Rapid decoherence in integrable systems: A border effect

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We show that rapid decoherence, usually associated with chaotic dynamics, is not necessarily a hallmark of nonintegrability: border effects in integrable systems may produce similarly drastic decoherence rates. These can be found when the subsystem under observation possesses an energy limitation as, e.g., in the *N*-atom Jaynes-Cummings model. We show for this model that special initial coherent wave packets exhibit entropy production rates strikingly similar to the chaotic case. Also, a (de)localization phenomenon is found to be a function of the proximity to the phase-space border. [S1063-651X(99)01611-6]

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The quantum entanglement process, decoherence, and the quantum $\leftrightarrow$ classical transition [1] have recently attracted much attention from physicists, both theoretical and experimental [2]. Creating entangled quantum states in the laboratory is now possible in ion trap experiments [3] and also with atoms in high-*Q* cavities [4]. Both of them realize a simple situation in which a two-level atom is coupled to a quantized harmonic oscillator by means of the Jaynes-Cummings model (JCM) [5]. This simple model has a long and frequent history as a convenient laboratory for testing theoretical predictions [6], being expected nowadays to serve also in practical implementations. The physics of this kind of system, where two or more atomic levels interact with a single-mode electromagnetic field, is usually explored by means of quantities such as the population inversion and the mean number of photons. These quantities revealed, among other phenomena, the existence of collapse and revival regions in the curves of population inversion [6,7]. One could then infer that the field and atom lose their identity in the collapse region, and most closely return to their initial states during the revival.

However, if one is concerned with the entanglement of the atomic and field subsystems, the population inversion can be a misleading quantity, particularly with respect to the purity of the quantum state. In fact, the works of Phoenix and Knight [8] and that of Gea-Banacloche [9] have shown that this system may greatly recover its purity during the very collapse interval, at half the revival time. In these papers, instead of population inversions, reduced density operators were used in calculating either the system's entropy or idempotency defect (linear entropy). Here we are interested in the decoherence process of systems constituted by subsystems with dissimilar Hilbert spaces. To this end, we take the *N*-atom JCM, where *N* two-level atoms interact with a single-mode field and, in view of the results just cited, we adopt the idempotency defect as a measure of the entanglement between the atomic subsystem and the field one. We note incidentally that, compared with the entropy, the calculations of the linear entropy are easier and convey essentially the same information.

From another point of view, we want to explore the quantum $\leftrightarrow$ classical connection and possible differences in the decoherence process in integrable and nonintegrable situations. For these reasons, we choose conditions which can be mimicked over the classical phase space. At this point a distinction is noteworthy between the here-called ''quasiclassical'' treatment of the JCM [10]—where one treats the atomic part quantum mechanically and the field one classically—and our ''semiclassical'' treatment—where the classical limit is taken for both atomic and field quantum subsystems. In the latter context, theoretical investigations using several models suggest that systems which are chaotic in the classical limit decohere rapidly [11,12]. This is also true for the *N*-atom JCM [13] in its nonintegrable version [14]. Moreover, for the specific case of Ref. [14], a connection between the entanglement process and the associated classical structures has been investigated. One of the main results of that investigation is the presence of some sensitivity to where in the classical phase space one places the center of the initial quantum coherent wave packets. It is now a well accepted fact that the decoherence rate is larger for chaotic systems than for integrable ones. We argue here that this belief that the fastest decoherence is to be attributed to chaotic regimes can be misleading in some cases. Such a phenomenon is particularly conspicuous when a (smaller) subsystem under observation has a finite Hilbert space, a (larger) subsystem coupled to it does not have such a restriction, and the global system is prepared in a state with mean energy larger than the amount allowed for the smaller subsystem. A phase-space description then reveals the presence of a ''border'' associated with the degrees of freedom of the smaller subsystem. In such a situation, initial conditions that drive the classical motion to the proximities of this border lead to decoherence rates strikingly similar to those of typical chaotic situations, even if the system is completely integrable. Moreover, another interesting phenomenon related to this is shown to occur: the proximity of the border tends to delocalize the wave packet, whereas for times when the dynamical evolution dictates a departure from the border there is a clear tendency to relocalize the wave packet in the sense that it recovers quantum coherence. These results are shown by comparing the classical and quantum description of the *N*-atom Jaynes-Cummings model, whose experimental realization is feasible in cavity QED setups. We present arguments according to which the experimental realization of this model would permit the effects here presented to be observable with $N\approx 20$ atoms, provided they could be prepared in a coherent state [15].

