Resonances and localization of classical waves in random systems with correlated disorder

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An original approach to the description of classical wave localization in weakly scattering random media is developed. The approach accounts explicitly for the correlation properties of the disorder, and is based on the idea of spectral filtering. According to this idea, the Fourier space (power spectrum) of the scattering potential is divided into two different domains. The first one is related to the *global (Bragg) resonances* and consists of spectral components lying within a limiting sphere of the Ewald construction. These resonances, arising in the momentum space as a result of a self-averaging, determine the dynamic behavior of the wave in a typical realization. The second domain, consisting of the components lying outside the limiting sphere, is responsible for the effect of *local (stochastic) resonances* observed in the configuration space. Combining a perturbative path-integral technique with the idea of spectral filtering allows one to eliminate the contribution of local resonances, and to distinguish between possible *stochastic* and *dynamical* localization of waves in a given system with arbitrary correlated disorder. In the one-dimensional (1D) case, the result, obtained for the localization length by using such an indirect procedure, coincides exactly with that predicted by a rigorous theory. In higher dimensions, the results, being in agreement with general conclusions of the scaling theory of localization, add important details to the common picture. In particular, the effect of the high-frequency localization length saturation is predicted for 2D systems. Some possible links with the problem of wave transport in periodic or near-periodic systems (photonic crystals) are also discussed. [S1063-651X(99)08010-1]

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I. INTRODUCTION

The localization of waves in random media is a topic of increasing current interest, owing to its fundamental role in wave-matter interactions, and also by the significance of possible applications. Localization appears in systems governed by time-reversible wave equations, and contradicts the usual intuition-grounded ideas that form the basis of radiative transfer theory. The concept of localization, originally developed for electrons in disordered solids [1], was transferred later to classical waves, in particular, to electromagnetics and acoustics (for a review see Refs. [2–4]). The enhanced backscattering of waves, which is sometimes referred to as weak localization and can serve as a precursor of strong localization, was observed experimentally in the mid 1980s by several groups [5]. Being anticipated in some earlier works as a correction to the conclusions of radiative transfer theory [6], the effect has been analyzed recently in a great number of papers as a counterpart of its electronic analog, with a rich collection of the results that now constitute a well-developed theory.

In the case of strong localization, our understanding of the phenomenon has, as a matter of fact, a qualitative character only. It is well known that the randomness of the potential leads to the appearance of localized states in disordered systems (see, e.g., Ref. [7]). The localization manifests itself most strongly in the one-dimensional (1D) case where even an arbitrary weak disorder causes exponential localization of all states of the system [8]. In two-dimensional (2D) systems, according to the scaling theory of localization [9], all states are also localized, whereas in three dimensions (3D) the situation seems to be much more complicated and depends essentially on the relative strength of the disorder. The absence of extending states in the regime of strong localization influences, in a most radical manner, the transport properties of the system [10], which is used in practice to search for the localization. The strong (exponential type) localization was observed experimentally in 1D and 2D random systems with classical waves [11,12], but in 3D no experimental confirmation of strong localization in structureless media yet exists beyond doubt (see, however, Refs. [13] and [14], where some signatures of possible localization in three dimensions have been reported).

Despite a huge number of related investigations, there is no unified theory that is able to describe consistently all the details of the wave localization beyond the existing general picture, and, therefore, a quantitative analytical description of the phenomenon still presents a challenge. The most important question here concerns the relation between the localization and its characteristics, say, localization length, on the one hand, and the correlation properties of the potential, on the other. At the same time, most of the existing approaches, with only few exceptions, treat the problem phenomenologically, for instance, describing the potential as a δ-correlated field, which obviously puts such formulations entirely outside the realm of the question.

The study of wave localization is most advanced in one-dimensional systems, for which it has been possible not only to prove the existence of localization, but also to estimate the localization length [8]. It can be shown, in particular, that the localization length is determined by those frequencies in the power spectrum of the potential that are known as Bragg resonances, and which we will also call *global resonances* hereafter. In the lowest order, with respect to the strength of disorder, the localization length is described by the frequencies of the main Bragg resonance in an effective periodic lattice, when the wavelength is twice the lattice constant.

In multidimensional systems, relations between the localization and the correlation properties of the disorder have not been understood completely. However, in recent years it has been conjectured that the localization of classical waves is not simply the result of a higher degree of disorder, for instance, due to Mie resonances of independent uncorrelated scatterers (a *microscopic* point of view), but rather the byproduct of an interplay between order and disorder. This idea, formulated by John [15], underlines the importance of structure to set up the localization in random media. Moreover, strong localization in 3D seems to be observed until now only in periodic or near-periodic composite materials, which were proposed by Yablonovitch [16] and are usually referred to as photonic crystals, although they are not limited to the optical, but, rather, are possible for other ranges of the electromagnetic spectrum, or for waves of other natures, such as acoustic (elastic) waves [17–19]. Starting from the above idea, in the present paper we go further in this *macroscopic* mode of thinking, suggesting that the localization in multidimensional systems is determined by the power spectrum of the potential taken somehow at the same frequencies of global resonances, in close analogy with what has been shown in one dimension.

Unlike the 1D case, to which the concept of self-averaging can be directly applied, in multidimensional media we are usually confined to estimating the mean value of a fluctuating quantity, i.e., by averaging over the ensemble of all possible realizations of the potential. The difference between self-averaging and ensemble averaging is of great importance for the localization phenomenon. To understand this, let us consider the problem of wave transmission through a slab of a randomly inhomogeneous medium. Even in the 1D case, where any degree of disorder leads to localization, and, hence, the transmission is exponentially small for almost all realizations, inside the medium, along with the general decrease and natural oscillations, there are enhancements of the wave intensity observed at some random points [20]. These spikes, which we will call *local resonances* hereafter, can exceed any given level and, obviously, are inherent to multidimensional systems as well.

