Thermodynamics of a two-dimensional unbounded self-gravitating system

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(Received 8 June 1999)

The thermodynamics of a two-dimensional self-gravitating system occupying the whole plane is considered in the mean-field approximation. First, it is proven that, if the number N of particles and the total energy E are imposed as the only external constraints, then the entropy admits the least upper bound $S^{+}(N,E)=2E/N+N\ln(e\pi^2)$ (in appropriate units). Moreover, there does exist a unique state of maximum entropy, which is characterized by a Maxwellian distribution function with a temperature $T=N/2$ independent of E. Next, it is shown that, if the total angular momentum J is imposed as a further constraint, the largest possible value of the entropy does not change, and there is no admissible state of maximum entropy, but in the case $J=0$. Finally, some inequalities satisfied by a class of so-called H functions and related generalized entropies are given. [S1063-651X(99)02011-5]

PACS number(s): 05.20.Gg, 05.20.Dd, 95.30.Sf

I. INTRODUCTION

The problem of the mean-field thermodynamics of a two-dimensional self-gravitating system confined inside a bounded plane domain D has been considered by Katz and Lynden-Bell [1] and more recently by Aly [2] (paper I hereafter). In this last paper, it was shown in particular that the Boltzmann entropy of an isolated system constituted of N particles and having an energy E, is bounded from above by a number $S^*(N,E)$ independent of D. Moreover, it was proven that, at least when D is a disk, the maximum of the entropy is reached by only one distribution function, which is naturally of Maxwellian type. Then the system admits a unique thermodynamically stable equilibrium state.

The main purpose of this paper is twofold. First, we want to extend the results quoted above to an unconfined—i.e., occupying the whole plane $\mathbf{R}^2$—isolated system of N particles having a prescribed energy E. Secondly, we want to discuss how the equilibrium state is affected when the total angular momentum J is imposed as a further constraint. Fixing the value of J is certainly natural, as angular momentum is a constant of motion for an unconfined system whose evolution is governed by any one of the classical kinetic equations (J is also conserved when the system is confined inside a disk; in that case, the constraint $J=0$ was assumed in Ref. [1] and in paper I).

The paper is organized as follows. The problems we want to solve are given a precise mathematical formulation in Sec. II. The existence of an entropy maximum and that of an entropy maximizer are discussed in Secs. III and IV when N and E, and N, E, and J, respectively, are imposed to take fixed values. Finally, the bound derived in Sec. III for the Boltzmann entropy is shown in Sec. V to imply upper bounds for a class of so-called H functions and related generalized entropies [3,4].

II. STATEMENT OF THE PROBLEMS

A. Assumptions and important physical quantities

We consider a two-dimensional self-gravitating system constituted of particles of mass m occupying the whole plane $\mathbf{R}^2$. We work with dimensionless variables, the units of length, velocity, and mass being taken to be, respectively, L_0 (an arbitrarily chosen quantity), $V_0:=(Gm)^{1/2}$ (with G the two-dimensional gravitational constant), and $M_0:=m$.

A state of the system is assumed to be entirely described by the one-particle distribution function $f(\mathbf{w})$ [unit $f_0:=(L_0V_0)^{-2}$] defined over the phase space $\mathbf{R}^4=\{\mathbf{w}=(\mathbf{x},\mathbf{v})\}$, where $\mathbf{x}$ and $\mathbf{v}$ denote the position and the velocity of a particle in a Galilean frame of origin O. $f(\mathbf{w})$ gives the density of particles in the phase-space, while

$$n(\mathbf{x})=\int f(\mathbf{x},\mathbf{v})d\mathbf{v} \tag{1}$$

gives the particle density in the physical space [an integral with respect to $d\mathbf{w}$ (respectively, $d\mathbf{x}$, $d\mathbf{v}$) is taken over the whole $\mathbf{R}^4$ (respectively, $\mathbf{R}^2$)]. Without loss of generality, we impose O to coincide with the center of mass of the system, i.e., we assume that

$$\int n(\mathbf{x})\mathbf{x}\, d\mathbf{x}=0. \tag{2}$$

f generates the mean gravitational potential (unit $\Phi_0:=V_O^2=Gm$)

$$\Phi(\mathbf{x}):=2\int \ln|\mathbf{x}-\mathbf{x}'|f(\mathbf{w}')d\mathbf{w}'=2\int \ln|\mathbf{x}-\mathbf{x}'|n(\mathbf{x}')d\mathbf{x}'. \tag{3}$$

