# Decay of unstable equilibrium and nonequilibrium states with inverse probability current taken into account

N. V. Agudov* and A. N. Malakhov
*Radiophysical Department, State University of Nizhny Novgorod, 23 Gagarin Avenue, Nizhny Novgorod 603600, Russia*
(Received 30 March 1999)

We study the causes of noise delayed decay of unstable states in nonlinear dynamic systems within the framework of the overdamped Brownian motion model. For the analysis, we use the exact expressions for the decay times of unstable states, which take into account the inverse probability current in contrast to the well-known mean first passage time method. These expressions are valid for any intensity of fluctuations and for arbitrary potential profiles. The effect of delay is shown to arise under the decay of unstable nonequilibrium states due to the action of two different mechanisms. These mechanisms are caused by the inverse probability current and by the nonlinearity of potential describing an unstable state. [S1063-651X(99)10111-9]

PACS number(s): 02.50.−r, 05.40.−a

## I. INTRODUCTION AND FORMULATION OF THE PROBLEM

The decay of unstable states is related to fundamental phenomena of statistical physics. The characteristic feature of this problem is the crucial role of fluctuations. Some unstable states, namely, unstable equilibrium states or marginal states, can decay only due to the action of fluctuations. This is the main reason for intensive investigation of this problem by many authors [1–22]. In the present paper, our main concern is with the kinetics of the decay. It was assumed (see, e.g., Ref. [11]) that the basic features of the decay of unstable nonequilibrium states can be obtained from macroscopic laws. It means that one can find a deterministic trajectory of the system evolution from the unstable state to a stable one, and, then, the fluctuations are assumed to be an insignificant correction to this macroscopic path (i.e., the fluctuations constitute only a minor perturbation). In addition, it was assumed that the fluctuations can only accelerate the escape from the unstable state [14]. However, in Refs. [16,23–34] it was found that there are systems that may drop out of these rules. In particular, in the systems considered in Refs. [16,23–30] the fluctuations can considerably increase the decay time of unstable and metastable states. These are the effects of noise delayed decay (NDD) of unstable states and noise enhanced stability (NES) of metastable states, depending on the parameters of the system considered. The NES effect was obtained in Refs. [23–27] for periodically modulated metastable nonlinear systems. The modulation was so intense [26,27] that the system was unstable in a short interval of time of the period of the driving force and metastable in the remaining time interval. In accordance with Refs. [26] and [27], the NES effect implies that the stability of metastable state can be increased by the fluctuations. In Refs. [16,28–30] and in the present paper, the unstable states without periodical driving are investigated. The NDD effect shows that fluctuations can delay the decay of purely unstable states. The effect of NDD is very similar to that of NES but the name NDD must be used when the state is unstable, because one cannot consider the ‘‘stability’’ of an unstable state.

Note, that the effect of partial or even full stabilization of unstable states by noise was known before. In Refs. [35] and [36], it was shown that in this case the fluctuations must be parametric (multiplicative), but in Refs. [16,23–30] and in the present paper, only additive fluctuations are considered. It turns out that the additive fluctuations can delay the decay of unstable states too. These results are in a contradiction with some usual notions on the decay of unstable states under the action of additive fluctuations. Therefore, these concepts must be corrected and supplemented, which is the main aim of this paper.

One may see that the NDD and NES are similar to the phenomena of stochastic resonance, since in all these cases one obtains a system response which has the resonant dependence on the noise intensity. However, the difference is in the nature of response considered: In the case of NDD and NES the response is the decay time of unstable and metastable states, while in the case of stochastic resonance it is the signal to noise ratio.

The commonly accepted and simplest model of the decay of unstable states is the model of one-dimensional overdamped Brownian motion in the potential field of force [1,9]:

$$\frac{dx}{dt} = -\frac{d\Phi(x)}{\eta dx} + \xi(t). \tag{1}$$

Here, $x$ is the representative phase point denoting the state of the system, $\Phi(x)$ is the potential describing the system itself, $\xi(t)$ is the white Gaussian noise, $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t+\tau) \rangle = 2q\delta(\tau)/\eta$, $2q/\eta$ is the intensity of fluctuations, $\eta$ is the coefficient of equivalent viscosity, and $q$ is the energy temperature of fluctuations. In the case of thermal fluctuations, $q = kT$.

Let the potential be as follows:

$$\Phi(x) = -ax^2/2. \tag{2}$$

*Present address: Dipartimento di Energetica ed Applicazioni di Fisica, Universita di Palermo, viale delle Scienze, 90128, Palermo, Italy.

 

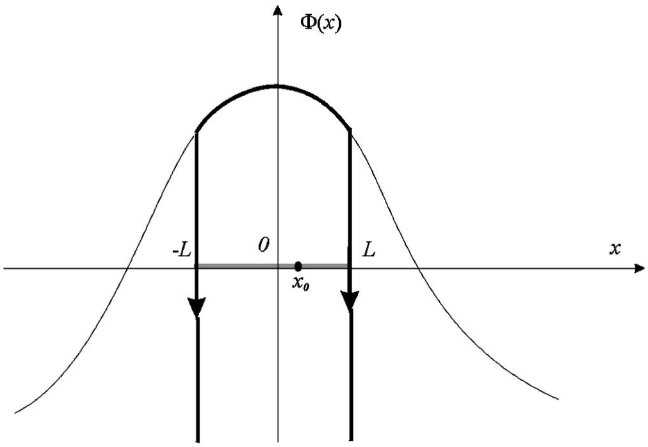


FIG. 1. Two potential profiles describing the same unstale state: potential profile with two absorbing boundaries used in MFPTM (thick curve) and the real potential profile (thin curve).

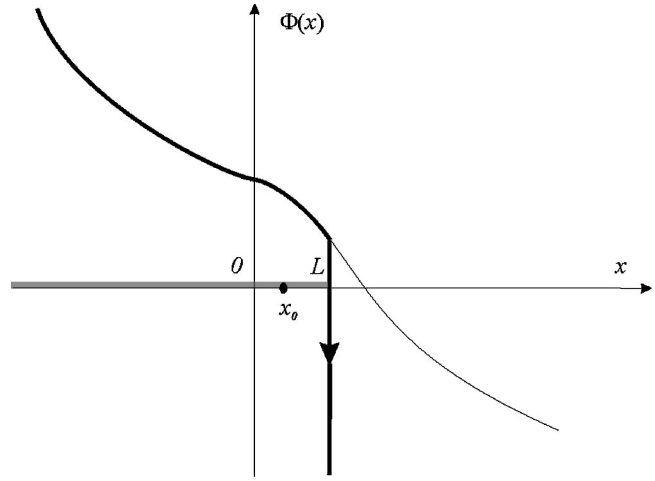


FIG. 2. Potential profile with one absorbing boundary used in MFPTM (thick curve) and the real potential profile (thin curve).

One usually considers this potential profile when analyzing the decay of unstable states [4–11]. In this case, the unstable equilibrium state is located at the top of the parabola ($x=0$), and the unstable nonequilibrium states are at the points $x\neq 0$. In the initial instant, let the phase point be located in the locally horizontal part of the parabolic potential profile: $x_0=0$. Due to the action of fluctuations, the point begins to move randomly. When the point is shifted by fluctuations from the top of the parabola, it is subjected to the action of the regular force and goes away from the unstable state down the parabola. It is the so-called effect of enhancement of fluctuations.