As a result, when one performs, as usual, ensemble averaging for the wave intensity, which is not a self-averaging quantity, one cannot say anything definite about the behavior of the wave in a typical realization. The reason for this situation is that the intensity may be exponentially large due to local resonances and, in spite of the rarity of these events, they give an essential contribution to the mean value. Therefore, we have to distinguish between the notion of *dynamical localization*, describing the transport properties of the medium in typical realizations and determined by global resonances only, on the one hand, and that of *stochastic localization*, i.e., localization of the average wave intensity, related to the properties of the whole statistical ensemble of realizations, and determined by all components of the spectrum, on the other. Consequently, to study a possible dynamical localization in the system, we must extract somehow the hidden global resonances or eliminate the contribution of local ones. This is not so simple to do as to say, but it can be facilitated in the lowest-order approximation, when the mixing of different spectral components is absent, and, as we conjecture, the procedure can be reduced to a filtering in the Fourier space.

According to this idea, which represents a central point of the present work, the algorithm that is utilized here is reduced to the two-step procedure as follows. First, to analyze the behavior of the system for an ensemble of all possible realizations, we use the perturbative path-integral approach developed recently in Refs. [21] and [22]. Then, to study dynamical localization, we perform a filtering of the result thus obtained for the mean intensity, in the Fourier space, keeping the contribution of global resonances only. Though such an indirect and rather involved procedure, making some fine distinctions between self-averaging and ensemble averaging, or, respectively, dynamical and stochastic behaviors, is used, the final results are very simple. The wave transport through the system is described by a functional of the power spectrum of the scattering potential, i.e., the correlation properties of the medium are accounted for in an explicit form. The functional may be easily evaluated for any given power spectrum, and, what is more, its form allows one to explain the localization as a complex interaction of different global resonances, i.e., in terms of some regularity hidden in any realization of the random system. The sign of this functional can serve as a test for the localization to be possible, and the absolute value is related to the localization length, when the localization is achieved. This functional depends crucially on the dimensionality of the problem. In the 1D case the result coincides exactly with that obtained previously by making use of an independent rigorous procedure based on the concept of self-averaging, that may be appraised as an indirect confirmation of the validity of the proposed approach. In all higher dimensionalities the results, being generally consistent with known predictions of other theories, offer additional important details to the general picture. In particular, the effect of the high-frequency localization length saturation is predicted for 2D systems. Some possible links with the problem of wave transport in periodic or near-periodic systems (photonic crystals) are also discussed.

The outline of the paper is as follows. In Sec. II we consider the differences between self- and ensemble averaging, exemplified by a 1D problem of wave transmission through a slab of disordered medium. Next, in Sec. III we introduce the notions of global and local resonances, and discuss their intimate relevance to various aspects of the localization phenomenon. In Sec. IV we give a brief description of the mathematical procedure aimed at the asymptotic calculation of the mean intensity radiated by a point source in a random medium. The results, which characterize stochastic and dynamical behaviors of the wave, are formulated in Secs. V and VI, respectively. The final section contains a summary and some concluding remarks.

II. SELF-AVERAGING VS ENSEMBLE AVERAGING

To analyze the behavior of classical waves in random systems we use the simplest model, based on the reduced Helmholtz equation,

$$\nabla^2 U(\mathbf{R}) + k^2[1+\tilde{\varepsilon}(\mathbf{R})]U(\mathbf{R}) = 0, \qquad (2.1)$$

where $k=2\pi/\lambda$ is the wave number associated with a homogeneous medium, and $\tilde{\varepsilon}(\mathbf{R})$ is the permittivity distribution (scattering potential). The statistical properties of the system are described by a set of correlation functions of the scattering potential, which is supposed to be a random function of coordinates, with zero mean value. In particular, we will use the correlator

$$B_{\varepsilon}(\mathbf{R})=\langle\tilde{\varepsilon}(\mathbf{R}')\tilde{\varepsilon}(\mathbf{R}'+\mathbf{R})\rangle, \tag{2.2}$$

that, being Fourier-transformed, determines the power spectrum $\Phi_{\varepsilon}(\mathbf{K})$ of the scattering potential,

$$\Phi_{\varepsilon}(\mathbf{K})=(2\pi)^{-m}\int d^{m}R\exp(-i\mathbf{K}\cdot\mathbf{R})B_{\varepsilon}(\mathbf{R}). \tag{2.3}$$

We assume that the system is statistically uniform on average, with correlations decreasing at infinity. This means in fact that in the infinite system *all* realizations of the random function $\tilde{\varepsilon}(\mathbf{R})$ are identical (with probability one) up to a spatial shift [7]. The latter property results in the existence of self-averaging quantities that may be associated with such random systems. For these quantities the average with respect to volume (size) of the system becomes (when the volume tends to infinity) the average with respect to realizations of the potential. The behavior of the mean value of a non-self-averaging quantity is completely different. To understand this let us mention that in a finite system the measure of the ''typical'' realizations is close to unity but does not reach unity. In the general case, only these typical realizations contribute to the mean value of a non-self-averaging quantity and its qualitative behavior reflects, more or less, the behavior of the same quantity in a typical realization. However, the averaging could sometimes drastically change the character of the quantity. This happens usually when the nontypical, low probability realizations give an essential contribution to the mean value; this fact explains naming these realizations as representative ones.

The difference between self- and non-self-averaging quantities manifests itself in the clearest manner in the wave localization phenomenon. To illustrate this let us consider the 1D problem of a plane wave transmission through a slab of disordered medium with thickness L. The wave field inside the slab is described by the 1D version of the Helmholtz Eq. (2.1). For an incident wave with a unit amplitude, the reflected and transmitted waves are defined by complex reflection $r(L,k)$ and transmission $t(L,k)$ coefficients, respectively, which are functions of the thickness and the wave number, on the one hand, and depend on the realization of the scattering potential, on the other. Although the transmittivity $T(L,k)\equiv|t(L,k)|^{2}$ of the system is a random function, it can be shown explicitly that $-L^{-1}\ln T(L,k)$ is the self-averaging quantity which tends to the inverse localization length $\xi^{-1}(k)$ [8]. In other words, the transmittivity decreases typically as

$$T(L,k)\propto\exp[-L/\xi(k)],\quad L\to\infty. \tag{2.4}$$

This behavior is related to the property of *dynamical localization* of the wave function, i.e., the localization that characterizes the transport of the wave in a typical realization.