A choice of gauge is implicit in this relation: the potential created by a particle is taken to vanish at a unit distance. Of course, Φ satisfies Poisson equation

$$\nabla^2\Phi=4\pi n=4\pi\int f(\mathbf{x},\mathbf{v})d\mathbf{v}. \tag{4}$$

From f, we can compute the following global quantities:
(1) *Number of particles*:

$$N[f]:=\int f(\mathbf{w})d\mathbf{w}=\int n(\mathbf{x})d\mathbf{x}. \tag{5}$$

(2) *Energy* (unit $E_0:=M_0V_0^2=Gm^2$):

$$E[f]:=\frac{1}{2}\int v^2 f(\mathbf{w})d\mathbf{w}+\frac{1}{2}\int f(\mathbf{w})\Phi(\mathbf{x})d\mathbf{w} \tag{6}$$

$$=\frac{1}{2}\int v^2 f(\mathbf{w})d\mathbf{w}+\frac{1}{2}\int n(\mathbf{x})\Phi(\mathbf{x})d\mathbf{x}, \tag{7}$$

where the first term in each of the two last members represents the kinetic energy $E_c[f]$ and the second one the potential energy $E_p[f]=E_p[n]$.

(3) *Angular momentum with respect to the center of mass* (unit $M_0V_0L_0$):

$$J[f]=\int r v_\phi f(\mathbf{w})d\mathbf{w}, \tag{8}$$

where we have used polar coordinates (r,ϕ).

(4) *Boltzmann entropy* (unit $S_0:=k_b$, Boltzmann constant):

$$S[f]:=-\int f(\mathbf{w})\ln[f(\mathbf{w})]d\mathbf{w}. \tag{9}$$

B. The problems

We are interested in this paper in determining the stable equilibria—i.e., the maximum entropy states—to which an isolated collisional system should relax when it evolves by conserving its number N of particles, its energy E and possibly its angular momentum J. We thus need to consider the two following problems.

(i) Consider the set $\mathcal{G}(N,E)$ of all the distribution functions which have well-defined kinetic energy, potential energy and entropy, and a given number N of particles and a given energy E. Is there among them one for which the entropy is a global maximum?

(ii) Consider the subset $\mathcal{G}(N,E,J)$ of $\mathcal{G}(N,E)$ constituted of all the functions having a well defined angular momentum equal to J. Is there among them one for which the entropy is a global maximum?

III. UPPER BOUND ON THE ENTROPY AND ENTROPY MAXIMIZER IN $\mathcal{G}(N,E)$

A. The case of a bounded system

For a system confined in a bounded domain $D\subset\mathbf{R}^2$, it is proven in paper I that the entropy is bounded from above in the set $\mathcal{G}(D,N,E)$ containing the functions of $\mathcal{G}(N,E)$ vanishing for $\mathbf{x}\notin D$, with

$$S^+(D,N,E)\underset{f\in\mathcal{G}(D,N,E)}{:=\sup} S[f]\leqslant\frac{2E}{N}+N\ln(e\pi^2)=:S^*(N,E). \tag{10}$$

[Eq. (10) is given in paper I as Eq. (5.9), but without proof. Owing to its importance here, we indicate in the Appendix how it can be derived].

Moreover, it is shown therein that there is a unique entropy maximizer f_R^+ when D is a disk of radius R and area $V=\pi R^2$, f_R^+ and its potential Φ_R^+ being given, respectively, by

$$f_R^+=\frac{N\beta}{\pi^2R^2}\frac{(2-\beta N)}{[(2-\beta N)+\beta N r^2/R^2]^2}e^{-\beta v^2/2} \tag{11}$$

and

$$\Phi_R^+=2N\ln R+\frac{2}{\beta}\ln\left[\frac{2-\beta N}{2}+\frac{\beta N}{2}\frac{r^2}{R^2}\right], \tag{12}$$

where $\beta(N,E,V)<2/N$ is the unique solution to the equation

$$E=\frac{N^2}{2}\left[\ln\frac{V}{\pi}+\frac{4}{\beta N}+\left(\frac{2}{\beta N}\right)^2\ln\left(1-\frac{\beta N}{2}\right)\right]. \tag{13}$$

B. Upper bound on the entropy

Inequality (10) has the remarkable property that its right hand side (RHS) does not depend on the particular domain D under consideration. This suggests that the entropy of an unconfined plane system having N particles and an energy E should also admit $S^*(N,E)$ as an upper bound. That this is the case can be seen as follows.