Thus, in the case of a small intensity of fluctuations, the decay process can be separated into two stages (see, e.g., Ref. [4]): the first stage is the movement under the action of the random force near the top, and the second stage is the drift under the action of the regular force on the slope of the parabola. In the first stage (unstable region), the influence of the regular force is insignificant. In the second stage (extensive region), one can neglect the random force. This concept is used in Refs. [4–9] to obtain the approximate expression (valid under $q\rightarrow 0$) for the decay time of the unstable equilibrium states (the scaling method). According to this method, if, initially, the representative point was located on the slope of the parabola ($x_0\neq 0$), then its further motion is defined mainly by the regular force, and the action of fluctuations is insignificant. At the same time, it is shown in Ref. [30] that it is just the case ($x_0\neq 0$), when the fluctuations can delay the decay of the unstable state. It means that the influence of fluctuations on the system in the extensive region is significant, and there are cases when one can not ignore it. Consequently, the scaling methods are not valid in each case. Therefore, their application range must be refined.

On the other hand, the results obtained in Refs. [5–7,30] are also approximate because they are restricted by the mean first passage time method (MFPTM), which requires the use of the absorbing boundaries. Let us explain this situation by an example. Consider a dynamic system described by an arbitrary potential profile similar to that depicted in Fig. 1. The unstable state of the system is near $x=0$. Let the decay time of the unstable state be the escape time from the given decision interval $R$: $[-L;L]$, $x_0\in R$. When one uses the MFPTM, the absorbing boundaries must be located in the ends of the decision interval $x=\pm L$, i.e., the real potential profile is replaced by another one (see the thick curve in Fig. 1).

The potential profile describing the unstable state may have a different shape, similar to that depicted, for example, in Fig. 2. In this case, to analyze the decay time, one usually considers the following decision interval $R$: $[-\infty;L]$. Evidently, in this case the MFPTM (the setup of the absorbing boundary) distorts the real potential profile too.

Therefore the following question arises: How will the effect of the NDD be changed, if the absorbing boundaries are absent? In this case the representative points can return into the decision interval after they have left it. In other words, an inverse probability current appears. Thus, to answer this question, we need to take into account this inwardly directed inverse probability current, which is neglected by the MFPTM.

To do this we use the new method proposed in Ref. [37]. This method allows one to obtain the exact expressions (which are valid for any intensity of fluctuations and for an arbitrary potential profile) for decay times of unstable states with the inverse probability current taken into account. In the present paper the analysis and comparison of the exact decay times with the approximate ones proposed by other authors are presented. Various unstable states described by polynomial potential profiles are considered.

The decay time of an unstable state is defined as follows: Let us consider the probability $P(t)$ for the representative phase point to fall within the given decision interval $R$ (see Figs. 1 and 2). Initially the point always falls within the interval $R$ and we have: $P(0)=1$. With time, the survival probability $P(t)$ decreases to some equilibrium value $P(\infty)=P_{eq}$. The value $P_{eq}$ depends on the type of the potential profile. One can distinguish two types of the potential profiles:

1. The potential profile $\Phi(x)\rightarrow+\infty$ with $x\rightarrow\pm\infty$ (See Fig. 3). In this case, at $t\rightarrow\infty$, the following equilibrium Boltzmann distribution is established in the system:

$$W_{eq}(x)=N\exp\left(-\frac{\Phi(x)}{q}\right),$$

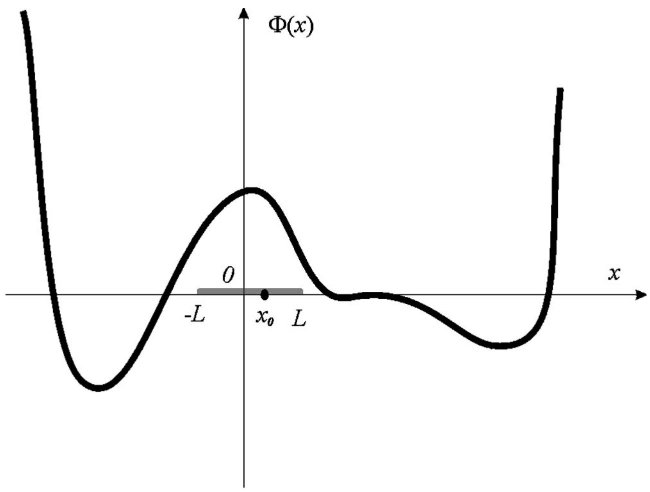


FIG. 3. A sketch of the potential profile of type 1.

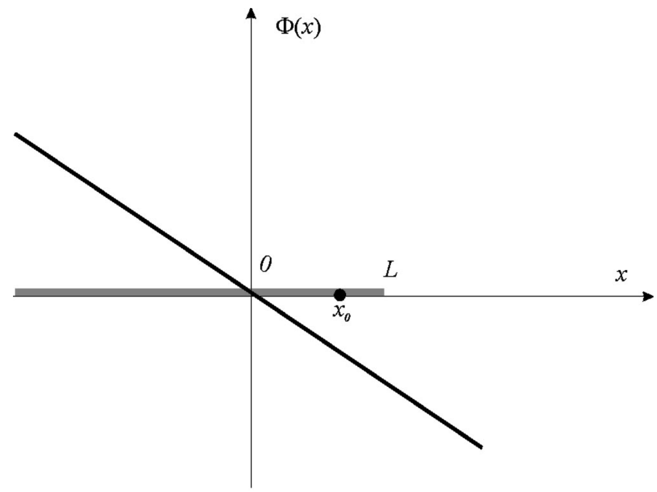


FIG. 4. Linear potential profile (4).

where $N$ is the normalization factor. Therefore, the equilibrium value $P_{eq}$ in the decision interval is

$$P_{eq}=N\int_R W_{eq}(x)dx.$$

2. The potential profile $\Phi(x)\to-\infty$ with $x\to\infty$, or with $x\to-\infty$, or with $x\to\pm\infty$ (See Figs. 1,2). In this case, $W_{eq}(x)=0$ and, consequently, $P_{eq}=0$.

Thus, under any type of potential profiles, the survival probability decreases from $P(0)=1$ to $P(\infty)=P_{eq}$, where $0\leq P_{eq}<1$. We define the decay time $\tau$ of unstable states as the relaxation time of the survival probability to the equilibrium value:

$$\tau=\frac{1}{1-P_{eq}}\int_0^\infty (P(t)-P_{eq})dt. \tag{3}$$

If the absorbing boundaries are given at the ends of the decision interval $R$, then the decay time (3) coincides with that obtained by the MFPTM (See Ref. [37]). In all other cases, the expression (3) takes into account the inverse probability current across the boundaries of the decision interval and therefore it differs from the MFPT. Thus, the MFPT is a particular case of the decay time of unstable state (3), namely, the case in which we neglect the inverse probability current.

The idea to take into account the inverse probability current is not new and the various definitions of decay and relaxation times can be used, as it is discussed, for example, in Refs. [14] and [38]. In particular, the decay (or relaxation) time (3) first proposed in Ref. [38] is similar to that considered in Refs. [14,43, and 44], however, it is not exactly the same. Nevertheless, following Refs. [14,43, and 44] further, we call the time (3) the nonlinear relaxation time (NLRT), in order to differ it from the MFPT which does not take into account the inverse current. (The word ''nonlinear'' refers to the fact that this is not a case which involves small fluctuations around equilibrium, where linear response theory would be valid.)

