

Influence of compressibility on scaling regimes of strongly anisotropic fully developed turbulence

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A statistical model of strongly anisotropic fully developed turbulence of the weakly compressible fluid is considered by means of the field theoretic renormalization group. The corrections due to compressibility to the infrared form of the kinetic energy spectrum have been calculated in the leading order in the Mach number expansion. Furthermore, in this approximation the validity of the Kolmogorov hypothesis on the independence of the dissipation length of velocity correlation functions in the inertial range has been proved.

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I. INTRODUCTION

One of the oldest open problems in theoretical physics is that of describing fully developed turbulence on the basis of a microscopic model. The latter is usually taken to be the stochastic Navier-Stokes equation subject to an external random force which mimics the injection of energy by the large-scale modes; see, e.g., Ref. [1]. The aim of the microscopic theory is to verify the basic principles of the celebrated Kolmogorov-Obukhov phenomenological theory [2], study deviations from this theory, and find the dependence of various Green functions (velocity correlation and response functions) on the times, distances, external (integral), and internal (viscous) turbulence scales. Most results are obtained within the framework of numerous semiphenomenological models which cannot be considered to be the basis for construction of a regular expansion in a certain small (at least formal) parameter, see Ref. [2].

One of the exceptions is provided by the renormalization-group (RG) method earlier applied successfully in the theory of critical behavior to explain the origin of critical scaling and calculate universal quantities (critical dimensions and scaling functions) in the form of ϵ expansions; see Ref. [3].

The RG was applied to the stochastic Navier-Stokes equation in Refs. [4–7]. For the isotropic homogeneous turbulence of incompressible viscous fluid, it allows one to prove the existence of the infrared (IR) scale invariance with exactly known “Kolmogorov” scaling dimensions and the independence of the correlation functions of the viscous scale (the second Kolmogorov hypothesis), and calculate a number of principal constants in a reasonable agreement with the experiment. The detailed exposition of the RG theory of turbulence and the bibliography can be found in the review paper [8].

As the model of isotropic incompressible fluid provides only a simplified description of real turbulent flows, it is interesting to generalize the model by taking into account anisotropy, compressibility, inhomogeneity, real geometry, and so on. In particular, in a number of papers the turbulence with the weak [9–11] and strong [12] uniaxial anisotropy has been studied. It was shown that, in the three-dimensional space, the IR scaling regime characteristic of the isotropic case survived also if the anisotropy were included (in the

language of the RG, this means that the corresponding fixed point remains IR stable).

In Refs. [13–16], the isotropic turbulence of compressible fluid was considered. The main difficulty is that the corresponding field theoretic model is not multiplicatively renormalizable, so that the RG technique is not directly applicable to it (for this reason, the results obtained in Ref. [13] cannot be considered reliable, see the discussion in Refs. [14,16]).

In Ref. [14], the problem of nonrenormalizability was solved in the frame of expansion procedure in small Mach number $Ma = v_c/c$ (where v_c is the characteristic mean-square velocity and c is the speed of sound). In the first nontrivial order $(Ma)^2$, the problem was reduced to the calculation of scaling dimensions of certain nonlocal composite fields (composite operators in the language of field theory), constructed from the fields of the renormalizable model of incompressible fluid.

Calculation of the scaling dimensions of composite operators is quite a cumbersome task. As a rule, their renormalization involves their mixing with each other, and in order to find the scaling dimension of a given operator, one has to consider the entire family of operators that admit to it in the renormalization procedure. The use of functional Schwinger equations and Ward identities, which express the Galilean symmetry of the model, simplifies the problem and in many cases allows us to find the dimensions exactly, see Refs. [6,8,17,18]. Using this technique for isotropic turbulence, the authors of [14] have calculated all the relevant scaling dimensions and, with the aid of these results, proved the validity of the second Kolmogorov hypothesis (independence of the velocity correlation function of the viscosity) in the leading order of $(Ma)^2$. This is in agreement with the result obtained previously in [19] within the approach based on the self-consistent equations. In Ref. [15], this proof was generalized to all orders of the formal expansion in $(Ma)^2$.

It should be stressed that the stability of the Kolmogorov fixed point in the presence of anisotropy is obviously not *a priori*: the analysis of the d -dimensional case shows that the stability is violated for $d < 2.68$ [11,12] (the two-dimensional case requires special care; see Ref. [20]). The stability of the Kolmogorov regime is also destroyed for the anisotropic magnetohydrodynamic turbulence [10] and for the strongly compressible fluid [16].

In this paper, we study the effect of compressibility in the first nontrivial order of the expansion in the small $(\text{Ma})^2$ within the framework of a more realistic model of the uniaxial anisotropic turbulence. The anisotropy is not supposed to be small. As in the isotropic case [14], the problem is reduced to the calculation of the scaling dimensions of a class of nonlocal composite operators in the model of incompressible strongly anisotropic turbulence considered in [12]. However, the set of relevant operators in this case is much wider than in Ref. [14]. Using the technique developed in [6,8,17,18] and the results obtained in [12], we have found all the scaling dimensions exactly. The main result of the paper is the substantiation of the validity of the second Kolmogorov hypothesis mentioned above, for strongly anisotropic, weakly compressible developed turbulence in the first nontrivial order in the Mach number.

II. THE MODEL

In the stochastic theory of fully developed turbulence, the motion of a viscous fluid is described by the Navier-Stokes equation

$$\varrho[\partial_t \mathbf{v}_i + \mathbf{v}_j \partial_j \mathbf{v}_i] = \nu_0 \Delta \mathbf{v}_i + \nu'_0 \partial_i \partial_j \mathbf{v}_j - \partial_i \mathcal{P} + f_i, \quad (2.1)$$

the continuity equation

$$\partial_i \varrho + \partial_j (\varrho \mathbf{v}_j) = 0, \quad (2.2)$$

and the equation of state $\mathcal{P} = \mathcal{P}(\varrho)$. Here $\partial_t \equiv \partial/\partial t$, $\partial_i \equiv \partial/\partial x_i$, $\mathbf{v}_i(\mathbf{x}, t)$ are the coordinates of the velocity field, $\varrho(\mathbf{x}, t)$ is the density of the fluid, $\mathcal{P}(\mathbf{x}, t)$ is the pressure, and ν_0 and ν'_0 are the molecular viscosity coefficients. Here and henceforth, summation over repeated indices is implied.

Following the tradition of stochastic models of turbulence, the randomness in Eq. (2.1) is introduced by the large-scale random force $f_i(\mathbf{x}, t)$ with Gaussian statistics with zero mean and matrix of the correlation functions $D_{ij} \equiv \langle f_i f_j \rangle$, which will be specified later.

We shall consider the weakly compressible fluid when the fields of the density and pressure can be written as sums of the mean values $\bar{\rho}$, \bar{p} and small fluctuations ρ , p : $\varrho = \bar{\rho} + \rho$, $\mathcal{P} = \bar{p} + p$. Without loss of generality, we take $\bar{\rho} = 1$. Due to the smallness of the fluctuations, the equation of state can be taken in the adiabatic approximation:

$$p = c^2 \rho, \quad (2.3)$$

where c is the adiabatic speed of sound in the turbulent medium. In the incompressibility limit one has $c^2 = \infty$ or, equivalently, $\text{Ma} = 0$.