On the contrary, the transmittivity itself is not a self-averaging quantity. In fact, its mean value $\langle T(L,k)\rangle$ evaluated for the same system decreases much more slowly, with the decrement four times less, up to logarithmic accuracy [8]. The reason for this discrepancy is very simple: the representative realizations for the mean transmittivity are small probable transparent realizations with $T\sim1$. Although the mean transmittivity also decreases exponentially with L, leading to a *stochastic localization*, i.e., the localization that reflects the behavior of an ensemble of all possible realizations when a positively defined quantity (transmittivity, wave intensity, etc.) is implied, it shows mainly an exponentially small measure of representative realizations. In spite of the obvious fact that the localization in a stochastic sense is less meaningful as compared to its dynamical partner, the evidence of stochastic localization allows one to draw two conclusions, at least. First, the existence of stochastic localization is sufficient for the fact of dynamical localization to be proved in a given system. In this case, the decrement of the exponential decay of a mean value, obtained in an ensemble of all possible realizations, may serve as an estimate of the lower boundary for the inverse localization length. Second, the existence of dynamical localization, while being necessary, at the same time is not sufficient for the stochastic localization to be observed. One can imagine, in principle, that the existence of local resonances in a dynamically localizing system leads to the situation where the decay of the mean value becomes, for instance, of a polynomial type, or the function even ceases to decrease at all.

III. RESONANCES

Now we proceed with the analysis of the localization length in the simplest 1D system described by the 1D version of Eq. (2.1). In the lowest order of randomness, the procedure of self-averaging leads to the classic result [8] for the inverse localization length:

$$\xi^{-1}(k)=\frac{\pi}{2}k^{2}\Phi_{\varepsilon}(2k). \tag{3.1}$$

The nature of this approximation is worthy of special discussion. It is clear, first of all, that Eq. (3.1) is a variant of the multiple scattering theory, because the perturbative approach is applied here not to the wave function U itself, but to the decrement of the field, and in this sense the result sums a subset of the terms in the multiple scattering series. Also, in this approximation the correlation function plays the role of scattering potential, and the spectrum of the scatterer, usually applied to the calculations of deterministic scattering, is replaced here by the *power spectrum*.

The relation (3.1) shows that in the lowest-order approximation, only $\pm2k$ components of the spectrum are responsible for the localization of wave with wave number k. This fact allows a clear physical explanation to be applied to the localization of waves in 1D systems. Indeed, the wave propagating initially with wave vector $\mathbf{k}$ [see Fig. 1(a)] is transformed by the component $\mathbf{K}=-2\mathbf{k}$ of the scattering potential into the wave with wave vector $-\mathbf{k}$, and after that back to $\mathbf{k}$ by the component $\mathbf{K}=2\mathbf{k}$. It is such subsequent interchange of the momenta between the two channels, of forward and backward propagation, coupled by the disorder, which forms a localized state in the 1D system. This effect, which can be recognized as a simple Bragg resonance, is

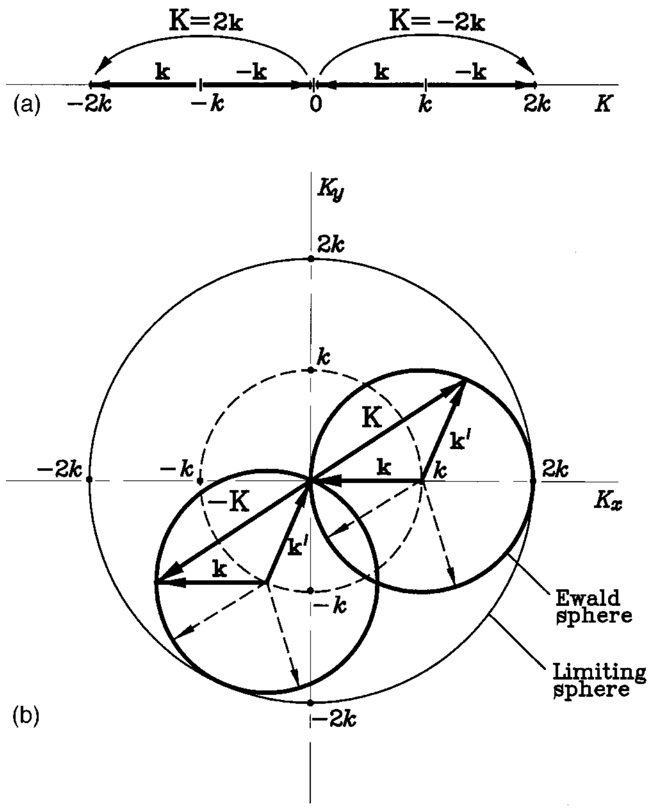

FIG. 1. Momentum diagram, representing schematically the process of resonant scattering in a weakly disordered medium. (a) One-dimensional case. The wave vectors of two counter-propagating waves, $\mathbf{k}$ and $-\mathbf{k}$, are mapped onto the K space in such a way that their endpoints lie at the origin, $K=0$. The exchange by the momenta between these two waves is governed in the lowest-order approximation by the relevant Bragg components ($\mathbf{K}=\pm 2\mathbf{k}$) of the scattering potential. (b) Multidimensional case (2D example is shown). The points of the Ewald sphere for a given wave vector $\mathbf{k}$ determine all possible spectral components that could resonantly transform the incident wave into a scattered one. The limiting sphere encircles all spectral components coupling any two wave vectors in the process of elastic scattering.

used widely in modern optics technologies dealing with periodic and near-periodic structures (fiber Bragg filters, distributed feedback lasers, etc.). Obviously, in the higher orders of the perturbative approach, other potential harmonics are also involved in this process. For example, in the second order one has to take into account all possible pairs of harmonics $(\mathbf{K}_1,\mathbf{K}_2)$ with the total wave vector $\mathbf{K}\equiv\mathbf{K}_1+\mathbf{K}_2=\pm 2\mathbf{k}$, etc. However, having in mind a possible generalization of this picture to multidimensional systems, we will restrict ourselves by considering the lowest-order approximation only.

