Two-particle dispersion by correlated random velocity fields

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(Received 25 May 1999; revised manuscript received 6 July 1999)

We consider the two-particle dispersion in a velocity field, where the relative two-point velocity scales according to $v^2(r) \propto r^{\alpha}$ and the corresponding correlation time scales as $\tau(r) \propto r^{\beta}$. We show that for $\alpha/2+\beta < 1$ the diffusion approximation holds, and the increase in the interparticle distances is governed by the distance-dependent diffusion coefficient $K(r) \propto r^{\alpha+\beta}$. The possible regimes outside of the validity of diffusion approximation are discussed. The Kolmogorov scaling in turbulent flow $\alpha=\beta=2/3$ corresponds to a borderline situation. The experimental data for this case suggest that the separation regime is probably ballistic. [S1063-651X(99)05011-4]

PACS number(s): 47.27.Qb, 05.40.$-$a

The story of scaling concepts in turbulent flows starts from the seminal work of Richardson [1], who observed that the mean square relative separation between two particles, initially in close vicinity, evolves in time according to $R^2(t)=\langle r^2(t)\rangle \propto t^3$. He moreover formulated a differential equation for the evolution of the distribution function of the two-particle distances, being of the form of diffusion equation with the distance-dependent diffusion coefficient $K(r) \propto r^{4/3}$, giving a heuristic picture of particles' separation. The problem of correct statistical description of Richardson's dispersion was continuously attacked during more than 70 years, but still did not found a satisfactory solution. It was Batchelor [2] who first demonstrated that Richardson's law follows from the same scaling argument that leads to the Kolmogorov-Obukhov energy spectrum. Later on, he proposed a different form of a diffusion operator, in which the diffusion coefficient is not distance, but time dependent [3]. The mixed forms were proposed in Refs. [4,5] (see Ref. [6] for the review of early work). References [7,8] dispense from the attempts to describe the dispersion by differential equations and propose an essentially integral-equation description based on a Levy-walk picture.

It is clear that the reasonable results, which can be compared to experiments, can be only obtained in computer simulations taking into account realistic properties of the turbulent velocity fields. On the other hand, the interest in the qualitative understanding of the mixing properties of random flows put forward the models which do not closely follow the statistics of turbulence flow (such as white-in time Gaussian fields of Refs. [9–11]), but which have advantages of being easier treated analytically or numerically. Thus, in kinematic simulations the velocity field is typically built up from the structures (plane waves [12], eddies [13], or combination of plane-waves and wavelets [14], etc.), each of which is characterized by its own spatial scale r and the scale-dependent correlation time. The amplitudes of the structures are chosen to mimic the known spectral properties of the velocity field. For typical spectra used the locality assumption holds: The values of relative velocity at distance r are mostly determined by the structures of the scale r, so that the typical Eulerian correlation time at this distance is of the order of the correlation time of the corresponding structure. For example, to mimic the δ-correlated field, one can take all these correlation times equal and small. In this case, the differential form of the relative diffusion operator is exact [9,10], but the relative diffusion itself follows the law $R^2(t) \propto t^{2/3}$ [14], with the exponent twice smaller than the Richardson's 3. The simulations of Ref. [13] reproducing the Richardson's law take the correlation time to scale as $\tau(r) \propto r^{2/3}$.

In what follows we consider qualitatively the situation of the particles' separation in a field whose two-time correlation function of relative velocities $v(\mathbf{r},t)=[V(\mathbf{r}+\mathbf{x},t)-V(\mathbf{x},t)]\mathbf{r}/r$ behaves as $\langle v(r,t_1)v(r,t_2)\rangle \propto v_r^2(r)G[t_2-t_1,\tau(r)]$. The temporal correlation part will be assumed to follow a universal scaling, $G(t,r)=g[t/\tau(r)]$, where $\tau(r)$ is the correlation time, which depends on the distance between the points. Note that this correlation time is evaluated in a reference frame attached to one of the particles, just as it is done in a Lagrangian approach of Ref. [13]. To be exact we shall define $g(t)$ in such a way that $g(0)=1$ and $\int_0^\infty g(s)ds=1$. We first proceed along the lines of the Taylor-type analysis of the situation. Thus we consider velocity field whose mean squared relative velocity at two points separated by the distance r scales as