A measure of the entanglement between two subsystems is given by the linear entropy or idempotency defect (or degree of purity) of the subsystem of interest, say subsystem a [16,17]. This can be calculated after evaluating the reduced density operator for the corresponding subsystem

$$\rho_a(t)=\mathrm{Tr}_f\{|\psi(t)\rangle\langle\psi(t)|\}, \tag{1}$$

and then tracing over its variables to get

$$\delta(t)=1-\mathrm{Tr}_a\{\rho_a^2(t)\}. \tag{2}$$

This quantity describes the degree of purity of the subsystem in a scale from zero (pure state) to one (statistical mixture). In these equations $|\psi(t)\rangle$ is the quantum state of the full system, which in the present work is composed of the N-atom (spin) degree of freedom denoted by the index a, and of the field (bosonic) degree of freedom denoted by f.

The linear entropy $\delta(t)$ will be calculated as a function of time for some initial conditions to be chosen from the corresponding classical phase space: namely, a minimum uncertainty wave packet in spin (atom) and oscillator (field) bases is constructed as

$$|\psi(0)\rangle=|w\rangle\otimes|v\rangle\equiv|wv\rangle, \tag{3}$$

where $|w\rangle$ stands for an SU(2) coherent state and $|v\rangle$ for a bosonic one [18],

$$|w\rangle=(1+w\bar{w})^{-J}e^{wJ_+}|J,-J\rangle, \tag{4}$$

$$|v\rangle=e^{-v\bar{v}/2}e^{vb^+}|0\rangle \tag{5}$$

with

$$w=\frac{p_a+iq_a}{\sqrt{4J-(p_a^2+q_a^2)}}, \tag{6}$$

$$v=\frac{1}{\sqrt{2}}(p_f+iq_f), \tag{7}$$

$|J,-J\rangle$ being the state with spin J and $J_z=-J$,$|0\rangle$ being the harmonic oscillator ground state, and p_a,q_a,p_f,q_f describing the phase space as will be seen below. The full initial state $|\psi(0)\rangle$ is evolved by the quantum Hamiltonian

$$H=\hbar\omega_0 b^+b+\varepsilon J_z+\frac{G}{\sqrt{2J}}(bJ_++b^+J_-)+\frac{G'}{\sqrt{2J}}(b^+J_++bJ_-), \tag{8}$$

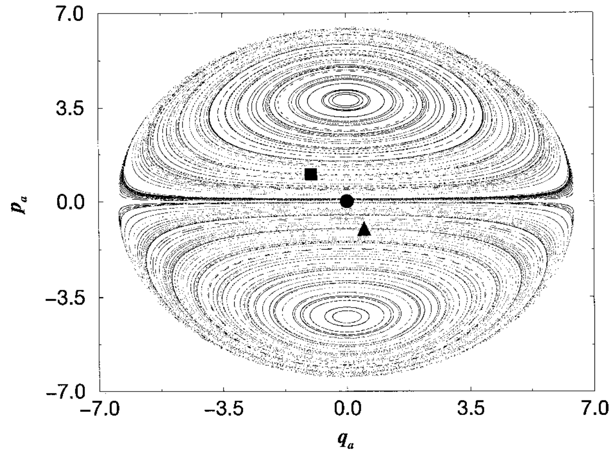

FIG. 1. Poincaré section for the spin degree of freedom (section with $q_f=0.0$ and $p_f>0.0$) in the resonant case ($\varepsilon=\omega_0=1$), energy $E=19.83$, $J=21/2$ in the integrable case ($G=0.5$ and $G'=0.0$). The marks represent the various choices for the centers of the coherent states: circle for i.c. ($q_a=0.01$, $p_a=0.01$, $q_f=0.0$, $pf=7.7834$), triangle for ($q_a=0.5$, $p_a=-1.0$, $q_f=0.0$, $pf=8.2160$), and square for ($q_a=-1.0$, $p_a=1.0$, $q_f=0.0$, $pf=7.1865$).

where the first term corresponds to the energy of the free single-mode quantized field with frequency ω_0, and the second term corresponds to the energy of the $N=2J$ atoms with energy separation $\hbar\varepsilon$ ($\hbar=1$ hereafter). The last two terms correspond to the interaction energy between the atomic system and the single-mode field. The second of these terms is the one responsible for the nonintegrability of the model; here we shall set it equal to zero and work within the so-called rotating-wave approximation.

