].

For more general driving terms $F[\phi]$ one must keep the functional determinant. Nevertheless it is clear that for large classes of stochastic partial differential equations, including many of the most important and interesting cases, the Jacobian can be safely ignored.

For differential operators D that are not of the form D_n discussed above, or driving forces F more complicated than those discussed above, one has to use other means of evaluating the functional determinant.

For the noisy Burgers equation (KPZ system), the functional determinant can be shown to be a field-independent constant that can be discarded. A proof of this will be provided in [31]. For the reaction-diffusion-decay system, on the other hand, we find it more convenient to explicitly keep the Jacobian determinant [30].

3. Jacobian functional determinant via Feynman diagrams

Let us now suppose that $D=D_n$, as discussed in the first section of this Appendix. Then $D(\vec{k},\omega)=(-i\omega)^n - D_0(\vec{k})$ and the ghost propagator is

$$G_{\mathrm{ghost}}(\vec{k},\omega) = \frac{1}{(-i\omega)^n - D_0(\vec{k})}, \tag{A27}$$

where the retarded nature of the Green function now implies that all poles in the ω plane occur in the lower half of this plane. For each ghost loop (assuming $F[\phi]$ contains no time derivatives) we must perform an integral over both frequency and momenta of the type

$$\mathcal{I}_n^m \equiv \int \frac{d\omega\, d^d\vec{k}\, P(\vec{k},\vec{k}_i)}{(2\pi)^{d+1}\prod_{i=1}^{m}\{[-i(\omega-\omega_i)]^n - D_0(\vec{k}-\vec{k}_i)\}}, \tag{A28}$$

where all the poles (in ω) lie in the lower half-plane and the ω_i and $\vec{k}_i$ are linear combinations of the momenta flowing into the ghost loop. [The function $P(\vec{k},\vec{k}_i)$ is some possibly complicated function of the momenta, typically a polynomial, derived from $\delta F/\delta\phi$. There is also a set of external legs (derived from $\delta F/\delta\phi$) attached to each vertex of the ghost loop, but we do not need to know the detailed structure of these vertices to derive the expression above.]

Since all the poles are known to lie in the lower half-plane, the contour of integration can be pushed out to infinity in the upper half-plane via the replacement $\omega \to \omega + i\Lambda(\Lambda > 0)$, without changing the value of the integral. Thus we can write

$$\mathcal{I}_n^m = \int \frac{d\omega\, d^d\vec{k}\, P(\vec{k},\vec{k}_i)}{(2\pi)^{d+1}\prod_{i=1}^{m}\{[-i(\omega-\omega_i)-\Lambda]^n - D_0(\vec{k}-\vec{k}_i)\}} \quad \forall\ \Lambda>0\,. \tag{A29}$$

Now take the limit $\Lambda \to +\infty$ to deduce $\mathcal{I}_n{}^m = 0$.

The only place that this argument fails is when the ω integral does not converge. This happens only for $n=1$ (first order in time) and $m=1$ (tadpole diagram), in which case we need to consider

$$\mathcal{I}_1^1 \equiv \int \frac{d\omega\, d^d\vec{k}\, P(\vec{k})}{(2\pi)^{d+1}\{-i\omega - D_0(\vec{k})\}}. \tag{A30}$$

This already reproduces the key results of the preceding section: The functional determinant can be ignored for $n>1$ and for $n=1$ it collapses to a single term. Performing the ω integral for this remaining term, we see

$$\mathcal{I}_1^1 = \int \frac{d^d\vec{k}}{(2\pi)^{d+1}}\, i\ln[-i\omega - D_0(\vec{k})]\Big|_{\omega=-\infty}^{\omega=+\infty} \tag{A31}$$

$$= \int \frac{d^d\vec{k}}{(2\pi)^{d+1}}\, i(i\pi) \tag{A32}$$

$$= -\frac{1}{2}\frac{\int d^d\vec{k}}{(2\pi)^d} \tag{A33}$$

$$= -\frac{1}{2}\delta^d(\vec{0}). \tag{A34}$$

We conclude, then, that the tadpole ghost diagram exactly reproduces the $\exp(-\frac{1}{2}\mathrm{Tr}(\delta F/\delta\phi))$ obtained by other means in the first section of this Appendix. In fact, Faddeev-Popov ghost techniques are in complete agreement with direct calculations of the functional determinant. This ghost-based analysis also makes clear why things are different in QFT. If one uses the Feynman propagator instead of the retarded propagator, there are poles on both sides of the real line and one cannot push the path of integration out to $+i\infty$.