Using the adiabatic relation (2.3), the continuity equation (2.2) can be rewritten in the form

$$\frac{1}{c^2} \partial_t p + \frac{1}{c^2} \partial_i (p \mathbf{v}_i) + \partial_i \mathbf{v}_i = 0. \quad (2.4)$$

For $c^2 = \infty$, the density becomes a constant, the velocity becomes transversal ($\partial_i \mathbf{v}_i = 0$), and we return to the case of incompressible fluid.

The velocity field \mathbf{v}_i can be expressed in the form $\mathbf{v}_i = \mathbf{v}_i^\perp + \mathbf{v}_i^\parallel$, where $\mathbf{v}_i^\perp \equiv P_{ij}^\perp \mathbf{v}_j$ is the transversal part satisfying

the condition $\partial_i \mathbf{v}_i^\perp = 0$, and $\mathbf{v}_i^\parallel \equiv P_{ij}^\parallel \mathbf{v}_j$ is the longitudinal part. The longitudinal P^\parallel and the transversal P^\perp projectors in the wave-vector (\mathbf{k}) space have the form $P_{ij}^\parallel = k_i k_j / k^2$ and $P_{ij}^\perp = \delta_{ik} - P_{ij}^\parallel$, respectively ($k \equiv |\mathbf{k}|$).

Since the velocity field has to become transversal in the incompressibility limit, its longitudinal part \mathbf{v}^\parallel has to be proportional to the inverse square of the sound speed, $|\mathbf{v}^\parallel| \sim c^{-2}$. Then from the Navier-Stokes equation (2.1) it follows that $|\mathbf{f}^\parallel| \sim c^{-2}$ for the longitudinal part of the random force. Hence, c^{-2} can be treated as a small formal parameter, and the compressibility corrections to the transversal part of the velocity field can be studied within the expansion in c^{-2} (or in Ma^2).

In the first order in c^{-2} , the continuity equation (2.4) takes the form

$$\frac{1}{c^2} (\partial_t + \mathbf{v}_i^\perp \partial_i) p + \partial_i \mathbf{v}_i^\parallel = 0. \quad (2.5)$$

In the leading approximation in c^{-2} (corresponding to the incompressible fluid), the Navier-Stokes equation (2.1) gives the well-known relation between the pressure p and the transversal velocity

$$\Delta p = -\partial_i \partial_j \mathbf{v}_i^\perp \mathbf{v}_j^\perp. \quad (2.6)$$

The last two equations allow us to express the pressure and the longitudinal part of the velocity via the transversal part \mathbf{v}^\perp ,

$$\mathbf{v}_i^\parallel = -\frac{1}{c^2} \partial_i \Delta^{-1} \nabla_t p, \quad p = -\Delta^{-1} \partial_i \partial_j \mathbf{v}_i^\perp \mathbf{v}_j^\perp, \quad (2.7)$$

where $\nabla_t \equiv \partial_t + \mathbf{v}_i^\perp \partial_i$ is the Lagrangian derivative for the transversal part of the velocity and Δ^{-1} is the Green function for the Laplace operator. In the field theory the quantities like the right-hand sides of Eqs. (2.7) are termed ‘‘composite operators.’’

Operating with the transversal projector P^\perp onto Eq. (2.1) and using relations (2.7), we arrive at the closed equation for the transversal part of the velocity (and, therefore, for all its statistical moments) in which the compressibility is taken into account to the order of c^{-2} , or, equivalently, Ma^2 ,

$$\begin{aligned} \partial_t \mathbf{v}_i^\perp &= \nu_0 \Delta \mathbf{v}_i^\perp - P_{ij}^\perp [\mathbf{v}_s^\perp \partial_s \mathbf{v}_j^\perp] - P_{ij}^\perp [\mathbf{v}_s^\perp \partial_s \mathbf{v}_j^\parallel + \mathbf{v}_s^\parallel \partial_s \mathbf{v}_j^\perp] \\ &\quad - c^{-2} \nu_0 P_{ij}^\perp [p \mathbf{v}_j^\perp] + f_i^\perp. \end{aligned} \quad (2.8)$$

To simplify the notation, we shall write \mathbf{v}_i instead of \mathbf{v}_i^\perp in what follows.

The positively definite ($d \times d$) square matrix of the pair correlation functions of the random force f_i^\perp will be taken in the form (see, e.g., [5,8])

$$\begin{aligned} \langle f_j^\perp(\mathbf{x}, t) f_s^\perp(\mathbf{0}, 0) \rangle &\equiv \varepsilon_0 D_{js}(\mathbf{x}, t) \\ &= \delta(t) \varepsilon_0 \int \frac{d^d \mathbf{k}}{(2\pi)^d} D_{js}^{st}(\mathbf{k}) \exp[i\mathbf{k} \cdot \mathbf{x}], \end{aligned} \quad (2.9)$$

$$D_{ij}(\mathbf{k}) = k^{4-d-\epsilon} \mathcal{P}_{ij}^\perp(\mathbf{k}) \quad (2.10)$$

(we recall that $\langle f_j^\perp \rangle = 0$). We see that temporal correlations of f_i^\perp have the character of white noise, while the spatial falloff of the correlations is controlled by the parameter ϵ and space dimension d . The functions (2.9) are translation invariant and for $\epsilon=2$ become scale invariant when the amplitude ϵ_0 acquires the dimension of the energy dissipation rate, ϵ , see Ref. [6]. The value $\epsilon=2$ is physically most acceptable, since it represents the assumption that random force acts at very large scales, which substitutes for the effect of boundary conditions. For simplicity, we use the force correlation function (2.9) without the usual infrared regularization. In this case, for $\epsilon=2$ the function (2.10) with the proper choice of the amplitude ϵ_0 in Eq. (2.9) can be considered as a powerlike model of the ‘‘ideal’’ pumping function $\delta(\mathbf{k})$, see [8]. The justification of this choice as well as the discussion of the central problem of the ϵ expansion, i.e., the continuation from $\epsilon=0$ to $\epsilon=2$, have been thoroughly discussed in Ref. [24].

The ratio $\epsilon_0/\nu^3 \equiv g_0$ plays the role of a bare coupling constant, i.e., the expansion parameter in the nonlinearity $(\mathbf{v}\partial)\mathbf{v}$ in the nonrenormalized perturbation theory. In the limit $\epsilon \rightarrow 0$, the constant g_0 becomes dimensionless, the diagrams of the Green functions become divergent in the ultraviolet (uv) region of the wave-vector space, and the problem of eliminating these divergences emerges. In the field theory this problem is solved by the well-known uv renormalization procedure, see, e.g., [21].