The classical Hamiltonian corresponding to Eq. (8) can be obtained by a standard procedure using the coherent states (4) and (5) as $\langle wv|H|wv\rangle$ [19], and in this case it results in a nonlinearly coupled two degrees of freedom function

$$\mathcal{H}(q_a,p_a,q_f,p_f)=\frac{\omega_0}{2}(p_f^2+q_f^2)+\frac{\varepsilon}{2}(p_a^2+q_a^2)-\varepsilon J+\frac{\sqrt{4J-(p_a^2+q_a^2)}}{\sqrt{4J}}(G_+p_ap_f+G_-q_aq_f), \tag{9}$$

where $G_\pm=G\pm G'$. The integrable situation corresponds to $G'=0$.

A Poincaré section of the integrable case is shown in Fig. 1 for the value $J=N/2=21/2$, which corresponds to a semiclassical regime [20]. Note that there exists a border associated with the spin degree of freedom, so the initial state can be prepared with total energy sufficiently high as to allow the classical variables q_a and p_a to reach this border of the phase space. Note also the presence of a separatrix of motion at the line $p_a=0$. The filled symbols indicated in the Poincaré section represent the centers of the quantum coherent states evolved by Hamiltonian (8). We have chosen one of the initial conditions (i.c.) very close to the separatrix of motion, whereas the other two not so close to it. The idempotency

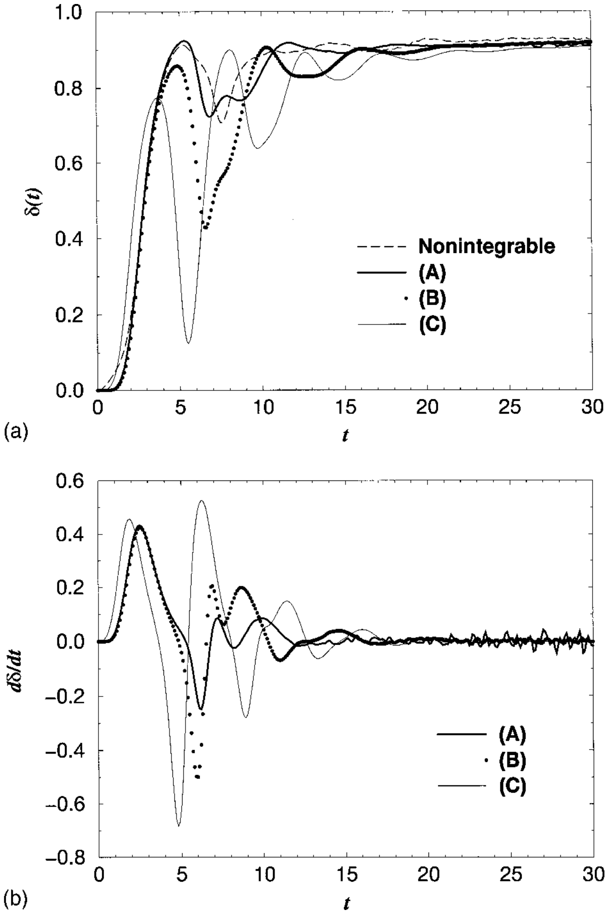

FIG. 2. (a) Linear entropy $\delta(t)$ as a function of time for the same parameters as in Fig. 1. The letters correspond to the i.c.'s chosen in the Poincaré section: the case (A) is for the circle, (B) is for the triangle, and (C) is for the square. The dashed curve represents the nonintegrable case ($G=0.5$ and $G'=0.2$) with the same energy and i.c. ($q_a=0.5, p_a=-1.0, q_f=0.0, pf=8.4280$); (b) rate of change of $\delta(t)$ as a function of time for the cases shown in (a).