Consider an arbitrary function f in $\mathcal{G}(N,E)$, and define $f_k=f$ for $r\leqslant k$, $k\in\mathbf{N}$, and $f_k=0$ for $k<r$. We can apply Eq. (10) to f_k, which characterizes a system having N_k ($\leqslant N$) particles, and well defined energy and entropy. Then we get the seeked inequality

$$S[f]\leqslant S^+(N,E)\underset{\mathcal{G}(N,E)}{:=\sup} S[f]\leqslant\frac{2E}{N}+N\ln(e\pi^2) \tag{14}$$

by taking the limit $k\to\infty$ and by applying a standard convergence theorem.

C. Existence of an entropy maximizer

Next we show the existence of an entropy maximizer, i.e., of a function f^+ such that

$$S[f^+]=S^+(N,E). \tag{15}$$

For that, we consider the formal Euler-Lagrange equation associated with our maximization problem and construct a particular solution to it belonging to $\mathcal{G}(N,E)$. Then we check *a posteriori* that the latter satisfies Eq. (15).

A solution to Euler-Lagrange equation is of the standard form

$$f^+=e^{-\alpha-\beta(v^2/2+\Phi^+)}, \tag{16}$$

where α and β are Lagrange multipliers relative to the particles number and energy constraints, respectively, and the potential Φ^+ of f^+ satisfies Poisson equation

$$\nabla^2\Phi^+=\frac{8\pi^2}{\beta}e^{-\alpha-\beta\Phi^+}. \tag{17}$$

The latter can be solved easily if we restrict our attention to circularly symmetric solutions, in which case we get the well known expression

$$\beta\Phi^{+}=2\ln\left(\lambda^{2}+\frac{\pi^{2}e^{-\alpha}}{\lambda^{2}}r^{2}\right), \tag{18}$$

where $r=|\mathbf{x}|$ and λ^2 is an arbitrary positive constant.

The values of the three parameters α, β, and λ^2 can be fixed in such a way that f^+ belongs to $\mathcal{G}(N,E)$ and that the gauge condition of Sec. II is fulfilled by Φ^+. Indeed we find the following.

The condition $N[f]=N$ gives

$$N=\frac{\lambda^{2}}{\pi^{2}}\int\frac{e^{-\beta v^{2}/2}}{(r^{2}+\lambda^{2})^{2}}d\mathbf{w}=\frac{2}{\beta}, \tag{19}$$

whence

$$\beta=\frac{2}{N} \tag{20}$$

and

$$\Phi^{+}=N\ln(\lambda^{2}+r^{2}), \tag{21}$$

$$f^{+}=\frac{\lambda^{2}}{\pi^{2}}\frac{e^{-v^{2}/N}}{(\lambda^{2}+r^{2})^{2}}. \tag{22}$$

After some straightforward algebra, the kinetic, potential and total energies of the function f^+ given by Eq. (22) are found to be given, respectively, by

$$E_c[f^{+}]=\frac{N^{2}}{2}, \tag{23}$$

$$E_p[f^{+}]=\frac{N^{2}}{2}(1+\ln\lambda^{2}), \tag{24}$$

$$E[f^{+}]=E_c[f]+E_p[f]=\frac{N^{2}}{2}(2+\ln\lambda^{2}). \tag{25}$$

Then the condition $E[f^+]=E$ is satisfied by taking

$$\lambda^{2}=e^{2(E-N^{2})/N^{2}}. \tag{26}$$

$\beta\Phi^+$ has the asymptotic behavior

$$\beta\Phi^{+}\underset{r\to\infty}{\sim}4\ln r+2\ln\frac{\pi^{2}e^{-\alpha}}{\lambda^{2}}+\frac{2\lambda^{4}}{\pi^{2}e^{-\alpha}r^{2}}+\cdots. \tag{27}$$