In many-dimensional systems, the lowest-order resonant component $\mathbf{K}$ of the spectrum $\Phi_\varepsilon(\mathbf{K})$ should satisfy the Bragg law,

$$\mathbf{k}'=\mathbf{k}+\mathbf{K}, \tag{3.2}$$

where the wave vectors $\mathbf{k}$ and $\mathbf{k}'$ are related to an incident and resonantly scattered waves, respectively. To illustrate the situation, we will use the momentum diagram [Fig. 1(b)] which is known as the Ewald construction and widely used in crystallography and related fields [23,24]. The Ewald construction is, in principle, a mapping of the wave vectors onto the K space (reciprocal lattice) of the crystal. Actually, we first draw the wave vector $\mathbf{k}$ of the incident wave such that its endpoint is at the origin of the Fourier space, the point $\mathbf{K}=0$. Assuming then that the initial point of the wave vector $\mathbf{k}'$ coincides with that of vector $\mathbf{k}$, we find that the endpoint of $\mathbf{k}'$ lies on a sphere of radius k due to the energy conservation in elastic scattering, $|\mathbf{k}'|=|\mathbf{k}|$. The points of this sphere, which is referred to as the Ewald sphere of reflection (Ewald circle in the 2D case), include all possible spectral components of the medium which could elastically transform the incident wave into a resonantly scattered one. For a perfect crystal, when the positions of atoms are periodic functions, a strong scattering occurs when the Ewald sphere passes through a point of a discrete reciprocal lattice. In structureless random media, which we deal with here, the spectrum is continuous and resonant scattering is produced by all points of the Ewald sphere. These points, therefore, define the channels coupled to a given one with wave vector $\mathbf{k}$. One of these channels, say, that is defined by the wave vector $\mathbf{k}'$ shown in Fig. 1(b), determines a new Ewald sphere, and, consequently, a set of possible spectral components $\mathbf{K}$ leading to scattering into all other coupled channels, only one of which is exactly the scattering in the initial direction $\mathbf{k}$.

Now we rotate the vector $\mathbf{k}$, such that its endpoint is fixed at the origin, and its initial point thus lies on a sphere of radius k. This operation covers all possible directions of the incident wave, and indicates that the components of the spectrum participating in the Bragg scattering, and, hence, determining the *global (Bragg) resonances*, are located within the limiting sphere of radius $2k$. It is worth mentioning that, in principle, the scheme of Fig. 1(a) is a degenerate version of this construction. Unlike 1D systems, with only two possible channels of counter-propagating waves, in many-dimensional systems there is an infinite number of different channels. It is clear *a priori* that the coupling between different channels (wave vector directions) should play a very important role in the scattering process leading to possible wave localization. Qualitatively, the stronger the coupling between any given channel and the backward or near-backward ones, the higher are the possibilities for the medium to localize the wave. On the contrary, coupling with lateral and, what is more, with a direction near the given one, should suppress or destroy the localization.

It would be very beneficial to study the multidimensional systems in the same manner as has been done for the one-dimensional case, and, when the localization is possible, to obtain the localization length as a functional of the power spectrum $\Phi_\varepsilon(\mathbf{K})$. However, we cannot directly apply the concept of self-averaging to multidimensional systems with an infinite number of scattering channels, and only an ensemble-averaged quantity, such as mean intensity of the wave, could be evaluated. The intensity is not a self-averaging quantity, and, moreover, unlike the transmittivity of a slab, it is not bounded from above. As was first shown by Frisch *et al.* in Ref. [20], where the 1D problem was considered, the intensity pattern inside the medium has a very complicated structure. Along with a general exponential decrease of the wave intensity and natural oscillations with spatial extent of the order k^{-1}, there exist rather extended ''dark'' regions of small intensity, with infrequent sharp spikes of local enhancements of the field at some random positions inside the medium. The amplitude of these *local (stochastic) resonances* can be arbitrarily large, and may exceed any given value, including the entry level. The effect of local resonances can be observed in systems of any dimensionality [11,25], and is clearly responsible for the lowering of the energy transport velocity and the enhancement of numerous nonlinear effects. The positions of these resonances, and even the fact of their existence itself, are extremely sensitive to the exciting frequency, and, as in the case of electromagnetic waves, depends strongly on their polarization.

At this point one essential remark is in order. Although we use the same term, *resonances*, for both global and local ones, it is important to understand the difference between these two notions. Whereas the local resonances are observed in the configuration space, the global ones are inherent for the momentum (wave vector) space. Moreover, unlike local resonances which are functions of a specific realization, the global resonances manifest themselves when a (self-) averaging is applied. Also, contrary to global resonances determining the transport properties of typical, macroscopically large, samples, the local ones mean the localized storage of energy inside the medium for some particular realizations of the disorder.

Some of the local resonances, namely, those with a large quality factor, are similar to the localized modes existing in infinite systems. In fact, going further with the analogy to a periodic structure, we may think about an effective Bragg lattice, within which some small defects are introduced. This leads to the appearance of the so-called defect (impurity) modes [26], with their centers distributed somehow within the medium. In finite systems, such modes can lead, when the concentration of the defects is sufficient for a percolation to be achieved, to an enhanced transmission for a corresponding resonant frequency [27,28], i.e., in our terminology, to some kind of untypical realizations with $T\sim 1$. In the case of ideal transparency, the scattering states within the slab show distinct features of localization, although, strictly speaking, it is senseless to talk about it because of their coupling to the propagating modes outside the slab.