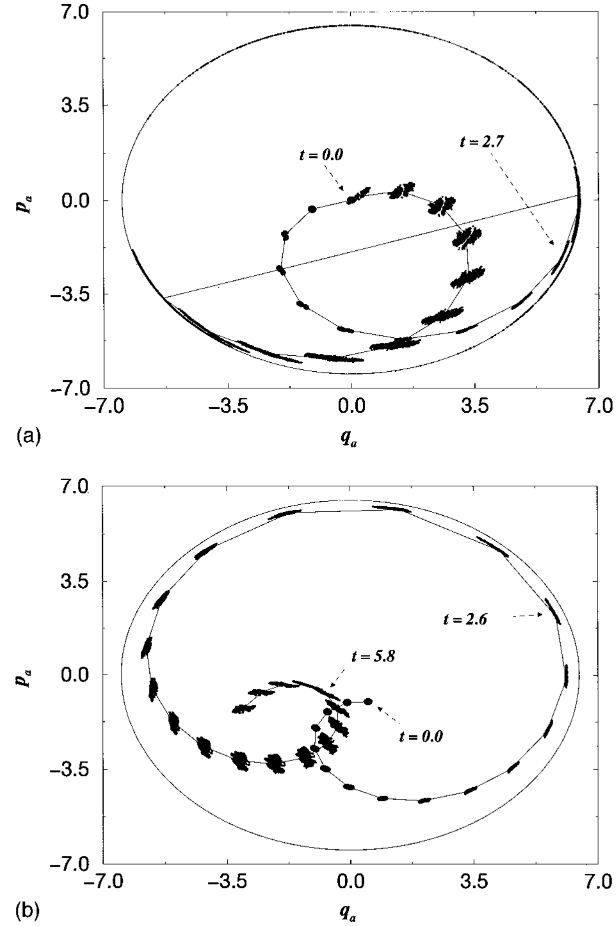

FIG. 3. Projection on the plane (q_a, p_a) of the time evolution for an ensemble of i.c.'s: (a) ensemble centered in the circle in Fig. 1, with $\Delta t=0.3$; (b) ensemble centered in the triangle, with $\Delta t=0.2$. The line follows the time evolution of the initial center of the ensemble. Also shown is the border of phase space for the spin degree of freedom.

defect $\delta(t)$ for these i.c.'s is shown in Fig. 2(a) as well as one dashed curve for the *nonintegrable* case ($G=0.5, G'=0.2$) with the i.c. centered in a chaotic region. Also shown in Fig. 2(b) are the corresponding increase rates for $\delta(t)$.

The idempotency defect for the integrable case with the i.c. closest to the separatrix of motion (thick solid curve) can be seen to increase as fast as the dashed curve, when reaching the value it takes at long times. The other two cases (dotted and thin solid curves) also show quite a fast increase at short times. It is rather unexpected that the short time behavior of the thick solid curve is amazingly similar to the dashed one which represents the evolution of a wave packet centered at a chaotic region. In order to understand this result, we study the classical counterpart of this effect. In Fig. 3 we plot the classical time evolution of ensembles of trajectories centered at the i.c.'s shown in Fig. 1, the time evolution of the patch being plotted at time intervals Δt. Clearly as the classical patch approaches the border of the spin phase space, it becomes very much spread over a large portion of phase space, which quantum mechanically corresponds to a large coherence-loss rate. When finally the border is reached, as is the case in Fig. 3(a), the patch is spread all over the circle representing the border of the spin phase space, which we have called the ''border effect'' in its classical version. As the figures show, the closer the separatrix is to the initial condition, the more pronounced is the effect. The initially coherent quantum wave packet whose center coincides with the initial classical patch can be imagined to follow roughly such a motion and lose coherence rapidly during this spreading on the border. In fact, the centers of Husimi distributions for these wave packets evolve in essentially the same manner as the classical patch [21]. Moreover, calculating the Lyapunov exponent for these regular trajectories confirms that near the separatrix, where exponential separation is really experienced, they are positive and finite. This ''border effect'' thus suggests that it is possible to exist in completely integrable systems decoherence processes which can be as fast as the chaotic ones. It suggests on the other hand that our previous result [14] showing that chaotic initial conditions lead to faster decoherence could possibly be seen, in a time scale of large decoherence, as chaos acting in the sense of driving the motion always to the proximities of the border whereas not all regular initial conditions do so (some of them remain far from the border). Detailed investigation in these directions will be presented elsewhere [22].

Also remarkable asre the oscillations in $\delta(t)$ which, as we comment on below, are not related to the fast Rabi oscillations. Note that large oscillations are accomplished before the plateau is reached in cases (B) and (C) indicating the presence of a mechanism through which purity is partially recovered in time. One can compare the time values indicated in Fig. 3(a) with the corresponding increase rate in $\delta(t)$ in Fig. 2(b) and see that the maximum rate occurs when the classical packet is close to the border. The classical patch is maximally delocalized by this time $(t\approx3.0)$. However, a few Δt's later the patch regains much of its original localization. This delocalization ↔ localization phenomenon has a quantum-mechanical counterpart, namely the oscillation in the linear entropy $\delta(t)$ occurs at the same time as the classical patch leaves the border region and tends to recover the initial (localized) shape. Indeed, following the dotted curve in Fig. 2(b) and its corresponding classical patch evolution in Fig. 3(b), one can see the patch approaching the border of the spin phase space around $t\approx2.6$, where the maximum decoherence rate is reached. Later on, the patch deviates to the central region and around $t\approx5.8$ the decoherence rate in Fig. 2(b) is at a minimum. The same pattern is found in connection with the case (C) shown in Fig. 2(a). This localization ↔ delocalization effect is more marked for initial conditions slightly off the separatrix because in these cases the oscillations in $\delta(t)$ have larger amplitudes, and in all cases examined they are related to this phenomenon, both qualitatively as well as quantitatively.