Our gauge condition on the gravitational potential imposes the vanishing of the constant term, which gives

$$\pi^{2}e^{-\alpha}=\lambda^{2}. \tag{28}$$

Therefore the Euler-Lagrange equation admits a unique circularly symmetric solution in $\mathcal{G}(N,E)$. It is given by

$$f^{+}=\frac{e^{2(E-N^{2})/N^{2}}}{\pi^{2}}\frac{e^{-v^{2}/N}}{(e^{2(E-N^{2})/N^{2}}+r^{2})^{2}} \tag{29}$$

and generates the potential

$$\Phi^{+}=N\ln(e^{2(E-N^{2})/N^{2}}+r^{2}). \tag{30}$$

We remark that (i) Our construction shows that the set $\mathcal{G}(N,E)$ is nonempty whichever be the values of N and E. (ii) f^+ and Φ^+ are the limits when $R\to\infty$ of the functions f_R^+ and Φ_R^+ given by Eqs. (11) and (12), respectively. For E and N fixed, Eq. (13) shows indeed that

$$\lim_{R\to\infty}(\beta N)=2, \tag{31}$$

$$\lim_{R\to\infty}[(2-\beta N)\ln R]=0, \tag{32}$$

$$\lim_{R\to\infty}[(2-\beta N)R^{2}/2]=e^{2(E-N^{2})/N^{2}}, \tag{33}$$

which implies for $R\to\infty$

$$\begin{aligned}\Phi_R^{+}&=N\left[-\frac{(2-\beta N)\ln R}{\beta N}+\frac{2}{\beta N}\ln\left(\frac{2-\beta N}{2}R^{2}+\frac{\beta N}{2}r^{2}\right)\right]\\&\to N\ln(e^{2(E-N^{2})/N^{2}}+r^{2})=\Phi^{+},\\ f_R^{+}&=\frac{\beta N}{2\pi^{2}}\frac{(2-\beta N)R^{2}/2}{[(2-\beta N)R^{2}/2+(\beta N/2)r^{2}]^{2}}e^{-\beta v^{2}/2}\\&\to\frac{1}{\pi^{2}}\frac{e^{2(E-N^{2})/N^{2}}}{e^{2(E-N^{2})/N^{2}}+r^{2}}e^{-v^{2}/N}=f^{+}.\end{aligned} \tag{34}$$

Consider now the entropy of f^+. A short calculation gives

$$S[f^{+}]=\frac{2E}{N}+N\ln(e\pi^{2}). \tag{35}$$

Comparing Eqs. (35) and (14), we see at once that f^+ has the largest possible value allowed by the latter equation. Therefore,

$$S[f^{+}]=S^{+}(N,E)=\frac{2E}{N}+N\ln(e\pi^{2}), \tag{36}$$

and f^+ maximizes the entropy over $\mathcal{G}(N,E)$.

D. Uniqueness of the entropy maximizer

The argument above gives a particular maximizer f^+, and the next question which needs to be addressed is that of the existence of other entropy maximizers. To discuss this problem, we first note that any maximizer needs to satisfy the Euler-Lagrange equation of our problem (see the discussion in Ref. [5]). Therefore, it is clear from the arguments above that f^+ is the only circularly symmetric maximizer. On the other hand, nonsymmetric maximizers do not exist. This can be proven by noting that the associated potential should be a solution to Eq. (17) and by applying to the latter the following result [6,7]: Any solution u to the equation

$$\nabla^2 u = e^{-u} \quad \text{in} \quad \mathbf{R}^2 \tag{37}$$

which satisfies the condition

$$\int e^{-u}\, d\mathbf{r} < \infty \tag{38}$$

is necessarily circularly symmetric.

To summarize this section, we can assert that, among the states of a system having prescribed values of the number of particles and energy, there is one and only one which maximizes entropy. The associated distribution function, given by Eq. (29), is a Maxwellian at the temperature

$$T = \beta^{-1} = N/2 \tag{39}$$

or, in dimensional form,

$$T = \frac{Gm^2 N}{2k_b}. \tag{40}$$

It is remarkable that this value is independent of the energy.

It should be noted that the existence of a unique temperature at which an equilibrium can exist can also be established by using virial-type relations (see paper I, Appendix C, for an expression of the virial theorem for a gravitational system confined in a bounded domain, which strongly suggests at first sight that the relation $2T = N$ should hold indeed for a system occupying the whole plane; and Ref. [7] for a virial-type relation valid for an unconfined two-dimensional non-neutral plasma—a system which has strong formal connections with the one considered here).