As is mentioned in Ref. [37], the definition (3) is legitimate if the variation of survival probability is sufficiently fast so that the integral converges, and if $P(t)$, during its time behavior, does not intersect the final value $P_{eq}$. In general, the function $P(t)$ does not need to be monotonic; however, for the fulfillment of the last condition, the monotonic variation $P(t)$ is sufficient. The intersection can appear only if $P_{eq}>0$, i.e., for the potential profiles of the first type. The potential of this type is considered in Sec. III, where the problem of intersection and monotony is discussed.

## II. LINEAR SYSTEMS

In order to understand better how the effect of NDD appears, we consider below two linear systems. Let the potential profile in Eq. (1) be a straight line with a given slope [the regular drift force $f(x)=-\Phi'(x)$ is a constant for the entire range of $x$]:

$$\Phi(x)=-kx \tag{4}$$

with unstable state at $x(0)=x_0<L$ (see Fig. 4). We take the following decision interval $R$:$[-\infty;L]$. It follows from Eq. (1) that in the absence of noise $(\xi(t)=0)$ the decay time is equal to

$$\tau(x_0,q=0)=\eta(L-x_0)/k=T_0(x_0). \tag{5}$$

In the presence of fluctuations, this time becomes a random value. Let us consider the decay time of the unstable state, first, as the MFPT of the boundary $L$. In accordance with Refs. [39–41], the MFPT is equal to

$$T(x_0,q)=\frac{\eta}{q}\int_{x_0}^{L} e^{\Phi(v)/q}\int_{-\infty}^{v} e^{-\Phi(u)/q}du\,dv. \tag{6}$$

Using Eq. (6), we obtain

$$T(x_0,q)=\eta(L-x_0)/k=T_0(x_0). \tag{7}$$

Thus, for $\Phi(x)=-kx$, the MFPT coincides with the escape time without noise. Fluctuations, on the average, do not affect the decay time of the unstable state. As is shown in Ref. [30], in order to obtain the dependence of the MFPT on fluctuations, it is necessary to take a nonlinear potential profile. In that case, the fluctuations can delay or speed up the decay of the unstable state depending on the kind of nonlinearity.

Let us take into account that the phase point can cross the point $x=L$ any number of times and in any direction (i.e., we take into account the inverse probability current). To do this, we must remove the absorbing boundary from the end of the decision interval $x=L$. Then, one can obtain the NLRT in several ways: Solving the Fokker-Planck equation for the probability density $W(x,t)$ (see, e.g., Ref. [1]), or for its Laplace transform [38], or using the above-mentioned exact expression in the quadratures obtained in Ref. [37]:

$$\tau(x_0,q)=T(x_0,q)+\frac{\eta}{q}\int_L^\infty e^{\Phi(v)/q}dv\int_{-\infty}^{L}e^{-\Phi(u)/q}du, \tag{8}$$

where $T(x_0,q)$ is the MFPT (6). Evidently, any of these ways leads to the following result:

$$\tau(x_0,q)=T_0(x_0)\left(1+\frac{q}{kL}\right)\geqslant T_0(x_0). \tag{9}$$

It follows from Eq. (9), that if we take into account the inverse probability current, the fluctuations will always delay the decay of the unstable state: the greater the fluctuations, the greater the NLRT, i.e., the effect of the NDD takes place.

Note, that for the considered potential the NDD appears for the NLRT, while it does not appear for the MFPT. It means that the first mechanism of NDD appearance, which was considered in Ref. [30] (where only the MFPTM were used), does not function here. In this case, the effect of the NDD appears because of the second mechanism: The inverse probability current, which is not taken into account by the MFPTM. Thus, in the general case, the NDD can appear due to the action of at least two mechanisms: First, caused by the nonlinearity of the potential profile and, second, caused by the inverse probability current. This example also shows that the MFPTM sometimes cannot reflect the real situation.

It was mentioned in Ref. [14] that one can neglect the inverse probability current, if decay times are less than the typical time required to reach the boundary of the decision interval from the absolute minimum of the potential. It follows from the above example that it is not true. Indeed, in this example the absolute minimum of the potential is infinitely far from $x_0$ at $x\to\infty$ and the return time of the phase point from the minimum is infinitely long, i.e., the decay time is always less. Nevertheless, we see that the inverse probability current caused by the fluctuations is so great that the effect of NDD appears. Further we show that the same situation takes place for many other linear and nonlinear systems described by various potential profiles. This is because the value of the inverse current is defined not only by the neighborhood of the absolute minimum but also by the shape of the entire part of the potential profile, which is beyond the decision interval. It is the inverse current that provides the delay of the unstable state decay regardless of the location of the potential minimum.

Let us return now to the classical view of the linear unstable system described by the parabolic potential profile (2) (Fig. 5). First we consider the case $x_0=0$ (unstable equilibrium state). As was mentioned above, this case was studied by many authors [1–9]. In particular, in Ref. [5] the MFPTM was used and the absorbing boundaries were arranged at $x=\pm L$. For the symmetric potential profile $\Phi(-x)=\Phi(x)$, the MFPT is equal to [2,39]

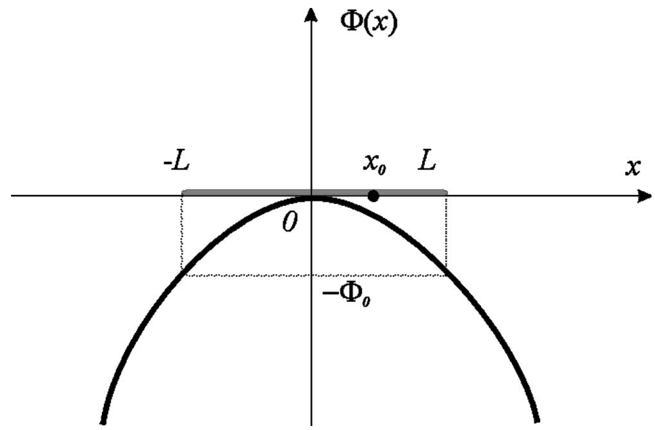


FIG. 5. Parabolic potential profile (2).

$$T(x_0,q)=\frac{\eta}{q}\int_{x_0}^{L}e^{\Phi(v)/q}\int_0^v e^{-\Phi(u)/q}du\,dv. \tag{10}$$

In Ref. [5], using scaling methods, the asymptotical expression for the MFPT (10) was obtained for $q\to 0$, when initially the representative point is exactly in the unstable equilibrium state $x_0=0$:

$$\tau_{as}=T_s\left[\ln\frac{1}{\sigma}-\psi\left(\frac{1}{2}\right)\right], \tag{11}$$

where $T_s=\eta/2a$ is the characteristic time of the system, $\Phi_0=aL^2/2$ is the value of potential at the point $x=L$ (See Fig. 5), $\sigma=q/\Phi_0$ is the dimentionless temperature of fluctuations, and $\psi(x)$ is the digamma function. In Ref. [5] it was assumed that for $q\to 0$ the MFPT must coincide with the NLRT, because under the small fluctuations the inverse probability current becomes negligible.