Before concluding, we comment on the absences of a revival region for the Rabi oscillations, and of the recovering of purity by the quantum state at half the revival time, a situation seen in our figures as a plateau attained by the linear entropy $\delta(t)$. As shown by Knight and Shore [23] and by Kudryavtsev *et al.* [24], the possibility of recovering purity at long times is smaller the greater the number N of atoms is, being also highly sensitive to initial conditions. Recovering of purity at the collapse interval is usually seen for systems with a small number of atoms (or atomic levels) and for special initial conditions (the atomic part is usually chosen either as the most excited atomic level or as the ground one). As an example, for one two-level atom in the cavity $(N=2J=1)$, the system shows itself at half the revival time as a Schrödinger-cat state. Later on, at the revival interval of time, one sees the usual rapid Rabi oscillations. In our case, we have $J=21/2$ and the initial atomic state is also in a coherent state, with occupation distributions for all the $2J+1$ atomic levels. This choice for the initial condition is guided by the classical phase space—from which we mimic the initial quantum state—and also by the fixed energy value, which links atomic initial conditions (q_a,p_a) and field ones (q_f,p_f). Starting in this situation, at sufficiently long times the atomic state tends to be uniformly distributed over the $2J+1$ levels and $\mathrm{Tr}_a\{\rho_a^2\}=1/(2J+1)$, indicating the plateau seen in the figures. On the other hand, if we increase the value of N, starting always with the atom in the most excited level, we see that the Rabi oscillations are still present but with smaller amplitudes. Thus we conclude that, besides the high value of N, the nonappearance in our cases of such Rabi oscillations has to do with our atomic initial conditions being also coherent [25]. This situation is consistent with one's expectation that Rabi oscillations should not be seen in the classical limit. All this summed up prevents the appearance here of any catlike behavior with appreciable recovering of purity at long times, but a final plateau instead.

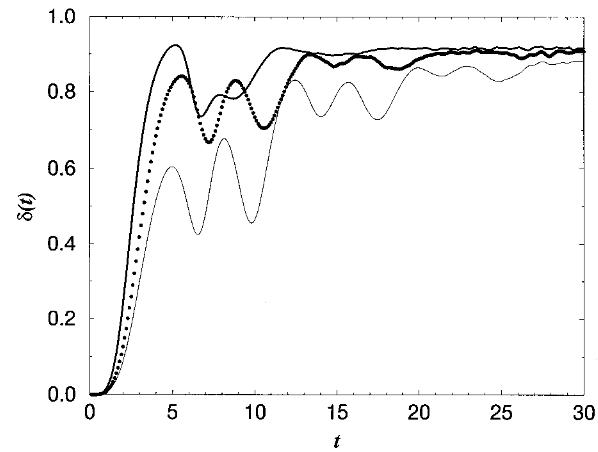

FIG. 4. Linear entropy $\delta(t)$ as a function of time for the same parameters as in Fig. 1 and various values of energy: thick solid curve for $E=19.83$, dotted curve for $E=8.5$, and thin solid curve for $E=4.5$.

In conclusion, we have found a new effect on the decoherence of a subsystem with finite Hilbert space coupled to a subsystem with infinite Hilbert space, when the mean energy of the initial quantum wave packet is much larger than the energy allowed for the ''small'' subsystem. In this case, the energy border has a striking effect on the linear entropy, especially in the short time behavior, when the border is ''felt'' in its time evolution. For energies such that the border is less effective, the Lyapunov exponents go to zero as they should. The quantum counterpart of this situation is seen in Fig. 4, where we fix the center of the quantum initial wave packet to be close to the separatrix (therefore testing the border for high enough energies) and vary the energy in such a way that the border is more (or less) effective. Note that for the highest energy we have the largest increase in $\delta(t)$ for short times and this effect is attenuated as the energy becomes smaller. Another interesting feature found in the present investigation is the localization ↔ delocalization phenomenon related to the proximity of the border followed by a relocalization which reflects a quantum coherence recovery.

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