In this paper we consider the uniaxial anisotropic turbulence. The transverse projector \mathcal{P}^\perp for the correlation matrix (2.10) is defined by the relations [9,11,12]

$$\mathcal{P}_{js}^\perp(\mathbf{k}) = (1 + \alpha_1 \xi^2) P_{js}^\perp(\mathbf{k}) + \alpha_2 R_{js}^\perp(\mathbf{k}), \quad (2.11)$$

$$P_{js}^\perp(\mathbf{k}) = \delta_{js} - P_{js}^\parallel(\mathbf{k}), \quad P_{js}^\parallel(\mathbf{k}) = k_j k_s k^{-2},$$

$$R_{js}(\mathbf{k}) = (n_j - \xi_k k^{-1} k_j) (n_s - \xi_k k^{-1} k_s), \quad \xi_k = (\mathbf{k} \cdot \mathbf{n}) k^{-1}, \quad (2.12)$$

where the unit vector \mathbf{n} yields the direction of the anisotropy axis and α_1, α_2 are free amplitudes. These amplitudes are not considered small in the present analysis, however restrictions to their values $\alpha_1 \geq -1$, $\alpha_2 \geq 0$ follow from positive definiteness of the matrix (2.9). For nonzero α_1, α_2 the random forcing describes differences in energy injection in the preferred direction and directions perpendicular to it with the subsequent generation of anisotropic structures in large-scale eddies.

III. FIELD THEORETIC FORMULATION AND THE RG EQUATION

As in the critical dynamics [22,23], the stochastic problem (2.8), (2.9), and (2.10) is mapped to a quantum-field model, which is determined by an effective De Dominicis–Janssen ‘‘action’’ $S(\mathbf{v}, \mathbf{v}')$ constructed on the basis of the original stochastic model. This action is a functional of the transversal velocity \mathbf{v} and an independent transverse auxiliary field \mathbf{v}' .

In this approach, the generating functional G of the veloc-

ity correlation and response functions is the functional integral

$$G(\mathbf{A}, \mathbf{A}') = \int D\mathbf{v} D\mathbf{v}' \det M(\mathbf{v}) \exp[S(\mathbf{v}, \mathbf{v}') + \mathbf{A}\mathbf{v} + \mathbf{A}'\mathbf{v}'], \quad (3.1)$$

with the effective action

$$S(\mathbf{v}, \mathbf{v}') = \frac{1}{2} g_0 \nu_0^3 \mathbf{v}' D \mathbf{v}' + \mathbf{v}' [-\partial_t \mathbf{v} + \nu_0 \Delta \mathbf{v} - (\mathbf{v}\partial)\mathbf{v} - (\mathbf{v}^\parallel \partial)\mathbf{v} - (\mathbf{v}\partial)\mathbf{v}^\parallel - c^{-2} \nu_0 \Delta \mathbf{v} p], \quad (3.2)$$

where \mathbf{A}, \mathbf{A}' are the source fields, which are equivalent to regular external forces. Here, the required integrations over the space-time arguments of the fields and sums over discrete indices are implicitly assumed.

The Jacobian $\det M$ in Eq. (3.1) ensures the cancellation of all the diagrams containing the self-contracted bare propagator $\langle \mathbf{v}\mathbf{v}' \rangle$, which arise along with other diagrams from the rules of the Feynman diagrammatic technique for the action (3.2), but do not arise in the construction of diagrams by direct iteration of the stochastic equation (2.8). Following [22,5,6], we simply define these superfluous diagrams as zero, and simultaneously set $\det M = 1$ in Eq. (3.1). We note that in our model such a definition is nontrivial because the interaction in Eq. (3.2) involves the derivatives with respect to the time variable. Nevertheless, this definition is feasible, as it has been shown in Ref. [14] using isotropic turbulence as an example. As a result, we arrive at a standard field theoretic model with action (3.2), and the standard renormalization theory is applicable to it.

The action (3.2) is not renormalized and the corresponding Green functions of the fields \mathbf{v}, \mathbf{v}' contain uv divergences for $\epsilon \rightarrow 0$. In order to analyze them, we rewrite the action (3.2) as a sum $S = S^I + S^C$:

$$S^I(\mathbf{v}, \mathbf{v}') = \frac{1}{2} g_0 \nu_0^3 \mathbf{v}' D \mathbf{v}' + \mathbf{v}' [-\partial_t \mathbf{v} + \nu_0 \Delta \mathbf{v} - (\mathbf{v}\partial)\mathbf{v}], \quad (3.3)$$

$$S^C(\mathbf{v}, \mathbf{v}') = a_{01} F_1 + a_{02} F_2, \quad (3.4)$$

where $a_{01} \equiv -c^{-2}$, $a_{02} \equiv c^{-2} \nu_0$, and the composite operators F_1, F_2 , according to Eq. (2.7) and using the relation $\partial_i v_j^\parallel = \partial_j v_i^\parallel$, can be represented in the form

$$F_1 = v_i' (\partial_j v_i - \partial_i v_j) \partial_l \Delta^{-1} \nabla_l \Delta^{-1} \partial_j \partial_j v_i v_j, \\ F_2 = v_i' (\Delta v_l) \Delta^{-1} \partial_i \partial_j v_l v_j. \quad (3.5)$$

In the limit $c^{-2} \rightarrow 0$, the action (3.3) describes the incompressible anisotropic turbulence. Renormalization of this model has been considered in [11,12]. It was shown that in order to ensure the multiplicative renormalizability, the model has to be extended by adding certain anisotropic dissipative terms with new viscosity coefficients $\nu_0 \chi_{0i}$, $i = 1, 2, 3$, where the dimensionless parameters χ_{0i} describe the relative impact of the different anisotropic structures on the viscous dissipation and play the role of additional coupling constants.

The renormalized action corresponding to the original nonrenormalized functional (3.3) is of the form

$$S = \frac{1}{2} g \nu^3 \mu^{2\epsilon} \mathbf{v}' D \mathbf{v}' + \mathbf{v}' [-\partial_t \mathbf{v} + \nu Z_\nu \Delta \mathbf{v} + \nu Z_\nu \chi_1 Z_{\chi_1} \mathbf{n} \Delta(\mathbf{v} \mathbf{n}) + \nu Z_\nu \chi_2 Z_{\chi_2} (\mathbf{n} \partial)^2 \mathbf{v} + \nu Z_\nu \chi_3 Z_{\chi_3} \mathbf{n} (\mathbf{n} \partial)^2 (\mathbf{v} \mathbf{n}) - (\mathbf{v} \partial) \mathbf{v}]. \quad (3.6)$$

Here, the renormalization mass μ is an additional arbitrary parameter of the renormalized theory, and the renormalized parameters g, ν, χ_i are related to their bare (unrenormalized) counterparts by the multiplicative renormalization formulas [12],

$$g_0 = g Z_g \mu^{2\epsilon}, \quad \nu_0 = \nu Z_\nu, \quad \chi_{0i} = \chi_i Z_{\chi_i}, \quad Z_g = Z_\nu^{-3}. \quad (3.7)$$

The renormalization constants Z are calculated within the perturbation theory. In the minimal subtraction scheme, they have the form “ $Z = 1 +$ only poles in ϵ ” and cancel all the uv divergences in the correlation functions of the primary fields in the model (3.6). The last relation in Eqs. (3.7) follows from the absence of the constants Z in the first and last terms of the action (3.6).