Now we can compare these results with the exact expression for the nonlinear relaxation time. From the results presented in Ref. [37], one obtains the NLRT for the symmetric potential of the second type (see Sec. I) as follows:

$$\tau(x_0,q)=T(x_0,q)+\frac{\eta}{q}\int_L^\infty e^{\Phi(v)/q}dv\int_0^L e^{-\Phi(u)/q}du, \tag{12}$$

where $T(x_0,q)$ is the MFPT (10). The plots $\tau(0,q)$, $T(0,q)$, and $\tau_{as}(q)$ calculated in accordance with Eqs. (12), (10), and (11) for the potential (2) are depicted in Fig. 6. One can easily see that these time scales coincide for $q\to 0$. This confirms the assumptions used in Ref. [5]. As in the previous example, the NLRT becomes greater than the MFPT with the fluctuations. The expression for the NLRT (12) differs from that for the MFPT by the second term, which tends to zero as $q\to 0$ and rises with $q$ for potential (2). Thus, the inverse probability current always enhances the NLRT compared with the MFPT for $q>0$. Let us note that the asymptotic expression (11) gives a rather good approximation for $\sigma$

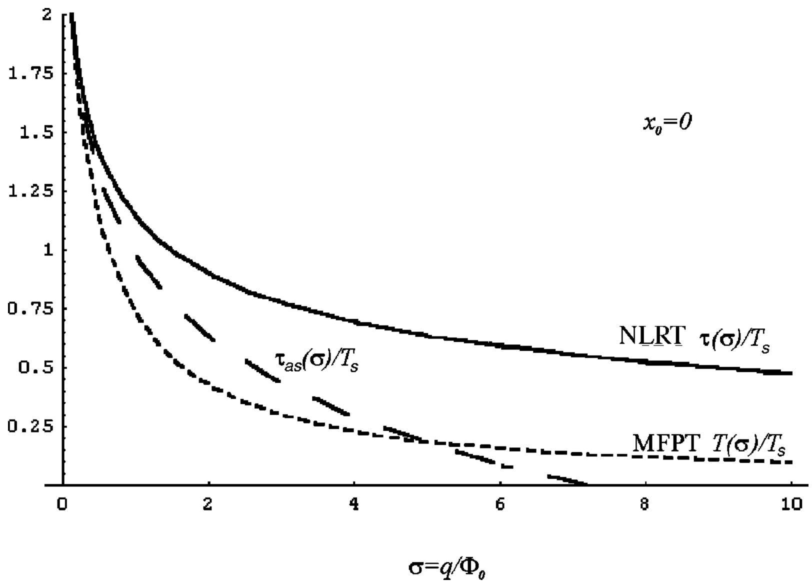


FIG. 6. Dimensionless NLRT, MFPT, and asymptotical decay time, versus the dimensionless temperature for the unstable equilibrium state described by parabolic potential (2) at $x_0=0$ (see Fig. 5).

$<0.5$. On the other hand, it can be shown that for $\sigma>3$ the NLRT given by Eq. (12) is represented as

$$\tau(0,q)=T_s\left(\sqrt{2\pi}\sigma^{-1/2}-\sigma^{-1}+\frac{1}{6}\sqrt{2\pi}\sigma^{-3/2}+\cdots\right). \tag{13}$$

Let us assume that the initial states are shifted from the top: $x_0\neq 0$ (unstable nonequilibrium state). The expressions for the MFPT (10) and for the NLRT (12) remain the same. The asymptotical expression (11) does not function in this case. The dependencies of the NLRT on the fluctuation temperature $q$ for $x_0/L=0.8$ (solid curve) and the MFPT (dashed curve) are presented in Fig. 7. First, one can see from Fig. 7 that in the case $x_0\neq 0$ the effect of the NDD appears for both time scales. Second, the inverse probability current enhances the NLRT compared with the MFPT, as for the case $x_0=0$. For the MFPT, the maximal magnification by noise is about 25% above its value without noise, while for the NLRT the magnification is greater than 250%. Note, that the dependence of the MFPT on the fluctuation intensity was studied earlier in Ref. [30].

As we already noted, the above Eq. (12) differs from Eq. (10) by the second term. This term is independent of $x_0$. It means that the conclusion made in Ref. [30] relative to the dependence of the MFPT on $x_0$ is also true for the NLRT: The maximum value of the function $\Theta(q)=\tau(x_0,q)/\tau(x_0,0)$ increases as $x_0$ approaches $L$.

Thus, if the representative phase point of the dynamic system is in the unstable nonequilibrium state, the effect of the NDD can be so significant that the NLRT can be many times greater than in a purely dynamic case without fluctuations. The expression for the decay time obtained by the scaling methods (11) is valid only for the unstable equilibrium state ($x_0=0$).

### III. SATURATION EFFECT. NONLINEAR SYSTEMS

Now we generalize the potential (2) including the saturation term (Fig. 8):

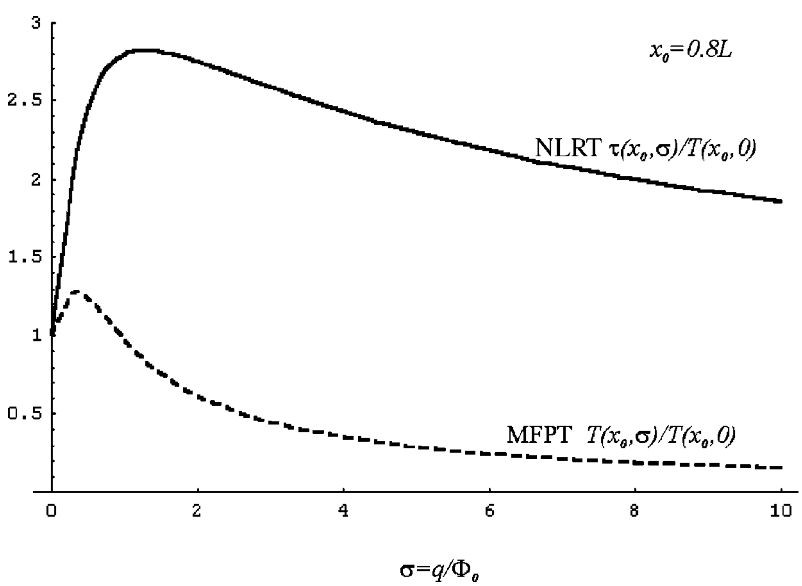


FIG. 7. Dimensionless NLRT and MFPT versus the dimensionless temperature for the unstable nonequilibrium state described by the parabolic potential (2) at $x_0=0.8L$ (see Fig. 5).