IV. UPPER BOUND ON THE ENTROPY AND NONEXISTENCE OF AN ENTROPY MAXIMIZER IN $\mathcal{G}(N,E,J)$

We now reconsider the previous problem in the set of admissible functions $\mathcal{G}(N,E,J)$. As

$$\mathcal{G}(N,E,J) \subset \mathcal{G}(N,E), \tag{41}$$

entropy is also bounded from above on $\mathcal{G}(N,E,J)$, with

$$S^+(N,E,J) := \sup_{\mathcal{G}(N,E,J)} S[f] \leqslant S^+(N,E) := \sup_{\mathcal{G}(N,E)} S[f]. \tag{42}$$

Let us determine the value of the least upper bound $S^+(N,E,J)$.

We first introduce the particular distribution function f_R defined as follows.

(1) f_R has the same number density n as the function given by Eq. (22), where we take λ to be a free parameter; therefore, f_R has N particles and a potential energy [see Eq. (24)]

$$E_p[f_R] = \frac{N^2}{2}[1 + \ln \lambda^2], \tag{43}$$

and the particles contained inside the disk $B_R := \{r < R\}$ of radius R have a moment of inertia

$$I_R[f_R] := \int_{B_R \times \mathbf{R}^2} r^2 f_R\, d\mathbf{w} = N\lambda^2 \left\{ \ln \frac{\lambda^2 + R^2}{\lambda^2} - \frac{R^2}{\lambda^2 + R^2} \right\} =: I(R,\lambda) \tag{44}$$

with respect to O.

(2) Outside B_R, f_R is a Maxwellian with an inverse temperature $\beta > 0$. Inside B_R, it is a Maxwellian with the same β when viewed in a frame rotating at the angular velocity

$$\Omega := \frac{J}{I(R,\lambda)}. \tag{45}$$

Setting $\Omega(r) := \Omega\Theta(R - r)$ (with Θ the usual Heaviside step-function) we thus have

$$f_R(r, v_r, v_\phi) := \frac{N\beta\lambda^2}{2\pi^2} \frac{1}{(r^2 + \lambda^2)^2} e^{-\beta\{v_r^2 + [v_\phi - r\Omega(r)]^2\}/2}. \tag{46}$$

Then the angular momentum of f_R is equal to J, and its kinetic energy is given by

$$E_c[f_R] = \frac{N}{\beta} + \frac{J^2}{2I(R,\lambda)}. \tag{47}$$

(3) The values of λ and β are fixed in such a way that

$$E[f_R] = \frac{N}{\beta} + \frac{J^2}{2I_R[f_R]} + \frac{N^2}{2}[1 + \ln \lambda^2] = E. \tag{48}$$

As for λ, we take the value given by Eq. (26), which is independant of R. Thus we need to choose β in such a way that

$$\frac{\beta N}{2} = \frac{N^2}{N^2 - J^2/I(R,\lambda)}, \tag{49}$$

which is possible if R is large enough for having $N^2 > J^2/I(R,\lambda)$ [note that $I(R,\lambda) \to \infty$ when $R \to \infty$].

Our function f_R thus belongs to $\mathcal{G}(N,E,J)$—which appears to be nonempty for any choices of N, E, and J. Its entropy is related to that of the maximizer f^+ in $\mathcal{G}(N,E)$ by

$$S[f_R] = S[f^+] - N \ln \frac{\beta N}{2} = S^+(N,E) - N \ln \frac{N^2}{N^2 - J^2/I(R,\lambda)} \leqslant S^+(N,E,J) \leqslant S^+(N,E), \tag{50}$$

where we have made use of Eq. (42) to write the last inequality. If we take the limit $R \to \infty$, we thus have

$$\lim_{R \to \infty} S[f_R] = S^+(N,E) \leqslant S^+(N,E,J) \leqslant S^+(N,E), \tag{51}$$

whence

$$S^+(N,E,J) = S^+(N,E). \tag{52}$$

Adding the angular momentum constraint does not change the least upper bound on the entropy.

Let us then suppose that there is a function $f_1^+ \in \mathcal{G}(N,E,J)$ which maximizes the entropy. The result above implies that it also maximizes the entropy in $\mathcal{G}(N,E)$. But we know by the results of Sec. III that there is only one maximizer in $\mathcal{G}(N,E)$, f^+, and the latter has zero angular momentum. Then, but for the case where $J=0$, the problem of maximizing entropy in $\mathcal{G}(N,E,J)$ has no solutions belonging to that set. Clearly, what happens here is that the angular momentum constraint is ''lost at infinity'' in the process of entropy maximization.