To determine the dependence of the renormalized correlation functions on the parameters a_{01} and a_{02} after the term (3.4) has been added to the action, let us consider the pair correlation function for the incompressible isotropic case (the detailed discussion can be found, e.g., in Refs. [8,24]),

$$\langle v_j(\mathbf{x}_1, t) v_m(\mathbf{x}_2, t) \rangle = \int \frac{d^d \mathbf{k}}{(2\pi)^d} G_{jm}^R(\mathbf{k}) \exp[i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)]. \quad (3.8)$$

The RG equation for the trace of its space Fourier transform $G^R(\mathbf{k}) = G_{ii}^R(\mathbf{k})$ is

$$\left[\mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} - \gamma_\nu \nu \frac{\partial}{\partial \nu} \right] G^R(\mathbf{k}) = 0, \quad (3.9)$$

where the β_g function and the anomalous dimension γ_ν are expressed via the renormalization constant Z_ν ,

$$\beta_g = -g(2\epsilon - 3\gamma_\nu), \quad \gamma_\nu = \bar{D}_\mu \ln Z_\nu. \quad (3.10)$$

Here \bar{D}_μ denotes the operation $\mu \partial / \partial \mu$ taken at fixed values of all the bare parameters.

The solution of the RG equations along with the dimensionality considerations gives

$$G^R(\mathbf{k}) = \bar{\nu}^2(s) k^{2-d} R(\bar{g}(s)), \quad s \equiv \frac{k}{\mu}, \quad (3.11)$$

where R is a “scaling function” of the invariant charge $\bar{g}(s)$, the effective variable satisfying the equations

$$s \frac{d\bar{g}}{ds} = \beta_g(\bar{g}(s)), \quad \bar{g}|_{s=1} = g. \quad (3.12)$$

The second effective variable, the invariant viscosity $\bar{\nu}(s)$, satisfies the equations

$$s \frac{d\bar{\nu}}{ds} = -\gamma_\nu(\bar{g}(s)), \quad \bar{\nu}|_{s=1} = \nu. \quad (3.13)$$

From the solution of Eqs. (3.12) it follows that $\bar{g}(s) \rightarrow g^*$ for $s \rightarrow 0$, where g^* is an infrared stable fixed point of the RG equations, i.e., the root of the equation $\beta_g = 0$ with the positive value of the correction exponent $\omega \equiv \partial \beta_g / \partial g$.

The solution of Eq. (3.13) is

$$\bar{\nu}(s) = \nu \exp \left[- \int_g^{\bar{g}(s)} dx \frac{\gamma_\nu(x)}{\beta_g(x)} \right]. \quad (3.14)$$

From Eq. (3.14) along with Eqs. (3.10) and (3.7) it follows that

$$\bar{\nu}(s) = \nu \left(\frac{g}{\bar{g} s^{2\epsilon}} \right)^{1/3} = \left(\frac{\epsilon_0}{\bar{g} k^{2\epsilon}} \right)^{1/3}. \quad (3.15)$$

For the spectrum of kinetic energy $E(k) \sim k^{d-1} G^R(\mathbf{k})$ in the asymptotic region $s \rightarrow 0$ we obtain from Eqs. (3.11) and (3.15) the expression $E(k) \sim \epsilon^{2/3} k^{-5/3}$, which is independent of the viscosity ν_0 and corresponds to the Kolmogorov value of the exponent.

When the anisotropic case is studied [action (3.6)], the new terms $\beta_{\chi_j} \partial G^R(\mathbf{k}) / \partial \chi_j$ are appended to the RG equation (3.9). The new β functions and the anomalous dimensions γ_{χ_i} corresponding to the new dimensionless parameters χ_i ,

$$\beta_{\chi_i} = -\chi_i \gamma_{\chi_i}, \quad \gamma_{\chi_i} = \bar{D}_\mu \ln Z_{\chi_i}, \quad (3.16)$$

are expressed via the renormalization constants Z_{χ_i} in the action (3.6). The additional invariant variables $\bar{\chi}(s)$ satisfy equations such as Eq. (3.12). In Ref. [12] it has been shown that those equations have an IR stable fixed point $\bar{g}(s), \bar{\chi}(s) \rightarrow g^*, \chi^*$, in which all the eigenvalues of the matrix of the correction exponents

$$\omega_{ij} = \frac{\partial \beta_{g_i}}{\partial g_j} \Big|_{g_i = g_i^*}, \quad g_i \equiv g, \chi_1, \chi_2, \chi_3, \quad i, j = 0, 1, 2, 3 \quad (3.17)$$

(to be precise, their real parts) are positive, i.e., the Kolmogorov asymptotic regime conserves the stability against the strong anisotropy.

The problem becomes more involved if the compressibility is taken into account. Let us suppose that we have managed to renormalize the action (3.4). Then, the new terms $\gamma_{a_i} \mathcal{D}_{a_i} G^R(\mathbf{k})$ appear in the RG equation (3.9), where γ_{a_i} are the anomalous dimensions of the renormalized parameters a_i . In contrast with the parameters χ , the renormalized counterparts of the parameters a_{01} and a_{02} have nonzero dimensions and, therefore, the scaling function R depends on the effective dimensionless variables

$$\bar{u}_1 = k^2 \bar{\nu}^2 \bar{a}_1, \quad \bar{u}_2 = k^2 \bar{\nu} \bar{a}_2. \quad (3.18)$$

The effective variables $\bar{a}_1(s)$ and $\bar{a}_2(s)$ satisfy equations such as Eqs. (3.13). In the infrared asymptotic region $k \rightarrow 0$, they take on the form $\bar{a}_i \sim k^{-\gamma_{a_i}}$, and the infrared asymptotic form of the dimensionless arguments (3.18) is given by the expressions $\bar{u}_i \sim k^{-\Delta_{a_i}}$ with the scaling dimensions Δ_{a_i} (for more details see, e.g., [8,25,26]). In the linear approximation with respect to the small parameters a_{01} and a_{02} in the functional (3.1), the leading correction to the scaling function R takes on the form $(1 + \text{const} k^{-\Delta_{\max}})$, where Δ_{\max} is the maximal dimension among Δ_{a_i} .

Therefore, the investigation of the dependence of the kinetic energy spectrum on the compressibility is related to the calculation of the scaling dimensions Δ_{a_i} , which, as we shall show below, can be expressed via scaling dimensions of the composite operators $F_{1,2}$ entering into the action (3.4).

IV. RENORMALIZATION AND SCALING DIMENSIONS OF THE COMPOSITE OPERATORS

The addition of the term (3.4) involving the operators F_1, F_2 (3.5) to the action (3.3) gives rise to new uv divergences (poles in ϵ) in the correlation functions. According to the generic rules, all the composite operators with the same canonical (naive) dimensions and tensor structure can be mixed in the renormalization procedure, i.e., a uv finite renormalized operator F^R has the form $F^R = F + \text{counterterms}$, where the contribution of the counterterms is a linear combination of F itself and other unrenormalized operators that ‘‘admix’’ to F . Therefore, to perform renormalization of the operators F_1 and F_2 , one has to consider a wider family of operators F_i which admix to F_1, F_2 .

The renormalized operators F_i^R are related to their nonrenormalized counterparts F_i by the well-known matrix formulas of multiplicative renormalization, see, e.g., Refs. [6,8],

$$F_i = Z_{ij} F_j^R, \quad (4.1)$$

where Z_{ij} is the matrix of the renormalization constants. In the minimal subtraction scheme, its diagonal elements have the form $1 + \text{poles in } \epsilon$ while the nondiagonal elements contain only poles. From the matrix Z_{ij} one calculates the matrix of anomalous dimensions $\gamma_{ij} = Z_{ik}^{-1} \bar{D}_\mu Z_{kj}$ and the matrix of scaling dimensions for the set of operators

$$\Delta_{ij}^F = D_{ij}^F + \gamma_{ij}. \quad (4.2)$$

The contribution $D_{ij}^F = [d_F - \gamma_\nu d_F^\omega]_{ij}$ is expressed via the anomalous dimension of the viscosity (3.10) and the total d_F and frequency d_F^ω canonical dimensions of the operator F [6,8], which are equal to the sums of the corresponding dimensions of the fields and derivatives that constitute F .