$$\langle v^2(r)\rangle \propto v_0^2\left(\frac{r}{r_0}\right)^{\alpha} \tag{1}$$

and the corresponding correlation time scales as

$$\tau(r) \propto \tau_0\left(\frac{r}{r_0}\right)^{\beta}. \tag{2}$$

For example, for Kolmogorov's scaling in a ''normal'' turbulent flow we always have $v^2(r) \propto \epsilon^{2/3} r^{2/3}$ where ϵ is the energy dissipation rate. Moreover, from the Kolmogorov scaling it also follows that $\tau(r) \propto \epsilon^{-1/3} r^{2/3}$ (i.e., the mean lifetime of the eddy must be proportional to its revolution time) [6,15] so that $\alpha=\beta=2/3$. This is just the situation simulated in Ref. [13]. In Ref. [14] another situation is con-

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sidered. Here one takes τ to be a sweep time, $\tau(r)\propto r/V_0$. The physical picture here is that the particles are transported together through the eddy region, by the mean flow velocity V_0. The eddy velocity is only a small perturbation on the background of the overall flow, and the proper lifetime of the eddy is large. In this case one clearly has $\beta=1$.

Let us concentrate on the behavior of the second moment of the particles' separation. Following the standard approach we start from the equation of motion for the interparticle distance

$$\frac{d\mathbf{r}}{dt}=\mathbf{v}(\mathbf{r}), \tag{3}$$

where $\mathbf{v}(\mathbf{r},t)$ is the fluctuating velocity and $\mathbf{r}(t)$ is the actual particle's position. Putting down an equation for the separation distance squared, $dr^2(t)/dt=2\mathbf{v}(\mathbf{r},t)\mathbf{r}(t)$, and averaging it over realizations of the process (many different particle pairs) we get

$$\frac{d\langle r^2(t)\rangle}{dt}=2\langle \mathbf{v}(\mathbf{r},t)\mathbf{r}(t)\rangle. \tag{4}$$

The values $\mathbf{v}(\mathbf{r},t)$ and $\mathbf{r}(t)$ are correlated, since $\mathbf{r}(t)$ is governed by Eq. (3), whose formal solution is given by the integral of the relative velocity of the particles along their Lagrangian trajectory, $\mathbf{r}(t)=\int \mathbf{v}(\mathbf{r}(t'),t')dt'$. Equation (4) thus reads

$$\frac{d\langle r^2(t)\rangle}{dt}=2\int \langle \mathbf{v}(\mathbf{r}(t),t)\mathbf{v}(\mathbf{r}(t'),t')\rangle dt'. \tag{5}$$

Imagine now that the *local correlation time* of the velocity field is so short that the relative displacements during this time are to some extent small. The changes in $\mathbf{r}$ can then be neglected, so that both velocities are evaluated in the same space point. Moreover, the lower boundary of time integration can be shifted to $-\infty$. One thus has

$$\frac{d\langle r^2(t)\rangle}{dt}=2\langle v^2(r)\rangle\int g\left(\frac{t-t'}{\tau(r)}\right)dt'=2\langle v^2(r)\rangle\tau(r) \propto v_0^2\tau_0\left(\frac{r}{r_0}\right)^{\alpha+\beta}. \tag{6}$$

This corresponds to diffusive behavior with position-dependent diffusion coefficient $K(r)\propto r^{\alpha+\beta}$. Taking, as a scaling assumption, $r\propto\langle r^2(t)\rangle^{1/2}=R$, one gets that the mean square separation R grows as

$$R^2\propto t^{2/[2-(\alpha+\beta)]}. \tag{7}$$

The reduction of a Lagrangian mean value $\langle \mathbf{v}(\mathbf{r}(t),t)\mathbf{v}(\mathbf{r}(t'),t')\rangle$ to a one-point quantity is based on a Taylor expansion for $\mathbf{r}(t')$ backwards in time starting from $\mathbf{r}(t)$: for $t-t'$ small $\mathbf{r}(t')=\mathbf{r}(t)-\mathbf{v}(\mathbf{r}(t),t')(t-t')+\cdots$. The corresponding expansion for the correlation function then reads