Despite the rarity of such realizations, the local enhancements of the field may be exponentially large and their contribution to the mean intensity may be significant. However, in the lowest-order approximation, when the mixing of different spectral components is absent, the elimination of local resonances may be reduced to a filtering in the K space. In fact, in this approximation the spectral content of these two phenomena is different: whereas the global resonances are related, as we have seen, to the frequencies within the limiting sphere of the Ewald construction, the local resonances are defined by spectral components lying outside this sphere, since only these harmonics could provide for spatially localized, narrow resonant spikes of the wave pattern. Consequently, our main assumption here is that in the lowest-order approximation we can extract the global resonances, leading to a dynamical localization, from the complete spectral ''portrait'' characterizing the stochastic localization of wave in a disordered system.

IV. CALCULATION TECHNIQUE

To study the resonances and localization in a multidimensional (m-dimensional) system, we should evaluate the mean intensity of the wave at a point $\mathbf{R}$ due to a point source located at $\mathbf{R}_0$,

$$\langle I(L,\mathbf{k})\rangle=\langle G(\mathbf{R}|\mathbf{R}_0)G^*(\mathbf{R}|\mathbf{R}_0)\rangle, \tag{4.1}$$

where $L=|\mathbf{R}-\mathbf{R}_0|$, and the wave vector $\mathbf{k}$ is directed along the line connecting the source with the observation point. Obviously, in statistically isotropic media the mean intensity depends only on the distance L, and, in the case of a homogeneous medium the intensity in the far field $kL\gg 1$ decays as $I_0(L)\sim L^{1-m}$. In a random medium, any deviation from this asymptotic behavior should reflect the coherent effects and serve as an indication to the wave localization, at least in the statistical sense.

The usual approach to the calculation of mean intensity in random media is based on the Bethe-Salpeter equation for the coherence function, which can be solved only perturbatively [29]. The first-order (ladder) approximation corresponds to a partial summation of the complete perturbation series, which retains terms of any order. However, the coherence function obtained in this approximation takes into account an essentially restricted class of scattering diagrams, specifically, those describing only single scattering of the wave by a given scatterer. When also the inhomogeneous waves are excluded, the ladder approximation for the coherence function reduces the problem to a phenomenological equation of radiative transfer in which the coherent effects are neglected [29]. At the same time just the coherence and constructive interference between multiply scattered waves gives rise to enhanced backscattering and strong localization. To account for these effects, it was proposed to include also into consideration, the maximally crossed (cyclic) diagrams, which correspond to the motion of the wave along the time-reversed paths with respect to the paths determined by the ladder diagrams. In fact, as was shown, the maximally crossed diagrams allow one to describe the enhanced backscattering, in particular, to obtain the enhancement factor ~ 2, that coincides exactly with known experimental results, obtained in the weak scattering regime [4]. However, for stronger scattering, when the mean-free-path reduces to the order of wavelength (a possible threshold of localization), the enhancement factor decreases essentially [30]. This means that all other diagrams come into play. In particular, the effect of recurrent paths, which describe multiple scattering on the same inhomogeneities, can be crucial. As a result, restricting ourselves to the ladder and maximally crossed diagrams only, we cannot describe correctly the phenomenon of strong localization.

Here we use an alternative approach, starting the analysis of the wave localization in multidimensional systems with the reduced Helmholtz equation for the Green's function $G(\mathbf{R}|\mathbf{R}_0)$. Next, we apply the embedding procedure originally proposed by Fock for the integration of quantum mechanical equations. The idea of the method is based on the introduction of an additional pseudotime variable τ and the transfer to a higher-dimensional space, in which the propagation process is described by a parabolic equation similar to the nonstationary Schrodinger equation in quantum mechanics [21]. The serious advantage of this approach is that it allows us to describe the recurrent events, dealing with an equation which satisfies the dynamic causality condition and, therefore, may be presented in a path-integral form.

We then use this representation for the asymptotic evaluation of the mean intensity in the case of weak disorder. The weakness of the disorder means here that the range of parameters which we are interested in is described by intermediate asymptotics

$$\lambda, l_\varepsilon \ll L \ll \ell_{\mathrm{t}}, \tag{4.2}$$

where l_ε is the correlation radius of the disorder, and ℓ_{t} is the conventionally defined mean-free path. Surprisingly, even in this approximation, the technique still does retain the main feature of the embedding procedure, which is capable of describing the recurrent events. In fact, this will allow us to go beyond the ladder approximation, and to obtain the correction term which is related to the phenomenon of strong localization [31].

The calculations of the mean intensity are reduced then to the self-consistent two-step perturbative procedure as follows. At the first step, the mean intensity is approximated by a double path integral of the functional [32]

$$\exp\left[\frac{k^2}{8}\int_0^L dt_1 \int_0^L dt_2 F(t_1,t_2;\mathbf{R}_1(t),\mathbf{R}_2(t))\right], \tag{4.3}$$

where the scattering function $F(\cdot)$ is given by

$$F(t_1,t_2;\mathbf{R}_1(t),\mathbf{R}_2(t)) = -\sum_{j=1}^{2} B_\varepsilon[\mathbf{R}_j(t_1)-\mathbf{R}_j(t_2)] + 2B_\varepsilon[\mathbf{R}_1(t_1)-\mathbf{R}_2(t_2)], \tag{4.4}$$

and the vector $R_j(t)$ denotes a ''causal'' trajectory in the $(\mathbf{R},\tau)$ space. At the next step, the path integral is evaluated perturbatively, which leads to a small correction to the value of $I_0(L)$. What is important is that the wave correction obtained describes the (exponential) decay of the mean intensity in the first order of the correlation function of the scattering potential, and conforms, in that sense, to our needs formulated in Sec. II.