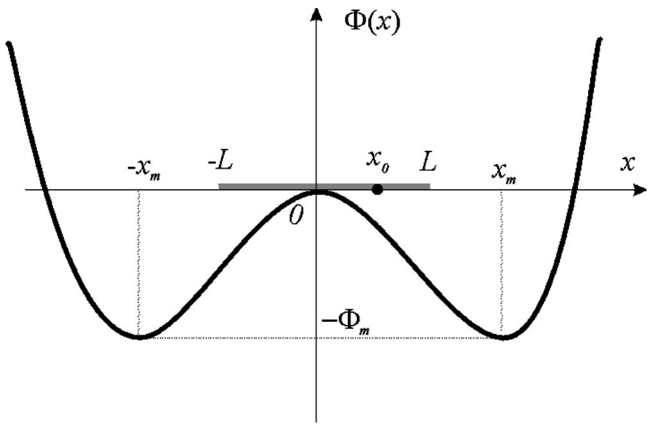


FIG. 8. Potential profile (14) describing the unstable system with saturation.

$$\Phi(x)=-\frac{1}{2}ax^2+\frac{1}{4}bx^4. \tag{14}$$

The unstable equilibrium state is still at the point $x=0$. The global equilibrium states of the system now are at points $x_m=\pm\sqrt{a/b}(\Phi(x_m)=-a^2/4b)$ but not at $x\to\pm\infty$ as it was previously for linear systems. Therefore, the nonequilibrium unstable states are everywhere except these three points. From the general expression presented in Ref. [37], it is obvious that in the case of the symmetrical potential $(\Phi(-x)=\Phi(x))$, the exact expression for the NLRT defined by Eq. (3) is as follows:

$$\tau(x_0,q)=T(x_0,q)-\frac{\eta}{q}\frac{C_2(q)}{C_1(q)}, \tag{15}$$

where

$$C_2(q)=\int_0^L e^{\Phi(v)/q}f^2(v)dv-\frac{P_{eq}}{1-P_{eq}}\int_L^\infty e^{\Phi(v)/q}(1-f(v))^2dv,$$

$$C_1^{-1}=\int_0^\infty e^{-\Phi(v)/q}dv, \quad f(x)=C_1\int_0^x e^{-\Phi(v)/q}dv,$$

$$P_{eq}=f(L),$$

where $\pm L$ are the boundaries of the decision interval $R$: $[-L;L]$, $x_0\in R$, and $T(x_0,q)$ is the MFPT (10), which, as in the linear case, represents the decay time approximately at $q\to0$. In addition, in Ref. [5] for $q\to0$ and $x_0=0$, the following asymptotical expression for the MFPT was obtained:

$$\tau_{as}=T_s\left(\ln\frac{2M^2\Phi_m}{(1-M^2)q}-\psi\left(\frac{1}{2}\right)\right), \tag{16}$$

where $M=L/x_m$, $\Phi_m=-\Phi(x_m)$ is the depth of potential wells. The plots $\tau(0,q)$, $T(0,q)$, and $\tau_{as}(q)$ are presented in Fig. 9. As expected, all these three times coincide for $q\ll\Phi(x_m)$. For $q>\Phi(x_m)$, the asymptotic formulas (16) is not correct. As it follows from Fig. 9, the NLRT is always more than the MFPT, since it takes into account the inverse probability current, which retards the escape of representative points from the decision interval.

Now let us consider the case $x_0\neq0$. The plots $\tau(x_0,q)$ and $T(x_0,q)$ for $x_0=0.8L$ and $M=0.5$ are presented in Fig. 10. As in the linear case, if $x_0\neq0$, the NDD effect appears. Comparing the plots in Fig. 10 and Fig. 7, one can conclude that the influence of the saturation term in Eq. (14) on the NDD phenomena is insignificant under small $q$. At large $q$, the NLRT for the system with saturation becomes less than the NLRT for the system without saturation. It can be explained as follows: The survival probability for the system with saturation varies with time from $P(0)=1$ to $P(\infty)=P_{eq}>0$, but not from 1 to 0, as it was for the system without saturation (2). Therefore, under the fluctuation temperature of the order of the depth of potential wells and greater $(q\geqslant\Phi(x_m))$, the equilibrium [defined by the evolution of $P(t)$] is attained earlier for the system with saturation, since in this case $P_{eq}$ is sufficiently large. That is why the NLRT for the system with saturation is smaller. On the other hand, it is nesessary to take into account that the NLRT defined by Eq. (3) can give rise to a wrong result if the function $P(t)$ is nonmonotonic and intersects the value $P_{eq}$. In this case, the real NLRT can be only greater than that defined by Eq. (3) and obtained from Eq. (15). Consequently, the real NDD effect can be greater. We do not know the function $P(t)$, because our method allows us to derive only the integral of it. Therefore, the monotony condition for this case must be verified separately elsewhere.

Let us now refer to another case of potentials (Fig. 11):

$$\Phi(x)=-kx-\frac{b}{2n+1}x^{2n+1}, \quad n=1,2,3,\ldots. \tag{17}$$

All the states of that system are nonequilibrium if $k\geqslant0$, $b>0$. The global equilibrium state is only at $x\to\infty$. In Ref. [7], the exact and approximate expressions for the MFPT were investigated for $k=0$ when $x_0\leqslant0$. As in the above cases, the NLRT and the MFPT for potential (17) coincide for $q\to0$ if the initial state of the system is that of unstable equilibrium (i.e., if $x_0=0$ and $k=0$). The asymptotical expressions are known [7]. Recently, in Ref. [30] it was shown that the NDD appears for these potentials as well. In accordance with Ref. [30], it takes place if $k=0$ and $0<x_0<L$ or if $k>0$ and $-L<x_0<L$. However, in Ref. [30] the decay time was obtained by the MFPTM, i.e., by an approximate method, which does not take into account the inverse probability current.

Now we can obtain the exact NLRT, using the abovementioned results (8) of Ref. [37]. Let $x_0\geqslant0$ and the decision interval be $R$: $[-\infty;L]$. In the absence of fluctuations, the ''dynamic'' decay time is equal to

$$\tau_0=\int_{x_0}^L\frac{\eta dx}{F'(x)}, \tag{18}$$

where $F(x)=-\Phi(x)$.

To analyze the influence of fluctuations on the nonlinear relaxation time, let us expand the NLRT (8) into a power series in $q$. To fulfill this program, we need to find an as-

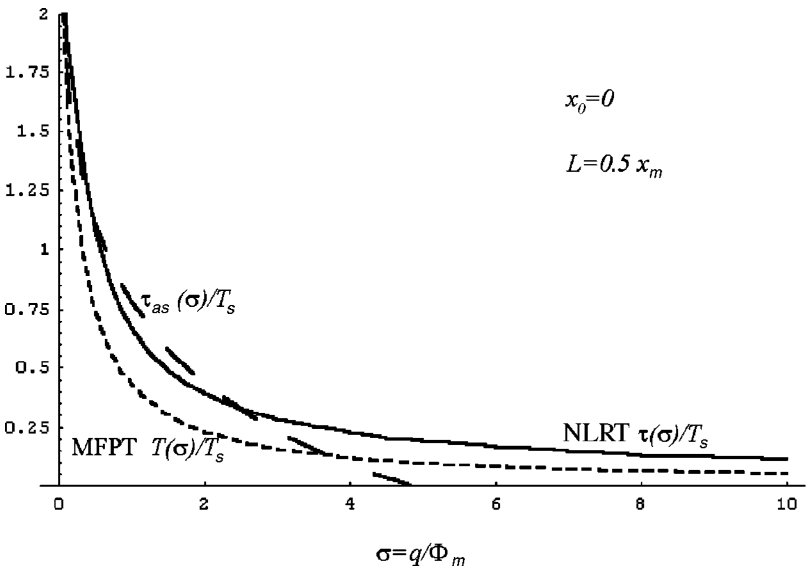


FIG. 9. Dimensionless NLRT, MFPT, and asymptotical decay time versus the dimensionless temperature for the unstable equilibrium state described by potential (14) at $x_0=0$ (see Fig. 8).