$$\int_0^t\langle \mathbf{v}(\mathbf{r}(t),t)\mathbf{v}(\mathbf{r}(t'),t')\rangle dt' =\int_0^t\langle \mathbf{v}(\mathbf{r}(t),t)\mathbf{v}(\mathbf{r}(t),t')\rangle dt' -\int_0^t\langle \mathbf{v}(r(t),t)\nabla_r\mathbf{v}(\mathbf{r}(t),t')\mathbf{v}(\mathbf{r}(t),t)(t-t')\rangle dt'\cdots. \tag{8}$$

Estimating the second term by the order of magnitude, we get it to be $\langle|v^3(r)|\rangle\tau^2/r$. This term is typically small compared with the first one if the particle displacement $l(r)=v(r)\tau(r)$ during the correlation time (mean free path) is small compared to r. The mean free path scales as

$$l(r)\propto(v_0^2)^{1/2}\tau_0(r/r_0)^{\alpha/2+\beta} \tag{9}$$

and grows parametrically slower than r if $\beta<1-\alpha/2$. The shifting of the lower integration boundary to $-\infty$ can be verified by the fact that the correlation time $\tau(r)$, typical for the distances of the order of mean square separation, grows as $R^\beta\propto t^{\beta/[2-(\alpha+\beta)]}$, i.e., for $\beta<1-\alpha/2$ slower than t.

In a short-correlation-time approximation one can obtain a closed differential equation for the probability density function (PDF) of relative displacements $p(\mathbf{r},t)$. Let us return to Eq. (3) and consider different realizations of the flow. According to Eq. (3), the relative distance at time $t>t_0$ for each realization of the flow is fully determined by its position at t_0. In different flow realizations $p(\mathbf{r},t)$ depends on $\mathbf{r}(t_0)$ only, so that the dispersion problem corresponds to a Markovian random process. The PDF of this process is governed by an integral Chapman-Kolmogorov equation, see Ref. [16]. Since the trajectories of the process are continuous, in a short-correlation-time limit the integral Chapman-Kolmogorov equation can be reduced to a differential one, i.e., to a Fokker-Planck equation

$$\frac{\partial p(\mathbf{r},t)}{\partial t}=-\sum_i\frac{\partial}{\partial x_i}[A_i(\mathbf{r},t)p(\mathbf{r},t)] +\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}[B_{ij}(\mathbf{r},t)p(\mathbf{r},t)]. \tag{10}$$

The assumptions are the existence of the transition moments, i.e., of the limits

$$B_{ij}(\mathbf{r},t)=\frac{1}{\Delta t}\langle[x_i(t+\Delta t)-x_i(t)][x_j(t+\Delta t)-x_j(t)]\rangle \tag{11}$$

and

$$A_i(\mathbf{r},t)=\frac{1}{\Delta t}\langle[x_i(t+\Delta t)-x_i(t)]\rangle \tag{12}$$

independent of Δt, for Δt small compared to the observation time t, see Ref. [16] for the in-detail discussion of mathematical requirements and their physical interpretation. The nonvanishing contribution in Eq. (11) stems from the first-order term; thus, fully parallel to Eq. (6) one has

$$B_{ij}(\mathbf{r},t)\approx\frac{d}{d\Delta t}\int_t^{t+\Delta t}\int_t^{t+\Delta t}\langle v_i(\mathbf{r}(t'),t')v_j(\mathbf{r}(t''),t'')\rangle dt'dt''$$

$$\approx 2\langle v_i(\mathbf{r})v_j(\mathbf{r})\rangle\tau(r) \tag{13}$$

for all $\Delta t \gg \tau(r)$. The nonvanishing contribution to A stems from the second-order term, since the first-order term $\int_t^{t+\Delta t}\langle\mathbf{v}[\mathbf{r}(t),t]\rangle dt$ vanishes. Thus,

$$A_i(\mathbf{r},t)\approx\frac{d}{d\Delta t}\int_t^{t+\Delta t}dt' \times\int_t^{t'}\left\langle\sum_j\frac{\partial}{\partial x_j}v_i[\mathbf{r}(t),t']v_j[\mathbf{r}(t),t'']\right\rangle dt'dt''. \tag{14}$$