V. STOCHASTIC LOCALIZATION

Applying a probabilistic interpretation to the path integral, and denoting the unknown wave correction by $\chi(\mathbf{k})$, we present the intensity of the wave in a moment,

$$\langle I(L,\mathbf{k})\rangle = I_0(L)[1+\chi(\mathbf{k})+\dots], \tag{5.1}$$

or cumulant,

$$\langle I(L,\mathbf{k})\rangle = I_0(L)\exp[\chi(\mathbf{k})+\dots], \tag{5.2}$$

form, respectively. Then, using a formal transition $L\to\infty$, i.e., the transfer to a plane wave expansion, we find that the wave correction is proportional linearly to the distance from the source and is given by

$$\chi(\mathbf{k}) = \frac{\pi}{2}k^3 L \int d^m K f(\mathbf{K})\Phi_\varepsilon(\mathbf{K}). \tag{5.3}$$

The filtering function $f(\mathbf{K})$ in Eq. (5.3) has the form

$$f(\mathbf{K}) = -K^{-1}\delta(K-|2\mathbf{k}\cdot\mathbf{K}/K|)+K^{-2}\vartheta(K-|2\mathbf{k}\cdot\mathbf{K}/K|), \tag{5.4}$$

where, $\delta(z)$ is the Dirac δ function, and $\vartheta(z)$ is the Heaviside step function. The spectral component $\mathbf{K}$ satisfying the condition $2\mathbf{k}\cdot\mathbf{K}=K^2$ is the frequency of the Bragg reflection resonance, that, as was demonstrated above, should play a central role in the mode coupling between different scattering channels.

Since we have used a first (linear) correction in the perturbative expansion, there is a direct connection between the two terms in Eq. (4.4) and the corresponding ones entering Eq. (5.4). The first term in the latter equation originates from the one-path correlations, and, being considered separately, gives a doubled extinction coefficient, which defines the decay of the coherent field, evaluated in the framework of the same approach. We deal, however, with the total intensity of the field, and the presence of the second term, which arises from the two-path term in scattering function, and contributes to the final expression with opposite sign, is of great importance here. As we will see below, accounting for this term allows one to obtain the dimensionality dependence, fitting, in general, the predictions of the scaling theory.

Although in both Eqs. (5.1) and (5.2) the wave correction is less than unity, which is simply the condition of validity of the result, these equations, being equivalent in this case, have a different behavior if we try to extrapolate the results to the case of non-small χ. In fact, for negative values of χ, we have an indication of the exponential localization, i.e., we present the results in the form of Eq. (5.2). If $\chi\geqslant 0$, we assume then that there is no (exponential) localization, and the mean intensity, being presented in the form (5.1), decreases far from the source more slowly than in a homogeneous medium.

For isotropic spectra $\Phi_\varepsilon(K)$, we arrive at the expression

$$\chi(k) = \pi k^2 L \int_0^\infty dK f_m(K)\Phi_\varepsilon(K), \tag{5.5}$$

where the form of the filtering function $f_m(K)$ depends on the dimensionality of the problem. In one, two, and three dimensions this function reads, respectively,

$$f_1(K) = -(k/K)\delta(K-2k)+(k/K^2)\vartheta(K-2k), \tag{5.6a}$$

$$f_2(K) = -[(1-K^2/4k^2)^{-1/2}-(2k/K)\arcsin(K/2k)] \times\vartheta(2k-K)+(\pi k/K)\vartheta(K-2k), \tag{5.6b}$$

$$f_3(K) = 2\pi k\vartheta(K-2k). \tag{5.6c}$$

The behavior of the filtering function is shown in Fig. 2. It is seen that in the lowest-order approximation the effect of disorder is split into two terms having opposite signs. Unlike the one- and two-dimensional cases, the negative term is absent in three dimensions.

To exemplify the results we use the simplest Gaussian correlation function

$$B_\varepsilon(R) = \sigma_\varepsilon^2 \exp(-R^2/l_\varepsilon^2). \tag{5.7}$$

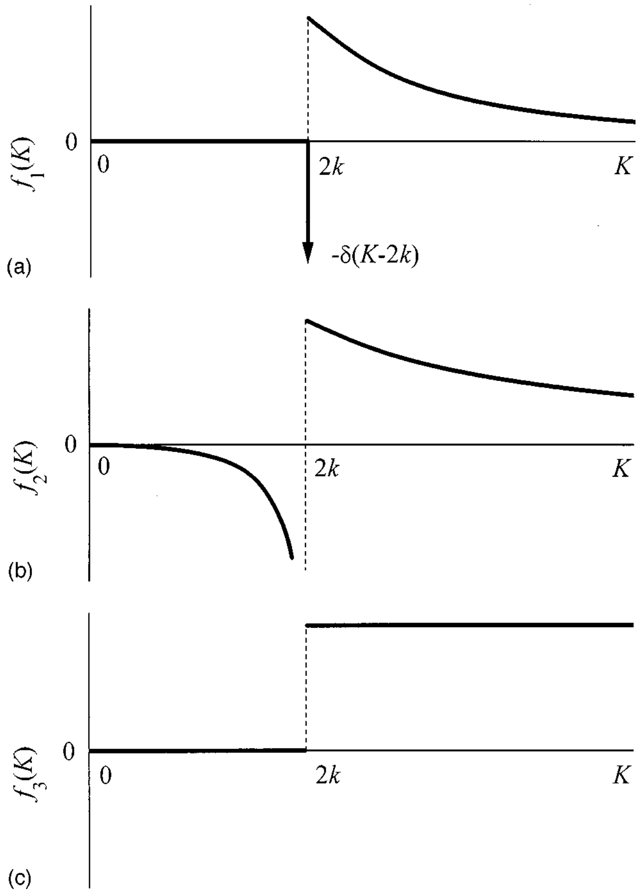

FIG. 2. Schematic representation of the filtering function $f_m(K)$ for 1D (a), 2D (b), and 3D (c) statistically isotropic media. It is seen that for $K>2k$ in all cases there is a positive tail proportional to K^{m-3}. For $K<2k$, the behavior of the filtering function, which is negative in general, depends essentially on the dimensionality of the problem: in the 1D case there is only one discrete component located at $K=2k$; in 2D it is a monotonically decreasing function with a singularity at $2k$ from the left; and in the 3D case the negative part is completely absent for isotropic media.

For this function we introduce two dimensionless parameters: the normalized wave number $\kappa=kl_\varepsilon$, and the normalized distance $l=L/l_\varepsilon$. In terms of these parameters, the wave correction $\chi(\kappa)$ in any dimension takes the form

$$\chi(\kappa)=s(\kappa)l\sigma_\varepsilon^2, \tag{5.8}$$

where the coefficient $s(\kappa)$ has to be evaluated numerically in the general case. The dependence of the coefficient $s(\kappa)$ on the normalized wave number κ is shown in Fig. 3 by a dashed line. The general property of the result is its crucial dependence on the dimensionality of the system.