ymptotical expansion of the typical integrals involved in the general formula (8). Let us present the typical indefinite integral as follows:

$$\int^{u} e^{F(v)/q} dv = e^{F(u)/q} K(u,q), \tag{19}$$

where $F(v)>0$, $F'(0)\neq 0$, the small parameter $q$ has any sign, and $K(u,q)$ is the unknown function, which is to be found for small $q$. Differentiating Eq. (19) with respect to $u$, one can easily find that

$$K'_u(u,q) = 1 - \frac{K(u,q)}{qz(u)}, \tag{20}$$

where $z(u)=1/F'(u)$. We assume that for small $q$, the right-hand side of Eq. (20) is also small. Then, using Eq. (20) as an iterative equation, as the first approximation, we obtain

$$K(u,q) = qz(u).$$

One can easily find the second approximation, substituting this value into the left-hand side of Eq. (20):

$$K(u,q) = qz - q^2 zz',$$

etc. It is evident that in the obtained equations the derivatives of all approximations tend to zero at $q\to 0$. It confirms the initial assumption.

As a result, we find the following expansion of the sought function $K(u,q)$ into the power series in $q$:

$$K(u,q) = qz\{1 - qz' + q^2(zz')' - q^3(z(zz')')' + q^4[z(z(zz')')']' + \cdots\}. \tag{21}$$

By doing so, for $A,B\geqslant 0$ and for small $q$ we have

$$\int_A^B e^{F(v)/q} dv = e^{F(B)/q} K(B,q) - e^{F(A)/q} K(A,q). \tag{22}$$

The expressions obtained are valid if $z(u)=1/F'(u)$ and all derivatives of $z(u)$ do not tend to infinity, i.e., if the derivative $F'(u)$ does not tend to zero in the integration interval. For the considered potential (17), this refers to the condition $k>0$.

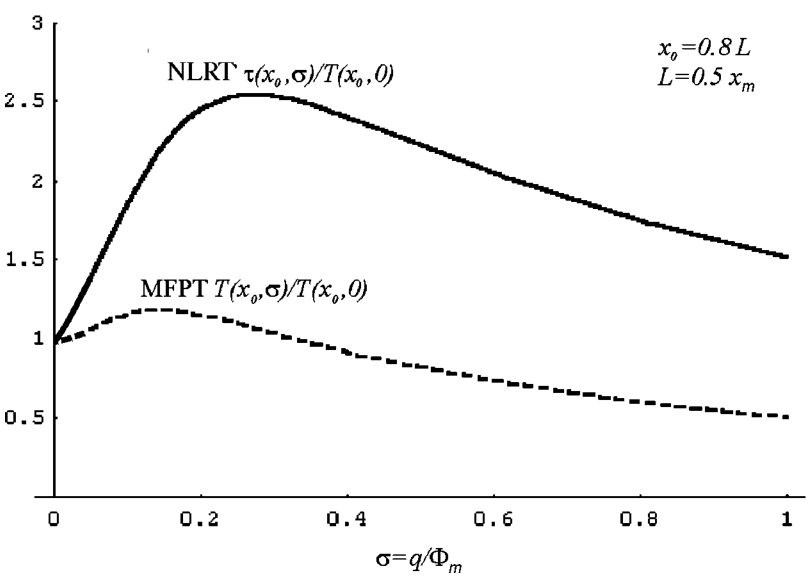


FIG. 10. Dimensionless NLRT and MFPT versus the dimensionless temperature for the unstable nonequilibrium state described by the potential (14) at $x_0=0.8L$ (see Fig. 8).

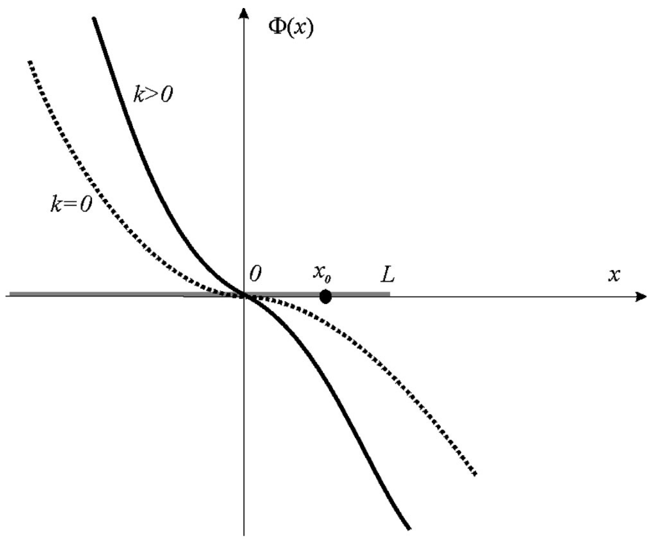


FIG. 11. Potential profile (30) describing the unstable state considered in Ref. [14].

Doing so, one can easily show that $(F(v)>0, L\geqslant 0)$ for small $q>0$,

$$\int_L^\infty e^{-F(v)/q}dv=-e^{-F(L)/q}K(L,-q), \tag{23}$$

where

$$K(L,-q)=-qz_L[1+qz_L'+q^2(z_Lz_L')'+q^3(z_L(z_Lz_L')')' +\cdots], \tag{24}$$

and $z_L=z(L)=1/F'(L)$, $z_L'=[(d/du)z(u)]_{u=L}$, etc.

Substituting the obtained Eqs. (22) and (23) into Eq. (8), we find that for small $q$ and $x_0>0$,

$$\int_{x_0}^L e^{\varphi(v)}dv\int_{-\infty}^v e^{-\varphi(u)}du=\int_{x_0}^L K(v,q)dv,$$

$$\int_L^\infty e^{\varphi(v)}dv\int_{-\infty}^L e^{-\varphi(u)}du=-K(L,-q)K(L,q).$$

Thus, for $x_0>0$

$$\tau=\frac{\eta}{q}\int_{x_0}^L K(v,q)dv-\frac{\eta}{q}K(L,-q)K(L,q). \tag{25}$$

Expanding function $K$ in accordance with Eqs. (21) and (24), and performing the partial differentiation, we find the following expansion of the NLRT into the power series in $q$ for $0<x_0<L$:

$$\tau(x_0,q)=\eta\int_{x_0}^L z(u)du+\tfrac{1}{2}\eta q[A_1(x_0)+A_1(L)] +\eta q^2\int_{x_0}^L z(zz')'du+\tfrac{1}{2}\eta q^3[A_3(x_0)+A_3(L)] +\eta q^4\int_{x_0}^L z[z(z(zz')')']'du+\tfrac{1}{2}\eta q^5[A_5(x_0) +A_5(L)]+\cdots, \tag{26}$$

where

$$A_1(u)=z^2, A_3(u)=z^2[-(z')^2+2(zz')'],$$

$$A_5(u)=z^2\{2[z(z(zz')')']'-2z'(z(zz')')'+((zz')')^2\}, \tag{27}$$

$$z=z(u)=1/F'(u).$$

One can easily see that the first term coincides with the ‘‘dynamic’’ decay time (18) as it must be. The second term is proportional to $q$ and is always positive. This indicates a rise of the NLRT $\tau(q)$ with $q$ from the point $\tau=\tau_0$ (for $q=0$). Thus, if the first derivative of the potential $\Phi'(u)$ is negative and is not zero, the fluctuations acting in dynamic systems *always increase* the decay time of the unstable state under small intensities. The linear system (4) considered above corresponds to the particular case of expression (26). Indeed, substituting Eq. (4) into Eqs. (26) and (27), we obtain the series with two terms, Eq. (9). For the nonlinear potential profiles the series (26) is infinite.