The total canonical dimensions of the fields and parameters of the model are found from the requirement that all the terms of the action (3.2) be dimensionless, see [6,8]: $d_t = d_\omega = -2$, $d_v = 1$, $d_{v'} = d - 1$ ($d_x = d_k = -1$ by definition). From these dimensions we then obtain the canonical dimensions of the operators F_1, F_2 equal to $d_{F_1} = d_{F_2} = d + 4$. We also note that these operators are Galilean invariant, scalar, and nonlocal.

Let $F \equiv \{F_i\}$ be a system of composite operators closed with respect to renormalization. The equation $a_i F_i^R = a_{0i} F$ (the summation over the subscript i is implied) can be regarded as a definition of the renormalized sources $a \equiv \{a_i\}$, which for the usual renormalization formulas $a_0 = a Z_a$, $F = Z_F F^R$ leads to the relations $Z_a = Z_F^{-1}$ for the renormalization constants and $\gamma_F = -\gamma_a$ for the corresponding anomalous dimensions. The requirement that the terms

$$\int dx a F^R = \int dx a_0 F, \quad x \equiv \mathbf{x}, t$$

be dimensionless then gives the ‘‘shadow relations’’ for the canonical and scaling dimensions of the operators F_i and sources a_i ,

$$d_a^k + d_F^k = d, \quad d_a^\omega + d_F^\omega = 1, \quad \Delta_a + \Delta_F = d + \Delta_\omega. \quad (4.3)$$

Due to Eqs. (4.3), the problem of finding the maximal dimension Δ_{a_i} for the sources corresponding to the operators $F_{1,2}$ in the action (3.4) is equivalent to the calculation of the minimal scaling dimension Δ_F associated with the operators $F_{1,2}$ and all the operators that admix to them in renormalization.

According to the general theory of renormalization, see, e.g. [21], counterterms in a field theory with a local interaction are also local. Therefore, the renormalization of the nonlocal operators F_1, F_2 is reduced to that of their local blocks (see below) and to the admixture of the local operators (i.e., monomials constructed of the fields and their derivatives at the same point \mathbf{x}, t) with the same canonical dimension and symmetry (Galilean invariant scalars). These local operators in our case are the following: $\bar{F} = \partial v' \partial v \partial v$, $\partial v' \nabla_i v \partial v$, $\partial v' \partial^3 v$, $n^2 \partial v' \partial v \partial v$, $n^2 \partial v' \nabla_i \partial v$, $n^2 \partial v' \partial^3 v$, $n^4 \partial v' \partial v \partial v$, $n^4 \partial v' \nabla_i \partial v$, $n^4 \partial v' \partial^3 v$, $n^6 \partial v' \partial v \partial v$, $n^6 \partial v' \nabla_i \partial v$, $n^6 \partial v' \partial^3 v$. The notation is symbolic and it implies all possible contractions of the vector indices of the fields \mathbf{v}' , \mathbf{v} , derivative ∂ , and unit vector \mathbf{n} . This set of operators is closed with respect to renormalization because the nonlocal operators F_1 and F_2 cannot admix to them. The first three types of the operators \bar{F} have been considered in [14]. It was shown that they did not affect the scaling dimensions of the nonlocal operators F_1, F_2 due to the fact that the corresponding renormalization matrix Z_{ij} was block-triangular. This feature of the renormalization matrix persists also in the other operators \bar{F} , which contain the vector \mathbf{n} , so that they also do not affect the scaling dimensions of F_1, F_2 . In contrast with the local operators \bar{F} , they contain additional factors of $\Delta^{-1} \partial v$ which have zero canonical dimension and negative scaling dimension $-4/3$ at $\epsilon = 2$ (we recall that the scaling dimension of the field \mathbf{v} equal to $-1/3$ at $\epsilon = 2$, see [6,8]). Therefore, the scaling dimensions of the operators \bar{F} are greater than the dimensions of the nonlocal operators F_1, F_2 , and the leading contribution to the IR asymptotic form of the spectrum is determined by the contributions of F_1 and F_2 . We note that due to renormalization, the scaling dimension of an operator F does not coincide in general with a naive sum of scaling dimensions of the fields and derivatives entering into F . But, for the incompressible case, the hypothesis that the scaling dimension of a nonlocal operator is the sum of scal-

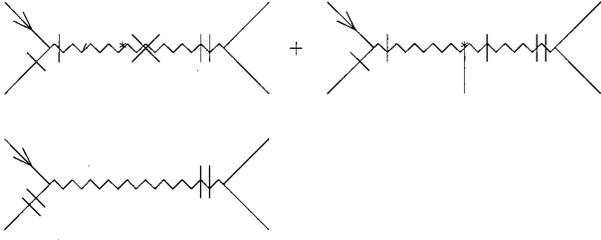


FIG. 1. Graphs of composite nonlocal operators $F_1 = v'_i(\partial_j v_i - \partial_i v_j)\partial_i \Delta^{-1} \nabla_i \Delta^{-1} \partial_i \partial_j v_i v_j$ and $F_2 = v'_i(\Delta v_i) \Delta^{-1} \times \partial_i \partial_j v_i v_j$ giving a leading correction to the infrared form kinetic energy spectra of weakly compressible developed turbulence.

ing dimensions of its local parts and of the factors of type $\Delta^{-1} \partial v$ has been confirmed in [25] by the explicit one-loop calculation of the scaling dimensions related to the local operators with the canonical dimension $d+4$, and we also accept it in what follows.

As a result, we obtain that the scaling dimensions of F_1 and F_2 are determined by their own renormalization. The latter is reduced to the renormalization of the local blocks entering into F_1 and F_2 .

Let us denote the field \mathbf{v} by the solid line, \mathbf{v}' by the oriented solid line, and the operator Δ^{-1} by the wave line. The derivative with respect to the coordinate is denoted by a slash, and the derivative with respect to time by a cross. Graphical representation of the operators (3.5) is depicted in Fig. 1, where the vector indices are omitted and the operator, containing the full time derivative ∇_t , is represented as a sum of the first two diagrams.

The contribution of the last operator from Fig. 1 to the correlation function $\langle \mathbf{v}' \mathbf{v} \mathbf{v} \mathbf{v} \rangle$ is depicted in Fig. 2. The shadowed rectangle denotes an arbitrary one-particle irreducible diagram with fixed external legs. One can show that the triangular subdiagram contains uv divergence and its elimination requires the renormalization of the local block of the nonlocal operator under consideration. Thus, for the complete renormalization of the operators (3.5), it is sufficient to study the renormalization of all their local blocks.