The nonexistence of an entropy maximizer—but not the equality (52)—can also be deduced by the following argument. If a maximizer f_1^+ existed in $\mathcal{G}(N,E,J)$, it would be related to its potential Φ_1^+ by

$$f_1^+ = e^{-\alpha-\beta[v_r^2/2+(v_\phi-r\omega)^2/2+\Phi_1^+-\omega^2r^2/2]}, \tag{53}$$

which can be derived by the same standard techniques as Eq. (16), α, β, and $\beta\omega$ being the Lagrange multipliers associated with the N, E, and J constraints. But clearly such an equation cannot have any solution of finite mass owing to the exponentially growing factor $e^{\beta\omega^2 r^2/2}$ present in its right-hand side (we have $\beta>0$ as a consequence of the kinetic energy being positive).

Note that an argument quite similar to the one reported here, although not presented in a completely formalized way, has been previously applied to three-dimensional systems [8].

V. UPPER BOUNDS ON *H* FUNCTIONS AND GENERALIZED ENTROPIES

H functions have been introduced in gravitational physics by Tremaine *et al.* [9] as a useful tool for studying the phenomenon of violent relaxation suggested by Lynden-Bell [10]. For future reference, we show here that the upper bound on the entropy derived above implies at once the existence of a nontrivial upper bound on a large class of *H* functions of two-dimensional systems.

A. Definitions

If $C(f)$ is a convex function such that $C(0)=0$, the functional

$$H_C[f] := -\int C[f(\mathbf{w})]d\mathbf{w} \tag{54}$$

is called an ''*H* function'' [9]. For instance, the quantities

$$H_q[f] := -\int f^q\, d\mathbf{w} \tag{55}$$

are *H* functions for any real number $q>1$ $[C(f)=f^q]$. Related to them are the so-called Renyi's and Tsallis's q entropies, defined, respectively, by [3]

$$S_{rq} = -\frac{N}{q-1}\ln\frac{|H_q[f]|}{N} \tag{56}$$

and [4]

$$S_{tq} = -\frac{|H_q[f]|-N}{q-1}. \tag{57}$$

Both quantities reduce to Boltzmann entropy (9) when $q\to 1^+$:

$$\lim_{q\to 1^+} S_{rq}[f] = \lim_{q\to 1^+} S_{tq}[f] = -\int f\ln f\, d\mathbf{w} = S[f]. \tag{58}$$

B. Upper bounds on H_q, S_{rq}, S_{tq}

Using the inequality (valid for $x,y\geq 0$): $x\ln(x/y)\geq(x-y)$, which is an immediate consequence of the convexity of the function f defined by $f(x)=x\ln x$ for $x>0$ and $f(0)=0$, we obtain

$$\int \frac{f}{N}\ln\frac{f/N}{f^q/|H_q[f]|}d\mathbf{w}\geq 0. \tag{59}$$

Combining this relation with Eq. (14), we obtain for any f in $\mathcal{G}(N,E)$

$$S_{rq}[f] = -\frac{N}{q-1}\ln\frac{|H_q[f]|}{N}\leq S[f]\leq\frac{2E}{N}+N\ln(e\pi^2), \tag{60}$$

whence

$$H_q[f]\leq -\frac{N}{(e\pi^2)^{(q-1)}}e^{-2(q-1)E/N^2}<0 \tag{61}$$

and

$$S_{tq}[f]\leq -\frac{N}{q-1}\left[\frac{e^{-2(q-1)E/N^2}}{(e\pi^2)^{(q-1)}}-1\right]<\frac{N}{q-1}. \tag{62}$$

Then all the quantities H_q, S_{rq}, and S_{tq} admit upper bounds over $\mathcal{G}(N,E)$. It must be noted that the boundedness of H_q is nothing but surprising, as H_q is the integral of a nonpositive function. What is more important regarding this quantity is that it admits a strictly negative upper bound.