If $x_0=0$, the expansion of the NLRT into the power series in $q$ is as follows:

$$\tau=\eta\int_0^L z(u)du+\tfrac{1}{2}\eta q[A_1(L)+A_1(0)] +\eta q^2\left[\int_0^L z(zz')'du+B_2(0)\right]+\tfrac{1}{2}\eta q^3[A_3(L)-A_3(0) +2B_3(0)]+\eta q^4\left[\int_0^L z[z(z(zz')')']'du+B_4(0)\right]+\cdots. \tag{28}$$

Here, the functions $A_k(u)$ are defined by Eq. (27), and the functions $B_k(u)$, are equal to

$$B_2(u)=2z^2z',\quad B_3(u)=z^2[2(zz')'+(z')^2],$$

$$B_4(u)=z^2[2(z(zz')')'+2z'(zz')'], \tag{29}$$

$$z=1/F'(u).$$

Thus, when $x_0=0$, the second term in Eq. (28) is the same as in the case $x_0>0$, i.e., it is positive and the NLRT increases for small $q$.

The series for the NLRT in $q$ for some different and more general cases of potential $\Phi(x)$ is obtained in Ref. [42].

The decay times of unstable equilibrium and unstable nonequilibrium states described by potential (17) for $n=1$,

$$\Phi(x)=-kx-\frac{1}{3}bx^3, \tag{30}$$

were considered in Ref. [14] in detail, and the NDD effect was not revealed. Therefore, we consider this case again and show that the decay time of the unstable state can increase with noise. Let $k>0$ and $b>0$. As is mentioned in Ref. [14], this case corresponds to the majority of experimental results. It follows from the general expression (28) that for $x_0=0$

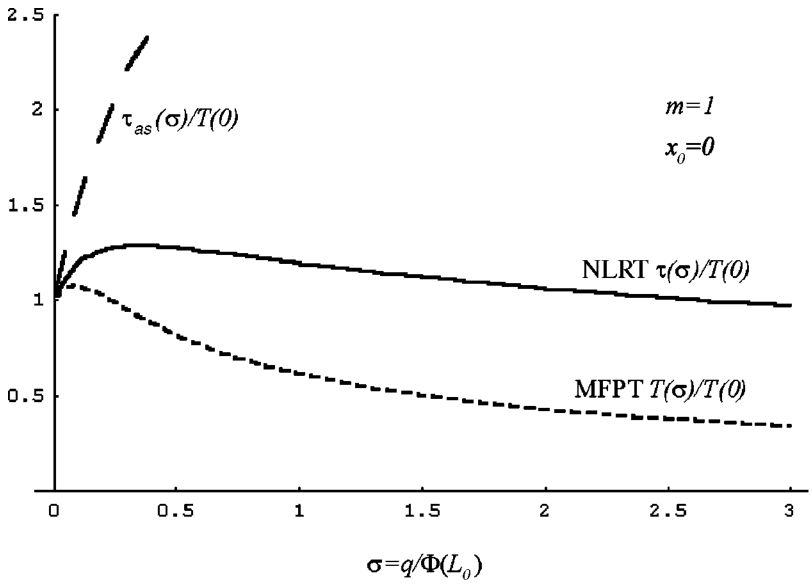


FIG. 12. Dimensionless NLRT and MFPT versus the dimensionless temperature for the unstable nonequilibrium state, described by the potential (30) at $x_0=0$, $k>0$, and $m=1$ (see Fig. 11).

$$\tau=\frac{1}{\sqrt{ab}}[\arctan m+\sigma E_1(m)-\sigma^2 E_2(m)+0(\sigma^3)], \tag{31}$$

where

$$E_1(m)=1+1/(1+m^2)^2,$$

$$E_2(m)=\frac{5}{32}\arctan m+\frac{5}{32}\frac{m}{1+m^2}+\frac{5}{48}\frac{m}{(1+m^2)^2}+\frac{1}{12}\frac{m}{(1+m^2)^3}+\frac{3}{2}\frac{m}{(1+m^2)^4}$$

$$m=\sqrt{3}L/L_0,\quad \sigma=2\sqrt{3}q/\Phi(L_0),\quad L_0=\sqrt{3k/b}.$$

The quantity $L_0$ characterizes the relative influence of the linear and nonlinear terms in the expression for potential profile (30). At the point $x=L_0$, these terms are equal. The plots of the NLRT (8), the MFPT (6), and the approximate expression (31) are presented in Fig. 12 versus the fluctuation temperature $q$. Thus, one can see that the NDD appears in this case as well and it is the main correction to the analysis carried out in Ref. [14] where the authors contend the reverse, namely, that for $k>0$ the fluctuations will accelerate the decay process.

The asymptotical formula (31) at $m\ll 1$ (in this case $E_2\approx 2m$) allows us to estimate the temperature $q_{max}$ for which the NLRT is maximal:

$$q_{max}\approx\frac{1}{24}\frac{L_0}{L}\Phi(L_0).$$

The expression (31) itself at $q\sim q_{max}$ gives a greater value than the exact Eq. (8). Evidently, it is because we have taken into account only the three first terms of the infinite series (26). Estimation of the decay time by the MFPTM leads, as in the above examples, to a reduced quantity, because it does not take into account the inverse probability current.

The analysis of expressions (6) and (8) shows that the difference between the MFPT and the NLRT increases with a decrease in parameter $m$. The dimensionless parameter $m$ characterizes the relative value of the nonlinear term compared with the linear one inside the decision interval for $x>0$. The decrease in $m$ means that the potential profile becomes closer to the linear one within the decision interval. In the purely linear case ($m=0$), the difference between the MFPT and the NLRT is maximal: the NLRT coincides with Eq. (9) and increases linearly with $q$, while the the MFPT is a constant value equal to $\eta L/k$. As $m\to\infty$, the NLRT becomes closer to the MFPT. The NDD also appears in this case, however, it is smaller than for $m\to 0$. Finally, for $m=\infty$ ($k=0$), when the considered unstable nonequilibrium state becomes equilibrium (the decay time without noise is infinite), the NDD disappears. As was mentioned above, it is this case which is well studied in the literature (see, e.g., Ref. [7]). However, if for $k=0$ we take $x_0>0$, then the NDD appears again and increases as the distance from $x_0$ to the boundary $L$ of the decision interval decreases.