The operator F_1 consists of two nonlocal factors Δ^{-1} , the full derivative ∇_t , and two local blocks

$$G_1 = v'_i(\partial_j v_i - \partial_i v_j), \quad G_2 = \partial_i \partial_j v_i v_j, \quad (4.4)$$

while F_2 contains one factor Δ^{-1} , the operator G_2 , and the local block

$$G_3 = v'_i \Delta v_i. \quad (4.5)$$

The scaling dimensions of the operators (3.5) are equal to the sums of the scaling dimensions of the above factors; among

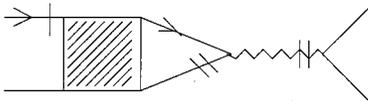


FIG. 2. The correlation function $\langle \mathbf{v}' \mathbf{v} \mathbf{v} \mathbf{v} \rangle$ with the contribution of the nonlocal operator $F_2 = v'_i(\Delta v_i) \Delta^{-1} \partial_i \partial_j v_i v_j$. The shadowed rectangle denotes an arbitrary one-particle irreducible diagram with four external legs.

them only the dimensions of the local blocks (4.4) and (4.5) require nontrivial calculation. In order to find them, one has to study the renormalization of the complete set of the operators that admix to G_i in renormalization. This set is rather big because of the anisotropy and the canonical dimension of G_i is high ($d_F=7$ for $d=3$). To simplify the analysis, we shall use some general rules for the operator mixing. Their proof and other examples can be found, e.g., in Refs. [6,8,17,18].

(a) In the action (3.6) the derivative in the interaction term can be moved onto the auxiliary field \mathbf{v}' using integration by parts: $v'_i v_j \partial_j v_i = -(\partial_j v'_i) v_i v_j$. Therefore, the derivative ∂ appears as an external factor for each external leg of the field \mathbf{v}' for any one-particle irreducible diagram, and the corresponding counterterm contains the factor $\partial v'$.

(b) Only Galilean invariant operators can admix to an invariant operator in the renormalization procedure.

(c) Let some operator G have the form of a total derivative of some other operator $[G]$, $G = \partial[G]$. In this case, the scaling dimension of G is simply given by the relation $\Delta_G = 1 + \Delta_{[G]}$.

(d) All the one-particle irreducible diagrams, containing closed circuits of the retarded propagators $\langle \mathbf{v} \mathbf{v}' \rangle$, vanish. We denote by \tilde{G} or $[\tilde{G}]$ the full sets of operators that can mix with a given G or $[G]$ in renormalization.

According to the item (c), instead of the operator G_2 from Eq. (4.5) it is sufficient to study the renormalization of the operator $[G_2] = v_i v_j$. Due to the transversality of the field v_i , the only operators that can admix to $[G_2]$ have the form $[\tilde{G}_2] = n_k n_l \partial_l v_k \delta_{ij} + n_k n_l \partial_l v_k n_i n_j$. Their scaling dimensions are equal to $\Delta_{[\tilde{G}_2]} = 1 + \Delta_v$. The scaling dimensions of the fields \mathbf{v} , \mathbf{v}' and the time have the form (see, e.g., [6])

$$\Delta_v = 1 - 2\epsilon/3, \quad \Delta_{v'} = d - 1 + 2\epsilon/3, \quad \Delta_t = -2 + 2\epsilon/3. \quad (4.6)$$

We then obtain $\Delta_{[\tilde{G}_2]} = 2 - 2\epsilon/3$, which gives $\Delta_{[\tilde{G}_2]} = 2/3$ at $\epsilon=2$. Since the operator $[G_2]$ itself is not renormalized, for the scaling dimension of G_2 we obtain [item (c)] $\Delta_{G_2} = 2 + \Delta_{[G_2]} = 2 + 2\Delta_v = 4 - 4\epsilon/3$, which gives $\Delta_{G_2} = 4/3$ at $\epsilon=2$.

The operator G_1 consists of two terms: $G_1 = G_{11} - G_{12}$. The term G_{12} is rewritten in the form $G_{12} = \partial_i(v'_i v_j)$; it is then sufficient to consider the operator $[G_{12}] = v'_i v_j$ [item (c)]. It can mix with the following operators: $\partial_i v'_j$, $n_i n_l \partial_l v'_j$, $\delta_{ij} n_k n_l \partial_l v'_k$, and $n_i n_j n_k n_l \partial_l v'_k$. They all are uv finite and their critical dimensions are simply given by $1 + \Delta_{v'} = d + 2\epsilon/3$, i.e., $13/3$ at $d=3$ and $\epsilon=2$. The diagonal element of the matrix Z_{ij} of the above set of operators equals 1 [item (a)] and, as in the case of the set associated with G_2 , this matrix is triangular. It then follows that $\Delta_{G_{12}} = 1 + \Delta_{[G_{12}]} = 1 + \Delta_v + \Delta_{v'} = d + 1$, which gives $\Delta_{G_{12}} = 4$ at $d=3$.

The operator G_{11} does not admix to itself due to item (a). Owing to the Galilean invariance, it does not mix with the operators of the same tensor structure which involve the vector \mathbf{n} [item (b)]. Furthermore, it does not mix with the invariant operators $n_j \nabla_i n_k v'_k$ [item (a)] and $n_i \nabla_t \partial_j v_i$ [item

(d)]. The set \tilde{G}_{11} , which can admix to G_{11} , includes the operators $\Delta \mathbf{v}'_j$, $n_j \Delta \mathbf{v}'_s n_s$, $n_s \partial_s n_l \partial_l \mathbf{v}'_j$, $\partial_j n_s \partial_s \mathbf{v}'_l n_l$, and $n_j n_k \partial_k n_s \partial_s \mathbf{v}'_l n_l$. All these operators are uv finite and their critical dimensions are equal to $2 + \Delta_{\nu'} = 16/3$. As in the case of the operators G_2 and G_{12} , these operators do not affect the critical dimension of $G_{11} = \mathbf{v}'_i \partial_j \mathbf{v}_i$. Since the latter is uv finite, its critical dimension is given by $\Delta_{G_{11}} = 1 + \Delta_{\nu'} + \Delta_{\nu} = 4$.

Now let us turn to the last operator, G_3 , from Eq. (4.5). The invariant operators $\mathbf{v}'_i \nabla_i \mathbf{v}_i$ and $\mathbf{v}'_i n_i \nabla_i \mathbf{v}_j n_j$ do not admix to G_3 due to item (a). Therefore, we are left with the three types of operators

$$\begin{aligned} \{\tilde{G}_{31}\} &= \{\mathbf{v}'_i (\mathbf{n}\partial)^2 \mathbf{v}_i, (\mathbf{n}\mathbf{v}') \Delta (\mathbf{n}\mathbf{v}), (\mathbf{n}\mathbf{v}') (\mathbf{n}\partial)^2 (\mathbf{n}\mathbf{v})\}, \\ \{\tilde{G}_{32}\} &= \{\partial_l (\mathbf{v}'_s \partial_l \mathbf{v}_s), \partial_l (\mathbf{v}'_i \partial_l \mathbf{v}_i), (\mathbf{n}\partial) [\mathbf{v}'_i (\mathbf{n}\partial) \mathbf{v}_i], \\ &\quad (\mathbf{n}\partial) [(\mathbf{v}' \partial) (\mathbf{n}\mathbf{v})]\}, \end{aligned} \quad (4.7)$$

$$\partial_l [(\mathbf{n}\mathbf{v}') \partial_l (\mathbf{n}\mathbf{v})], \quad \partial_l [(\mathbf{n}\mathbf{v}') (\mathbf{n}\partial) \mathbf{v}_i], \quad (\mathbf{n}\partial) [(\mathbf{n}\mathbf{v}') (\mathbf{n}\partial) (\mathbf{n}\mathbf{v})],$$

$$\{\tilde{G}_{33}\} = \{(\mathbf{n}\partial) \Delta (\mathbf{n}\mathbf{v}'), (\mathbf{n}\partial) (\mathbf{n}\partial)^2 (\mathbf{n}\mathbf{v}')\}.$$