In the one-dimensional case the wave correction is strictly negative, and it is true not only for the Gaussian model, but also for any medium with a monotonically decreasing spectrum. Therefore, we can predict an exponential-type decay of the wave intensity inside the medium even for an ensemble of all possible realizations. In the limiting case of a δ correlated potential (the power spectrum is the same constant for any frequency) the wave correction is zero, which can be easily verified by direct integration. In two dimensions, along with a negative sign at higher frequencies, there is a

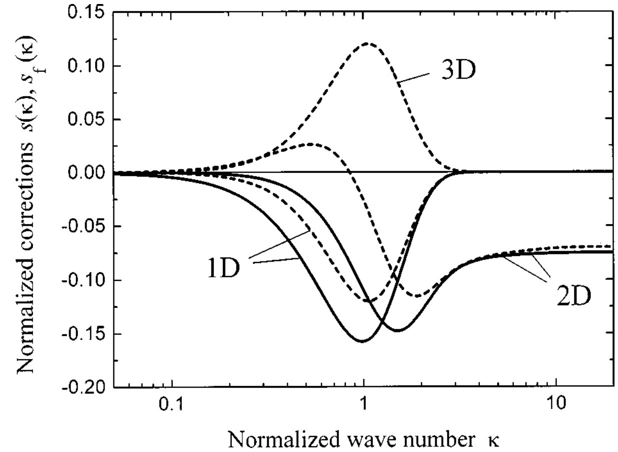

FIG. 3. Normalized wave correction $s(\kappa)=\chi(\kappa)/l\sigma_\varepsilon^2$ (dashed line) and its filtered version $s_{\rm f}(\kappa)=\chi_{\rm f}(\kappa)/l\sigma_\varepsilon^2$ (solid line) plotted as functions of the normalized wave number $\kappa=kl_\varepsilon$ for statistically isotropic media. The behavior of the correction $\chi(\kappa)$ reflects the properties of the medium for a statistical ensemble of all possible realizations. The correction $\chi_{\rm f}(\kappa)$ is related to the dynamical localization, i.e., to the behavior of the wave in a typical realization of the scattering potential. In the three-dimensional case, the filtered wave correction $\chi_{\rm f}(\kappa)$ vanishes for isotropic media.

low-frequency positive part of the correction. It means that at these frequencies the decrease of the intensity due to localization is slower than the increase of probability to excite a local resonance at any point of the medium. In three dimensions, the wave correction is positive, with the maximum observed near $\kappa\sim 1$. The positivity of the correction means that apart from the regular term which decays as L^{-2}, there is an additional term which arises from the local resonances and decays more slowly, as L^{-1}.

VI. DYNAMICAL LOCALIZATION

To extract the information about dynamical localization from the data obtained, we follow the approach proposed in Sec. II. Specifically, we have to eliminate the contribution of local stochastic resonances, keeping the Bragg resonances only. In the simplest case of 1D media, the filtering function $f_1(K)$ consists of two terms. One of them is discrete and extracts the main Bragg components $K=\pm 2k$. Another term is continuous and accounts for a high-frequency tail of the spectrum $\Phi_\varepsilon(K)$. According to our idea, this is related to local resonances excited inside the medium. To calculate the filtered version of the wave correction, $\chi_{\rm f}$, we keep the discrete term only, which leads to

$$\chi_{\rm f}(k)=-\frac{\pi}{2}k^2L\Phi_\varepsilon(2k). \tag{6.1}$$

Inasmuch as the negative sign of the correction is interpreted as a signature of exponential localization, then by using the relation between the filtered wave correction $\chi_{\rm f}$ and the inverse localization length ξ^{-1},

$$\xi^{-1}(k)=-\chi_{\rm f}(k)/L, \tag{6.2}$$

we arrive at nothing else than exactly Eq. (3.1), obtained within the framework of the perturbative self-averaging procedure.

Our idea, stimulated by the above consideration, is to extract only the spectral components lying within the limiting sphere, from the wave correction given by the expansion (5.3) in 2D and 3D systems. The result for the filtered version of the correction has the form

$$\chi_{\mathrm{f}}(\mathbf{k})=\frac{\pi}{2}k^3L\int d^mK\,\vartheta(2k-K)f(\mathbf{K})\Phi_\varepsilon(\mathbf{K}). \quad (6.3)$$

We see that in many-dimensional systems, where an infinite number of coupled channels exists, the result is determined by competition of two effects. The first one, which is related to the first term of the filtering function (5.4) appearing with a minus sign, tends to localize the wave. For any given vector **k** we integrate the power spectrum over the two Ewald spheres, with the weighting factor K^{-1}. The second effect, related to (the part of) the second term of the filtering function, which has an opposite sign, is to suppress or destroy the localization. To account for this contribution, we integrate over the area within the limiting sphere, but outside the Ewald spheres of reflection, with the weight K^{-2}. When the direction of **K** coincides with the direction that is backward to a given wave vector **k**, then the contribution of the delocalizing term vanishes. The farther the direction of **K** from the backward one and the closer it is to the initial propagation direction **k**, the greater the contribution of the delocalizing term to the filtered wave correction. This picture correlates very well with the qualitative consideration (Sec. II) of the relative role of the coupling between different channels in the Ewald construction.