The above dependence of the NDD on the parameter $m$ is explained by the fact that the noise delayed decay appears due to the action of two different mechanisms. The first mechanism (which was studied in detail in Ref. [30]) is caused by the nonlinearity of the potential profile. The second one is caused by the inverse probability current across the boundary $L$. The first mechanism leads to the increase in both MFPT and NLRT (8) (where the MFPT is involved as a term). The action of the second mechanism cannot be accounted for by the MFPTM. At $m\to 0$, the NLRT and MFPT differ greatly and the NDD appears only for the NLRT. It means that as $m\to 0$, the NDD appears only due to the action of the second mechanism. Indeed, for $m=0$, the potential profile becomes linear and in this case the first mechanism does not act (see Sec. II). With increasing $m$, the difference between the NLRT and MFPT disappears. It means that the first mechanism of the NDD comes into effect, while the action of the second one becomes weaker.

## IV. CONCLUSION

The above analysis shows that the effect of NDD is typical of all kinds of the considered nonequilibrium unstable states. In other words, the NDD phenomenon exists for a wide range of parameters and initial conditions of the nonlinear systems. The asymptotical expressions such as Eqs. (11) and (31) obtained earlier in Refs. [4,5,7, and 14] are valid only for the specific values of the parameters and initial conditions for which the unstable state becomes equilibrium. The shift from the unstable equilibrium state to the unstable nonequilibrium state by changing the system parameters or initial conditions, leads to the appearance of NDD.

The NDD phenomenon appears due to the action of two different mechanisms. One of them is caused by the nonlinearity of the potential profile describing the unstable state within the decision interval. This mechanism is responsible for the resonant dependence of MFPT on fluctuation temperature. It was considered in Ref. [30]. The other mechanism is caused by the inverse probability current directed into the decision interval. The latter one can not be accounted by the MFPT method. That is why the estimation of the decay times by the MFPT method gives a reduced value. Both these mechanisms are activated by fluctuations. Therefore, one can not contend that in the process of decay of unstable nonequilibrium states the action of fluctuations is insignificant: The effect of the NDD shows that we can both accelerate or slow down the decay of unstable nonequilibrium states when varying the intensity of fluctuations.

## ACKNOWLEDGMENTS

The authors are grateful to Dr. B. Spagnolo for fruitful discussions and useful comments. This work has been supported by RFBR (Project No. 99-02-17544 and 96-15-96718).


[1] H. Risken, *The Fokker-Planck Equation. Methods of Solution and Applications* (Springer-Verlag, Berlin, 1989).
[2] C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry, and the Natural Sciences* (Springer-Verlag, Berlin, 1985).
[3] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, New York, 1984).
[4] M. Suzuki, Prog. Theor. Phys. **56**, 77 (1976); **56**, 477 (1976); **57**, 380 (1977); Phys. Lett. A **67**, 339 (1978).
[5] F. Haake, J.W. Haus, and R. Glauber, Phys. Rev. A **23**, 3255 (1981).
[6] P. Colet, F. de Pasquale, and M. San Miguel, Phys. Rev. A **43**, 5296 (1991).
[7] F.T. Arecchi, A. Politi, and L. Ulivi, Nuovo Cimento B **71**, 119 (1982).
[8] A.V. Polovinkin, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **38**, 153 (1995) [Radiophys. Quantum Electron. **38**, 102 (1995)].
[9] S.A. Akhmanov and G.F. Antonov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **3**, 405 (1960) (in Russian).
[10] J.M. Casado and M. Morillo, Phys. Rev. A **42**, 1875 (1990).
[11] F. de Pasquale and P. Tombesi, Phys. Lett. A **72**, 7 (1979).
[12] G. Broggi, L.A. Lugiato, and A. Colombo, Phys. Rev. A **32**, 2803 (1985).
[13] J.P. Crutchfield, J.D. Farmer, and B.A. Huberman, Phys. Rep. **92**, 45 (1982).
[14] P. Colet, M. San Miguel, J. Casademunt, and J.M. Sancho, Phys. Rev. A **39**, 149 (1989).
[15] V.S. Anischenko and A.B. Neiman, Zh. Tekh. Fiz. **60**, 3 (1990) [Sov. Phys. Tech. Phys. **35**, 1 (1990)].
[16] J.E. Hirsch, B.A. Huberman, and D.V. Scalapino, Phys. Rev. A **25**, 519 (1982).
[17] P.S. Landa and R.L. Stratonovitch, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **30**, 65 (1987) [Sov. Radiophys. **30**, 53 (1987)].
[18] A. Careta and F. Sagues, J. Chem. Phys. **95**, 3399 (1991).
[19] V.E. Shapiro, Phys. Rev. E **48**, 109 (1993).
[20] S. Faetti, P. Grigolini, and F. Marchesoni, Z. Phys. B: **47**, 353 (1982).
[21] G.D. Lythe, Nonlinear Sci. B **7**, 147 (1994).
[22] S. Ciuchi, F. de Pasquale, and B. Spagnolo, Phys. Rev. E **47**, 3915 (1993).
[23] I. Dayan, M. Gitterman, and G.H. Weiss, Phys. Rev. A **46**, 757 (1992).
[24] M. Gitterman and G.H. Weiss, J. Stat. Phys. **70**, 107 (1993).
[25] J.M. Casado and M. Morillo, Phys. Rev. E **49**, 1136 (1994).
[26] R.N. Mantegna and B. Spagnolo, Phys. Rev. Lett. **76**, 563 (1996).
[27] R.N. Mantegna and B. Spagnolo, Int. J. Bifurcation Chaos Appl. Sci. Eng. **8**, 783 (1998).
[28] N.V. Agudov and A.N. Malakhov, Int. J. Bifurcation Chaos Appl. Sci. Eng. **5**, 531 (1995).
[29] A.N. Malakhov and A.L. Pankratov, Physica C **269**, 46 (1996).
[30] N.V. Agudov, Phys. Rev. E **57**, 2618 (1998).
[31] M. Frankowicz and G. Nicolis, J. Stat. Phys. **33**, 595 (1983).
[32] M. Frankowicz, M. Malek Mansour, and G. Nicolis, Physica A **125**, 237 (1984).
[33] E. Arimondo, D. Dangoisse, and L. Fronzoni, Europhys. Lett. **4**, 287 (1987).
[34] J. Iwaniszewski, Phys. Rev. A **45**, 8436 (1992).
[35] R. Z. Khasminski, *Stability of System of Differential Equations under Random Perturbations of their Parameters* (Nauka, Moscow, 1969) (in Russian).
[36] V. Horsthemke and R. Lefever, *Noise Induced Transitions* (Springer-Verlag, Berlin, 1984).
[37] A.N. Malakhov, Chaos **7**, 488 (1997).
[38] N.V. Agudov and A.N. Malakhov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **36**, 148 (1993) [Radiophys. Quantum Electron. **36**, 97 (1993)].
[39] L.S. Pontryagin, A.A. Andronov, and A.A. Vitt, Zh. Éksp. Teor. Fiz. **3**, 165 (1933) [in *Noise in Nonlinear Dynamics*, edited by F. Moss and P.V.E. McClintock (Cambridge University Press, Cambridge, 1989), Vol. 1, p. 329].
[40] R. L. Stratonovitch, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963), Vol. 1.
[41] P. Hanggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).
[42] A.N. Malakhov, Izv. Vyssh. Uchebn. Zaved. Radiofiz. **41**, 519 (1998) [Radiophys. Quantum Electron. **41**, (1998)].
[43] K. Binder, Phys. Rev. B **8**, 3423 (1973).
[44] J. D. Gunton, M. San Miguel, and P. S. Sahni, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 1983).