The operators $\{\tilde{G}_{33}\}$ do not affect the scaling dimensions of $\{\tilde{G}_{31}\}$ and $\{\tilde{G}_{32}\}$ [item (c)], they are uv finite, and their dimensions are equal to $\Delta_{\{\tilde{G}_{33}\}} = 19/3$ at $d=3$ and $\epsilon=2$. The operators $\{\tilde{G}_{32}\}$ do not affect $\{\tilde{G}_{31}\}$ [item (c)], they are also finite (like G_{12}), and their scaling dimensions are equal to $\Delta\{\tilde{G}_{32}\} = 5$.

Thus, we need to renormalize the remaining set that includes the operators G_3 and \tilde{G}_{31} . They are renormalized with mixing, and the corresponding matrix Z_{ij} is nontrivial. In the isotropic case the renormalization constant of G_3 is expressed via the known renormalization constant Z_{ν} in the action (3.6) and, therefore, the scaling dimension Δ_{G_3} is related to the known function γ_{ν} [14]. In the presence of anisotropy the situation becomes more complicated. However, even in this case it turns out to be possible to express the matrix Z_{ij} in terms of the known renormalization constants Z_{ν} and Z_{χ_i} from the action (3.6).

Consider the generating functional (3.1) with $\det M = 1$ and the renormalized action (3.6). It is uv finite and, therefore, its derivative with respect to the renormalized parameters $e = \{g, \chi_i, \nu\}$ (they are the generating functionals of the composite operators $\partial_e \mathcal{S}$) is also uv finite, as well as the operators $\partial_e \mathcal{S}$ themselves.

The functional $G(\mathbf{A}, \mathbf{A}')$ satisfies the RG equation

$$\begin{aligned} \mathcal{D}_{\text{RG}} G(\mathbf{A}, \mathbf{A}') = 0, \quad \mathcal{D}_{\text{RG}} = &\left(\mu \frac{\partial}{\partial \mu} - \gamma_{\nu} \nu \frac{\partial}{\partial \nu} \right. \\ &\left. + \beta_g \frac{\partial}{\partial g} + \beta_{\chi_i} \frac{\partial}{\partial \chi_i} \right) \end{aligned} \quad (4.8)$$

with the functions β_g and β_{χ_i} defined in Eqs. (3.10) and (3.16). Let us define the matrix ω_{ik} by the relation

$$\omega_{ik} = -g_i \frac{\partial \gamma_{g_i}}{\partial g_k}, \quad (4.9)$$

where $g_i \equiv \{g, \chi_i\}$ (we recall that $\gamma_g = -3\gamma_{\nu}$). Using Eqs. (3.10) and (3.16), we readily find that at the fixed point g_i^* the matrix (4.9) coincides with the matrix of correction exponents ω_{ik} defined in Eq. (3.17).

We define the commutator of two differential operators $\mathcal{D}_i, \mathcal{D}_j$ in a standard way, $[\mathcal{D}_i, \mathcal{D}_j] \equiv \mathcal{D}_i \mathcal{D}_j - \mathcal{D}_j \mathcal{D}_i$. The commutators of the operators \mathcal{D}_{RG} , $\mathcal{D}_{\nu} \equiv \nu \partial_{\nu} \equiv \nu \partial / \partial \nu$, and $\partial_{g_i} \equiv \partial / \partial g_i$ are of the form

$$\begin{aligned} [\mathcal{D}_{\text{RG}}, \mathcal{D}_{\nu}] = 0, \quad [\mathcal{D}_{\text{RG}}, \partial_{g_i}] = \omega_{ij} &\left(\delta_{i0} \frac{1}{3g} \mathcal{D}_{\nu} - \partial_{g_j} \right) \\ &- \delta_{i0} \frac{\beta_g}{g} \partial_{g_j}. \end{aligned} \quad (4.10)$$

Differentiation of the RG equation (4.8) with respect to ν and g_i along with the commutation relations (4.10) gives

$$\begin{aligned} \mathcal{D}_{\text{RG}} \partial_{g_i} G = \omega_{ij} &\left(\delta_{i0} \frac{1}{3g} \mathcal{D}_{\nu} - \partial_{g_j} \right) G - \delta_{i0} \frac{\beta_g}{g} \partial_{g_j} G, \\ \mathcal{D}_{\text{RG}} \mathcal{D}_{\nu} G = 0. \end{aligned} \quad (4.11)$$

The fact that the operators \mathcal{D}_{RG} and \mathcal{D}_{ν} are commutative allows the left-hand side of the first equation in Eq. (4.11) to be rewritten in the form

$$\mathcal{D}_{\text{RG}} \partial_{g_i} G = \mathcal{D}_{\text{RG}} \left(\partial_{g_i} - \delta_{i0} \frac{1}{3g} \mathcal{D}_{\nu} \right) G - \delta_{i0} \frac{\beta_g}{3g^2} \mathcal{D}_{\nu} G.$$

Using this relation, Eq. (4.11) is rewritten as

$$\mathcal{D}_{\text{RG}} X_i = -\omega_{ij} X_j - \delta_{i0} \frac{\beta_g}{g} X_0, \quad i, j = 0, 1, 2, 3, 4, \quad (4.12)$$

which, at the fixed point $g^* \neq 0$ along with $\beta_g = 0$, gives

$$\mathcal{D}_{\text{RG}} X_i = -\omega_{ij} X_j. \quad (4.13)$$

This is nothing more than the scaling equation for the quantities

$$X_i = [\partial_{g_i} - (3g)^{-1} \delta_{i0} \mathcal{D}_{\nu}] G(\mathbf{A}, \mathbf{A}'),$$

and ω_{ij} is the matrix of their anomalous dimensions. Its eigenvalues are positive (it follows from the IR stability of the fixed point, see [12]). According to Eq. (4.9), it is expressed via the renormalization constants Z of the action (3.6) calculated in the one-loop approximation in Ref. [12].

Using the explicit form of the generating functional (3.1), the quantities X_i are explicitly expressed via the derivatives of the renormalized action (3.6) with respect to the parameters g, ν , and χ_i ,

$$\begin{aligned}
X_i &= \int D\mathbf{v}D\mathbf{v}' \tilde{X}_i \exp[S(\mathbf{v}, \mathbf{v}') + \mathbf{A}\mathbf{v} + \mathbf{A}'\mathbf{v}'] \\
&= \int D\mathbf{v}D\mathbf{v}' [\partial_{g_i} S - (3g)^{-1} \delta_{i0} \mathcal{D}_v S] \\
&\quad \times \exp[S(\mathbf{v}, \mathbf{v}') + \mathbf{A}\mathbf{v} + \mathbf{A}'\mathbf{v}']. \quad (4.14)
\end{aligned}$$

Therefore, the quantities X_i represent the generating functionals of the correlation functions that involve not only primary fields \mathbf{v}, \mathbf{v}' but also renormalized composite operators \tilde{X}_i .