For incompressible flows, $\Sigma_j(\partial v_j/\partial x_j)=0$, one has

$$A_i(\mathbf{r},t)=\frac{1}{2}\sum_j\frac{\partial}{\partial x_j}B_{ij}, \tag{15}$$

so Eq. (10) reduces to a diffusion equation

$$\frac{\partial p(\mathbf{r},t)}{\partial t}=\sum_{i,j}\frac{\partial}{\partial x_i}\left(\frac{1}{2}B_{ij}(\mathbf{r},t)\frac{\partial}{\partial x_j}p(\mathbf{r},t)\right), \tag{16}$$

which, for statistically isotropic systems, takes Richardson's form

$$\frac{\partial p(r,t)}{\partial t}=\frac{1}{r^{d-1}}\left(\frac{\partial}{\partial r}r^{d-1}K(r)\frac{\partial}{\partial r}p(r,t)\right), \tag{17}$$

where $K(r)$ is the radial part of $\frac{1}{2}\mathbf{B}$ and is proportional to $(v_0^2\tau_0/r_0^{\alpha+\beta})r^{\alpha+\beta}$. The original Richardson equation corresponds to the values $\alpha=\beta=2/3$ following from the Kolmogorov scaling.

The existence of the domain of applicability of Eq. (17) relies on the time hierarchy $\tau(r)\ll\Delta t\ll t$ which must asymptotically hold for the separation distances typical for the observation time t, which, as already mentioned, is the case for $\beta<1-\alpha/2$. Note that Eq. (17) does not depend on the precise structure of the flow lines and on the higher moments of the velocity distribution, and rules out, in a short-correlation-time limit, the time-dependent and mixed forms of diffusion coefficient, which might still apply in alternative situations.

Let us consider the situation after the breakdown of diffusive regime. The mean square separation cannot grow faster than is allowed by Eq. (3), when taking $v(r)$ to be of the order of the particles rms velocity (this supposes the particles to undergo ballistic separation, without ever changing the direction of their outwards motion). This ballistic assumption gives

$$R^2\propto t^{4/(2-\alpha)} \tag{18}$$

which is independent on β. Comparison of Eqs. (7) and (18) shows that the transition takes place at $\beta=1-\alpha/2$, i.e., immediately after the breakdown of the diffusive approximation. Ballistic separation is the typical mechanism of dispersions in flows, where the flow lines of the relative motion are open. On the other hand, in isotropic and homogeneous two-dimensional (2D) flows these flow lines are typically closed, showing a ''cat's eye in a cat's eye'' structure as depicted in Ref. [12]. Since the larger eddies are persistent during long time, the particles gets trapped within those, and the separation distance at time t cannot exceed the characteristic radius of such eddies. The particles thus perform a spiraling motion and slowly increase the area visited. We shall term this regime as inflatory separation. The typical separation distance can be then estimated by reverting Eq. (2) and is

$$R^2\propto t^{2/\beta}, \tag{19}$$

which is now α independent. The transition from diffusion regime to the inflatory one again takes place exactly at $\beta=1-\alpha/2$; the inflatory and the ballistic regimes assume different flow structures but can coexist in flows of complex geometries. This is probably the situation in the numerical simulation of Ref. [14], with $\beta=1$, where the particles' separation relies on the rare ballistic events and not on the typical behavior. The same situation is observed in the particles' dispersion in two-dimensional flows generated by inverse cascade [17], see Ref. [18].