APPENDIX B: EQUATIONS OF MOTION IN THE PRESENCE OF NOISE

1. Effect of adding a small decay term

We start with the zero-loop equations of motion for the SPDE,

$$\left(D^\dagger - \frac{\delta F^\dagger}{\delta\phi}\right)\int g_2^{-1}(D\phi - F[\phi]) = J. \tag{B1}$$

In particular, for zero external source $(J=0)$ any solution of the nonstochastic bare equations of motion, $D\phi - F[\phi]=0$, is also a solution of the zero-loop equations of motion. (Zero loops *almost* correspond to setting the noise amplitude to zero and reducing the SPDE to its nonstochastic analog.) But there is a risk that the zero-loop equations may have *more* solutions than the nonstochastic bare equations. This potential problem arises if the operator $D^\dagger - (\delta F/\delta\phi)^\dagger$ is singular [so that it has a nontrivial null space (kernel)]. If this operator is singular, then there will be many different fields $\phi(\vec{x},t)$ that correspond to a given J, making the whole Legendre transform procedure invalid.

The best way to fix this is to add a small decay term in the system and then take the limit as the decay term vanishes. Specifically, take

$$F[\phi] \rightarrow F[\phi] - \epsilon\phi. \tag{B2}$$

This perturbed system has zero-loop equations given by

$$\left(D^\dagger - \frac{\delta F^\dagger}{\delta\phi} + \epsilon I\right)\int g_2^{-1}(D\phi - F[\phi] + \epsilon\phi) = J. \tag{B3}$$

Even if $D^\dagger - (\delta F/\delta\phi)^\dagger$ is singular, the perturbed operator will not be, and so the perturbed equations of motion will have a unique solution $\phi_{\mathrm{soln}}(J,\epsilon)$. It is appropriate to take the Legendre transform using this unique solution and consider the limit $\epsilon\rightarrow 0$ at the end of the calculation. If $D^\dagger - (\delta F/\delta\phi)^\dagger$ is nonsingular, this does not change anything. If $D^\dagger - (\delta F/\delta\phi)^\dagger$ is singular, this procedure provides a prescription for defining a unique solution to the zero-loop equations.

The above complication is not peculiar to SPDEs and their associated nonquantum field theories; the same sort of behavior also occurs in ordinary QFTs. For example, in QED, $J=0$ corresponds to arbitrary constant electromagnetic field. In order to assure that $J=0$ has the unique solution $F=0$, the easiest thing to do is to add a small photon mass.

2. A vanishing theorem

Suppose $D^\dagger - (\delta F/\delta\phi)^\dagger$ is nonsingular. Then the zero-loop equations of motion for $J=0$ are equivalent to the classical equations $D\phi - F[\phi]=0$. The effective action at zero-loop order, evaluated on solutions of the zero-loop equations of motion, is exactly zero.

In fact, the one-loop effective action evaluated on solutions of the zero-loop equations of motion is also exactly zero. This happens due to the explicit occurrence of $D\phi - F[\phi]$ in the one-loop contribution to the effective action [see Eqs. (63) or (64)], so that for solutions of the zero-loop equations of motion there is an exact cancellation between the Jacobian and the fluctuation operator S_2.

On the other hand, if $D^\dagger - (\delta F/\delta\phi)^\dagger$ is singular, just perturb the system with a small amount of ϵ decay. The previous argument goes through for $\epsilon\neq 0$ [technically as long as ϵ is not an eigenvalue of $D^\dagger - (\delta F/\delta\phi)^\dagger$]. Taking the limit $\epsilon\rightarrow 0$ justifies the extension of the vanishing result to the singular case.

Now consider solutions of the one-loop equations of motion. These one-loop equations of motion are of the form

$$\left(D^\dagger - \frac{\delta F^\dagger}{\delta\phi} + \epsilon I\right)g_2^{-1}(D\phi - F[\phi] + \epsilon\phi) = O(\mathcal{A}), \tag{B4}$$

with the right-hand side being a complicated expression. Nevertheless, we do not need to know exactly what this term is to deduce that evaluated at solutions of these equations of motion $\Gamma[\phi_{\mathrm{soln}}] = 0 + O(\mathcal{A}^2)$.

This vanishing of the effective action at solutions of the one-loop equations of motion provides a useful consistency check on specific calculations. The underlying reason for this vanishing theorem is most easily addressed in the MSR formalism. In fact, it can be shown that

$$\Gamma[\phi;\phi_0] = \tfrac{1}{2}\int\int \{(D_{\mathrm{eff}}\phi - F_{\mathrm{eff}}[\phi])g_2^{-1} \times (D_{\mathrm{eff}}\phi - F_{\mathrm{eff}}[\phi])\} d^d\vec{x}\, dt\, d^d\vec{y}\, dt', \tag{B5}$$

where D_{eff} and F_{eff} are some effective differential operator and effective driving force appropriate to the fully interacting theory.

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