Performing the differentiation in Eq. (4.14) explicitly, one obtains

$$\begin{aligned}
\tilde{X}_i &= e_{i0} \mathbf{v}' \Delta \mathbf{v} + e_{i1} (\mathbf{v}' \mathbf{n}) \Delta(\mathbf{v} \mathbf{n}) + e_{i2} \mathbf{v}' (\mathbf{n} \partial)^2 \mathbf{v} e_{i3} (\mathbf{v}' \mathbf{n}) \\
&\quad \times (\mathbf{n} \partial)^2 (\mathbf{v} \mathbf{n}), \quad (4.15)
\end{aligned}$$

where the coefficients e are expressed via the renormalization constants from Eq. (3.6):

$$\begin{aligned}
e_{00} &= \nu \left(\partial_g Z_\nu - \frac{Z_\nu}{3g} \right) \Big|_{g=g^*}, \quad e_{0i} = \nu \chi_i \left[\partial_g (Z_\nu Z_{\chi_i}) \right. \\
&\quad \left. - \frac{Z_\nu Z_{\chi_i}}{3g} \right] \Big|_{g=g^*}, \quad i=1,2,3, \\
e_{10} &= \nu \partial_{\chi_1} Z_\nu \Big|_{g=g^*}, \quad e_{1i} = \nu \chi_i \partial_{\chi_1} (Z_\nu Z_{\chi_i}) \Big|_{g=g^*}, \\
e_{20} &= \nu \partial_{\chi_2} Z_\nu \Big|_{g=g^*}, \quad e_{2i} = \nu \chi_i \partial_{\chi_2} (Z_\nu Z_{\chi_i}) \Big|_{g=g^*}, \\
e_{30} &= \nu \partial_{\chi_3} Z_\nu \Big|_{g=g^*}, \quad e_{3i} = \nu \chi_i \partial_{\chi_3} (Z_\nu Z_{\chi_i}) \Big|_{g=g^*}. \quad (4.16)
\end{aligned}$$

It is obvious from Eqs. (4.16) that \tilde{X}_i are given by linear combinations of the operators G_3 and $\{\tilde{G}_{31}\}$, and the matrix ω (4.9) determines their anomalous dimensions. The eigenvalues ω_i of the matrix ω have been calculated in Ref. [12] in the first order of the ϵ expansion. All the real parts of these eigenvalues are positive (two of the eigenvalues are complex). We calculate from Eq. (4.2) the scaling dimensions of the operators G_3 and $\{\tilde{G}_{31}\}$ $\Delta_{G_3} = 13/3 + \omega$ ($\omega \equiv \omega_0$), $\Delta_{\{\tilde{G}_{31}\}} = 13/3 + \omega_i$ for $i=1,2,3$. From the results of Ref. [12], it follows that the exponent ω is smaller than each of the eigenvalues ω_i , so that the main contribution of the operators in consideration of the IR asymptotic behavior of the kinetic energy spectrum is given by the operator G_3 .

Finally, from Eqs. (3.5), (4.4), (4.5), and ‘‘shadow relation’’ (4.3) we obtain the scaling dimensions for the original composite operators F_1, F_2 and the corresponding parameters a_1 and a_2 ,

$$\Delta_{F_1} = d + 4 - 2\epsilon, \quad \Delta_{F_2} = d + 4 - 2\epsilon + \omega, \quad (4.17)$$

$$\Delta_{a_1} = 4\epsilon/3 - 2, \quad \Delta_{a_2} = 4\epsilon/3 - \omega. \quad (4.18)$$

For $d=3$ and $\epsilon=2$ this gives $\Delta_{a_1}=2/3$, $\Delta_{a_2}=2/3 - \omega$ ($\Delta_{a_2} = -10/3$ in the first order in ϵ).

Since the parameter a_1 is not renormalized (see above), we have $\bar{a}_1 = a_1 = a_{01} = -c^{-2}$, which along with Eqs. (3.18) and (3.15) in the IR asymptotic region for the effective variable \bar{u}_1 gives $\bar{u}_1 \rightarrow u_1^* \sim c^{-2} \epsilon^{1/3} k^{-2/3}$. Using the well-known relation $\epsilon \sim \nu_c^3/L$ (where ϵ is the mean dissipation rate, ν_c is the mean-square velocity field, and L is the outer scale of turbulence), the expression for u_1^* can be rewritten as

$$u_1^* \sim (\text{Ma})^2 (kL)^{-2/3}. \quad (4.19)$$

In a similar way, one can find the k dependence of the variable u_2^* at $\epsilon=2$. From the relation $u_2^* \sim u_1^*(kl)^\omega$ [see Eq. 3.18], where $l = \epsilon^{-1/4} \nu_0^{3/4}$ is the Kolmogorov dissipation length, one obtains

$$u_2^* \sim (\text{Ma})^2 (kL)^{-2/3} (kl)^\omega, \quad (4.20)$$

and $u_1^* \gg u_2^*$ ($\omega > 0$) in the inertial range $kl \ll 1$. Therefore, the leading contribution to the small k behavior of the scaling function R from Eq. (3.11) is given by the term with the variable u_1^* . In the linear approximation in the Mach number, the leading correction to the kinetic energy spectrum is of the form

$$E(k) \sim \epsilon^{2/3} k^{-5/3} [1 + A(\text{Ma})^2 (kL)^{-2/3}], \quad (4.21)$$

where A is a numerical factor. This correction is independent of the viscosity coefficient ν_0 , which proves the validity of the second Kolmogorov hypothesis. The contribution of u_2^* gives rise to a ν_0 -dependent term, but in the inertial range it only determines a vanishing correction. For $Ma \ll 1$, the correction is rather small because in the inertial range one has $(kL)^{-2/3} \ll 1$. In contrast with the isotropic model, the amplitude factor in Eq. (4.21) and the coefficient A depend on the anisotropy parameters.

V. CONCLUSION

We have shown that in the statistical model of the fully developed turbulence in the presence of uniaxial anisotropy, the kinetic energy spectrum in the inertial range is independent of the viscosity coefficient (i.e., the second Kolmogorov hypothesis holds) in the leading approximation in the Mach number.

In this paper, we have dealt only with the dependence on the uv scale (or, equivalently, on the viscosity coefficient) and have not discussed the dependence on the integral scale L . The RG approach along with the operator product expansion are also applicable to this problem. The most singular L dependence is revealed by the different-time velocity correlations and physically is explained by the well-known sweeping effects, see, e.g., [27]. The RG treatment of this problem has been given in Ref. [24] (see also Ref. [8]) and it is readily generalized to our case. It is now generally accepted that the intermittency phenomenon leads to a singular L dependence of the equal-time correlations, see, e.g. Ref. [28]. In Ref. [29], it has been applied to the simple example of the so-called rapid-change model of passive scalar advection [30] in order to confirm the singular dependence of the equal-time correlation functions on L and calculate the cor-

responding anomalous exponents within the ϵ expansion; the results obtained are in agreement with the previous results obtained using the so-called zero-mode technique [31]. The generalization of these results to more realistic models such as the stochastic Navier-Stokes requires a considerable improvement of the existing technique and remains an open problem.

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