For a given spectrum $\Phi_\varepsilon(\mathbf{K})$ the final result depends somehow on both the modulus and direction of **k**. Here we will concentrate on the analysis of wave localization in structureless isotropic media, when the scattering is independent of the direction. In 2D isotropic media, similar to the 1D case, the filtered correction is always negative (see Fig. 3, solid lines), and the wave is exponentially localized, with the localization length $\xi(\kappa)$ shown in Fig. 4 as a function of the wave number. Although a well-defined minimal value of ξ at some intermediate frequency band is observed, in the high-frequency limit the localization length is unexpectedly constant, independent of the wave number k. This result contradicts a common belief that the high frequency behavior of the wave, being governed by the geometrical optics rules, exhibits only extending states. The argument, however, is not convincing by itself, because even high-frequency mechanisms, such as whispering-gallery resonances, combined with a constructive interference, may confine the wave within a finite domain, the effect that is reminiscent of the corresponding type high-Q resonances in circularly symmetric disks. Besides, unlike electronic systems, where at low energies the electron is always trapped in the wells of disordered potential, for classical waves the high-frequency limit corresponds to the increase of the scattering strength of the potential. On the other hand, we have to remember that all the results obtained in this paper are limited by the lowest-order approximation. The increase of the wave number leads clearly to a decrease of the mean-free-path ℓ_t, and the intermediate asymptotics (4.2) can quickly fail to be valid. Nevertheless, the same effect has been observed recently in numerical simulations [33], where the localization length for 2D strongly disordered systems was shown to saturate at relatively high frequencies.

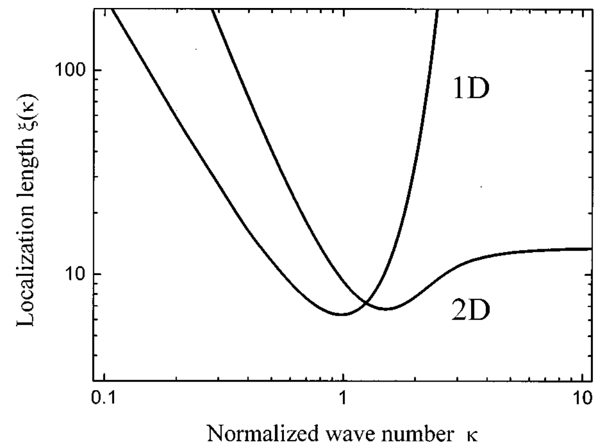

FIG. 4. Localization length $\xi(\kappa)$ plotted as a function of the normalized wave number $\kappa=kl_\varepsilon$ for 1D and 2D statistically isotropic media. The localization length is normalized to σ_ε^2 and is given in units of l_ε.

In three dimensions, for isotropic systems $\chi_{\mathrm{f}}(k)\equiv 0$, i.e., despite the existence of local resonances, enhancing the field at some random points and thus accumulating the energy inside the medium, the wave cannot be localized in the Anderson sense. Irrespective of the true existence of the mobility edge in 3D systems with classical waves, when the strength of the disorder is energy dependent, our approach, which is based essentially on a perturbative procedure, cannot describe such a transition, and other techniques have to be utilized. Nevertheless, the vanishing of χ_{f} in three dimensions correlates very well with the current situation, where to to the best of our knowledge, the exponential localization has not been observed until recently in 3D *structureless* systems, notwithstanding several claims of being very close to achieving it in such media.

VII. SUMMARY

In this work we have studied the localization phenomenon in the scalar wave approximation. By using the notions of ensemble and self-averaging, we could distinguish between the properties of stochastic and dynamic behavior. Combining then the perturbative path-integral technique with the idea of spectral filtering, we have analyzed the transport properties of a random system in the localization regime. The results obtained are limited, obviously, by the lowest-order approximation, i.e., by the weak disorder limit. However, even the short consideration of a self-avoiding-walk analogy given in Ref. [22], shows that the limitations of the perturbative procedure, which has been utilized, may be essentially different in various dimensions. In fact, when we transfer, in turn, from one to three spatial dimensions, there is more and more phase space available for the trajectory to avoid the intersection points, which contribute to the path integral, and the same perturbative result may account for a stronger disorder in higher dimensions. Therefore, the limitations that are very strict in 1D, may be weaker in 2D, and, perhaps, not so serious in the three-dimensional case.

Although the purpose of the present work was to study random media (scattering potentials with correlations decreasing at infinity), there is a naturally arising question of whether the results obtained could be related in any way to periodic or near-periodic structures (photonic crystals) exhibiting band gaps in their spectra. In periodic media the value of $\Phi_{\varepsilon}(\mathbf{K})$ has obviously the meaning of a static structure factor that can be easily calculated for any composite material, and then the value of $\chi_{\mathrm{f}}(\mathbf{k})$ can also be evaluated for a given $\mathbf{k}$ according to Eq. (6.3). Could one maintain that the wave would be localized if

$$\chi_{\mathrm{f}}(\mathbf{k})<0 \tag{7.1}$$

for all directions $\mathbf{k}/k$ (if it is possible in principle!), and could one consider this condition as a criterion of strong localization in such media? Irrespective of the exact answer to this question, it is clear that in any case, the form of Eq. (6.3) seems to remove the problem of the Mie-Bragg controversy in creation of gaps, by reducing it to a terminological difference only. In fact, when one speaks about *Bragg scattering* in periodic structures, one means usually that the spatial period of the structure (lattice constant) matches, in an appropriate way, the wavelength in the host material. In the case of the *Mie resonances* one implies that along with a high value of the dielectric contrast, the diameter of inclusion (say, of spherical form) fits the wavelength inside the inclusion. It is clear, at the same time, that only the combination of all these parameters (to which we can add the volume filling factor, which is not independent, however), form, on an equal footing, the static structure factor, which, in its turn, is likely to define the localizing properties of the medium. The same conclusion may be related also to the connectivity of inclusions and the symmetry of the resulting structure, the questions that have been explored intensively in many recent works by using a ''cut-and-try'' approach, either by computer simulations, or in real experiments [17,19].

If criterion (7.1) is actually relevant to a search for localization in periodic structures, some other questions arise: Could we predict the kinds of structures, and estimate the optimum values of their parameters, that are most favorable for the appearance of large absolute band gaps? What are the limitations of the results in various dimensions? etc. Although the work poses more questions rather than giving definite answers to them, we hope that both the approach itself and the final results offer a physical insight into the problem of wave localization, and may stimulate further research in this direction.

ACKNOWLEDGMENT

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