The values of $\alpha=\beta=2/3$, as following from the Kolmogorov scaling, correspond just to the borderline case, $\beta=1-\alpha/2$. This applies when any superscaling (cascading) assumption holds. If one, e.g., supposes, that there exists a unique kinematic parameter Ξ of dimension $[L^a/T^b]$, which determines the flow's behavior in some range of scales, then from scaling considerations it follows immediately that any velocity (if only scaling and coordinate-dependent) behaves as $v^2(r)\propto(\Xi r^{b-a})^{2/b}$ [so that $\alpha=2(1-a/b)$] and that any characteristic time, as a function of r, behaves as $\tau(r)\propto(\Xi^{-1}r^a)^{1/b}$ (so that $\beta=a/b$). From this an equality $\beta=1-\alpha/2$ follows. For the borderline case of the Richardson's dispersion the functional asymptotic smallness of the mean-free path does not hold anymore; it can be small only by some numerical parameter. From Eq. (9) it follows then that $l(r)=(v_0\tau_0/r_0)r$ and is small compared to r if a number parameter $Ps=v_0\tau_0/r_0$ (the persistence parameter of the flow) is small.

We stress that in the borderline case all three dispersion mechanisms, the diffusive (Richardson's) one, the ballistic one and the area inflation lead to the *same* functional time-dependence of the mean squared separation. The small values of Ps lead to a diffusive behavior for which the Richardson's diffusion equation is exact; for large values of Ps the other regimes are possible. The situations can be distinguished only on the ground of numerical prefactors and the behavior of the trajectories of the relative particles' motions. We note that the transition from a Richardson's to, e.g., ballistic regime when changing Ps can be abrupt, as suggested by a simple heuristic model considered in the Appendix.

Let us make some estimates for the value of Ps in a typical Richardson's regime, starting from the values of the Richardson's constant and the prefactor of the longitudinal velocity in a three-dimensional case. Thus, in tree dimensions one has $\langle v_r^2(r)\rangle=C_L\epsilon^{2/3}r^{2/3}$, where C_L is a numerical factor connected with the Kolmogorov constant. The generally adopted numerical value of this factor is $C_L\approx 2.0$. On the other hand, the typical value of the numerical factor G in the Richardson's law, $\langle R^2(t)\rangle = G\epsilon t^3$, is $G\approx 0.2$. Starting from the diffusion approximation we get for the diffusion coefficient $K(r)=v_0^2\tau_0(r/r_0)^{4/3}\approx Ps\, C_L^{1/2}\epsilon^{1/3}r^{4/3}$. From this $\langle R^2(t)\rangle = (2Ps\sqrt{C_L}/3)^3\epsilon t^3$ follows, i.e., $G\cong(2Ps\sqrt{C_L}/3)^3$. This gives us the numerical value of the persistence parameter of approximately 0.6, which is to no extent small. The strongly ballistic nature of the ''normal'' Richardson's dispersion can be seen also from the comparison of the particles' separation velocity and the rms Eulerian velocity difference at the distance r. Such a comparison gives

$$\frac{v_{\mathrm{sep}}(r)}{v_r(r)} \cong \frac{(3G/2)^{1/3}}{C_L^{1/2}} \approx 0.5 \tag{20}$$

which means (taking into account the possible curvature of the relative trajectories) that the particles' separation velocity has a strong ballistic component. Note that the closure approximation giving $G\approx 2$ leads to $v_{\mathrm{sep}}(r)/v_r(r)\approx 1$, i.e., corresponds to a purely ballistic behavior. The particles' dispersion in two-dimensional flows generated by inverse cascade [17], is not fully ballistic, but still possesses a considerable ballistic component [18].

In conclusion, we considered the two-particle dispersion in a velocity field, where the relative two-point velocity scales according to $v^2(r)\propto r^\alpha$ and the corresponding correlation time scales as $\tau(r)\propto r^\beta$. We show that for $\alpha/2+\beta<1$ the dispersion can be described within a diffusion approximation. The time evolution of interparticle distances is then governed by distance-dependent diffusion coefficient $K(r)\propto r^{\alpha+\beta}$. The Kolmogorov scaling in turbulent flow $\alpha=\beta=2/3$ corresponds to a borderline situation $\alpha/2+\beta=1$, where the type of stochastic process responsible for the dispersion depends on the numerical coefficients, for example, on the persistence parameter of the flow, $Ps = v_0\tau_0/r_0$. The experimental data suggest that in three-dimensional flows the particle separation is dominated by ballistic events.