The bound on $H_q[f]$ implies at once that

$$H_C[f]\leq aH_q[f]\leq -\frac{aN}{(e\pi^2)^{(q-1)}}e^{-2(q-1)E/N^2}<0 \tag{63}$$

for any *H* function associated to a *C* such that, for some constants $a>0$ and $q>1$,

$$as^q\leq C(s) \quad \forall \quad s\geq 0. \tag{64}$$

This provides us with a nontrivial upper bound for a fairly large class of *H* functions.

VI. CONCLUSION

Let us summarize and briefly comment on the results which have been obtained in this paper. Boltzmann entropy is bounded from above over the set $\mathcal{G}(N,E)$ of all the distribution functions having a given number N of particles and a given energy E. The maximum of the entropy is reached for only one distribution function, which is a Maxwellian at a temperature $Gm^2N/2k_b$. It should be noted that a similar result holds true in exact statistical mechanics: in the microcanonical ensemble, the temperature of a system characterized by N, E, and $J=0$ can be shown indeed to be given by [11]

$$k_bT=\frac{N-3/2}{N-1}\frac{Gm^2N}{2}. \tag{65}$$

The argument leading to the previous conclusion also shows that there is no equilibrium state for a system in contact with a thermostat, but if the temperature of the latter takes the peculiar value recalled above. This is in contrast with the case of a confined system, for which equilibria were found in paper I to exist for any temperature $T>Gm^2N/2k_b$. But it is in accordance with the nonexistence of the statistical mechanics canonical ensemble for an unconfined system [11].

The upper bound on the entropy is not changed if a further constraint fixing the total angular momentum J of the system is imposed. When $J\neq 0$, there is no distribution function in $\mathcal{G}(N,E,J)$ maximizing the entropy. Here, there is a difference with the microcanonical approach of exact statistical mechanics, which leads to well defined results for a system having N, E, and J fixed [11].

Each H function H_q $(q>1)$ admits a strictly negative upper bound $H_q^+(N,E)$ over $\mathcal{G}(N,E)$. This result can be reexpressed in terms of upper bounds for Renyi's and Tsallis's q entropies S_{rq} and S_{tq}, which have proven to be useful quantities in statistical physics and information theory. It also naturally extends to all the H functions associated to a convex function C satisfying $C(f)\geqslant af^q$ for some constants $a>0$ and $q>1$.

APPENDIX: DERIVATION OF THE EXPLICIT UPPER BOUND

Let D be some arbitrary bounded domain. As shown in paper I, the entropy of an arbitrary function $f\in\mathcal{G}(D,N,E)$ is bounded by

$$S[f_R^+]=\beta E+N\left[(1-\beta N)\ln\frac{V}{\pi}+1-\ln\frac{\beta N}{2}-\left(1-\frac{2}{\beta N}\right)\ln\left(1-\frac{\beta N}{2}\right)\right], \tag{A1}$$

with R the radius of a disk having the same area as D and f_R^+ given by Eq. (11). Combining the latter equation with Eq. (13)—which determines $\beta\in]0,2/N[$ as a function of N and E—and setting

$$\theta:=\frac{2}{\beta N}\in]1,+\infty[, \tag{A2}$$

we obtain

$$\begin{aligned}S^+(D,N,E)&\leqslant S[f_R^+]\\&\leqslant\frac{2E}{N}+N\left[\ln(e\pi^2)+2+\ln\theta-2\theta-(\theta-1)^2\ln\left(1-\frac{1}{\theta}\right)\right]\\&=:\frac{2E}{N}+N[\ln(e\pi^2)+g(\theta)].\end{aligned} \tag{A3}$$

After a little algebra, we get

$$g'(\theta)=(\theta-1)\left[\frac{2-3\theta}{\theta(\theta-1)}-2\ln\left(1-\frac{1}{\theta}\right)\right]=:(\theta-1)h(\theta), \tag{A4}$$

$$\lim_{\theta\to\infty}h(\theta)=0, \tag{A5}$$

$$h'(\theta)=\frac{1+(\theta-1)^2}{\theta^2(\theta-1)^2}>0. \tag{A6}$$

$h(\theta)$, being an increasing function vanishing at infinity, is negative on $]1,+\infty[$. Then $g'(\theta)<0$, and $g(\theta)$ decreases monotonically, which implies

$$g(\theta)<\lim_{\theta\to 1^+}g(\theta)=0,\quad\forall\quad\theta\in]1,+\infty[. \tag{A7}$$

Equation (10) thus follows immediately by reinjecting this result into Eq. (A3).

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