The hospitality of LMHD at the University Paris VI and the financial support by CNRS are gratefully acknowledged. The author is indebted to Professor P. Tabeling, Professor J. Klafter, and Dr. R. Reigada for helpful discussions.

APPENDIX

In order to elucidate the nature of the transition from diffusive to ballistic motion and the possible regimes of the Richardson's dispersion, let us consider a heuristic dispersion model that for small τ_0 behaves diffusively and for τ_0 large changes abruptly to a ballistic regime. Parallel to our Lévy-walk model of Ref. [8], we consider a motion of a particle on a line with a coordinate-dependent velocity. We take the magnitude of the velocity to be a function of r only and to be equal to $v(r)=v_0(r/r_0)^{\alpha/2}$. Moreover, we account for the temporal changes of the flow by letting the particle from time to time change the velocity's direction, keeping its magnitude constant. Distinct from Ref. [8], the probability of changing the velocity during the instant of time dt depends on the particle's position and is given by $dp=dt/\tau(r)=\tau_0^{-1}(r/r_0)^{-\beta}dt$. Different scattering events are considered to be independent (as they stem, so-to-say, from different eddies). From this expression the probability of being scattered while crossing a distance dx follows:

$$dp=\frac{dr}{v(r)\tau(r)}=\frac{1}{v_0\tau_0}(r/r_0)^{-(\beta+\alpha/2)}dr. \tag{A1}$$

The probability of not being scattered on the way from r_1 to r_2 follows then as a Hertz distribution,

$$P(r_2|r_1)=\exp\left(-\int_{r_1}^{r_2}\frac{1}{v_0\tau_0}(r/r_0)^{-(\beta+\alpha/2)}dr\right)$$
$$=\exp\left(-Ps^{-1}r_0^{(\beta+\alpha/2)-1}\int_{r_1}^{r_2}r^{-(\beta+\alpha/2)}dr\right). \tag{A2}$$

Performing the integration, we get

$$P(r_2|r_1)=\exp\left[-\frac{Ps^{-1}}{1-(\beta+\alpha/2)}\left(\frac{r_2}{r_0}\right)^{1-(\beta+\alpha/2)}\right]$$
$$\times\exp\left[\frac{Ps^{-1}}{1-(\beta+\alpha/2)}\left(\frac{r_1}{r_0}\right)^{1-(\beta+\alpha/2)}\right] \tag{A3}$$

for $\beta<1-\alpha/2$ and

$$P(r_2|r_1)=\left(\frac{r_2}{r_1}\right)^{-1/Ps} \tag{A4}$$

for $\beta=1-\alpha/2$. Note that this model (as long as only the spatial aspects of the motion are considered) leads to a Markovian process of asymmetric walks, where the step direction is chosen at random and a step length follows from Eqs. (A3) or (A4). The conditional step length distribution $P(r_2|r_1)$ in this model is strongly asymmetric, and we shall be interested mostly in the behavior of the outward steps, $r_2>r_1$. For $\beta<1-\alpha/2$ the distribution possesses the first and the second moments, both depending on r_1, and can be mapped on a diffusion process.

For the cascading case $\beta=1-\alpha/2$ the existence of moments depends on the value of Ps: The nth conditional moment of the outwards step length ($r_2>r_1$) in this case is

$$M_n(r_1)=\frac{r_1^{1/Ps}}{Ps}\,\frac{1}{n-1/Ps}\,r^{n-1/Ps}\Big|_{r_1}^{\infty}.$$

One readily infers that for $Ps=1/2$ the second moment disappears, thus indicating that the process gets to be of a nondiffusive nature, and starts to depend on long steps (ballistic events). For $Ps=1$ disappears the first moment, so that the process is *dominated* by the ballistic events. We note that the transitions between the regimes are sharp and not gradual, which could possibly be the case also for a genuine problem of transport in flows.

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