Stabilization of unstable fixed points in the dynamics of a laser with feedback

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We report theoretical and experimental results on the stabilization of unstable steady states in a CO_2 laser with feedback. Periodic and chaotic oscillations have been suppressed by means of a control loop consisting of a high-pass filter, known as washout filter. Although the filter characteristics are determined on the basis of linear analysis, taking into account the problem of robustness with respect to parameter changes, the present control strategy provides a large attractive domain in the phase space. [S1063-651X(99)00707-2]

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I. INTRODUCTION

Chaotic dynamics can be interpreted as a superposition of an infinite number of different unstable periodic orbits among which the system continuously switches [1]. Moreover, irregular oscillations are often related to the competition between different unstable fixed points. Thus the chaotic attractor, that is the region in the phase space visited by the trajectory, contains a rich variety of possible states that can be fruitfully exploited if any flexible control technique can be implemented, to stabilize the originally unstable behaviors.

In order to control chaos, that is to drive the dynamics to periodic orbits or steady states by applying small perturbations, several methods have been proposed [2,3]. The key idea in the early work by Ott, Grebogi, and Yorke (OGY) [4] is to use linear control theory and feedback on a system parameter to direct the motion along the stable manifold of an unstable state. A scalar version of the OGY control method, called occasional proportional feedback [5], and some variations of it, have been successfully used to stabilize unstable steady states and periodic orbits in a multimode Nd:YAG doubled laser, which is a high dimensional system [6]. The problem of the stabilization of an unstable steady state has also been faced in a multimode Nd doped optical fiber laser. In this system the steady state becomes unstable through an Hopf bifurcation, and it has been shown that a feedback proportional to the derivative of the intensity always stabilizes the dynamics [7]. Derivative control has been also successfully applied to the Chua's circuit operating in the double-scroll regime [8], in an electrochemical system [9] and in the Mackey-Glass model [10]. Large periodic modulations of the pump parameter have been demonstrated to be an alternative way to track unstable states in the two-level Lorenz-Haken model [11]. However, this nonfeedback method presents the disadvantage of inducing a modulated output.

In this paper we report theoretical analysis and experimental results about the stabilization of the unstable steady states in a class-B laser. Since the dynamics of a pure class-B laser is ruled by the interplay between intensity and population inversion, it is necessary to introduce a third degree of freedom to observe chaotic behaviors. A CO_2 laser fits such requirements, provided the laser output is fed back to an intracavity electro-optic modulator in a regenerative configuration [12]. This system presents a very rich dynamics, including self-pulsing and deterministic chaos. The control method hereafter described is based on a washout high-pass filter which allows robust stabilization of the unstable fixed points in a wide range of control parameter values. At variance with other methods based on a derivative feedback [7–10] with infinite bandwidth (or at least higher than the characteristic frequencies of the dynamics), here the cut-off frequency of the filter is set to a value much lower than the characteristic frequency, in order to optimize the control. Moreover, the simplicity and speed of this method make it suitable for several experimental applications, not restricted to the field of laser optics.

The paper is organized as follows: Sec. II contains the description of the theoretical model and the discussion yielding the stabilization through a derivative control. In Sec. III we report the experimental results on the stabilization of unstable fixed points in the CO_2 laser with electro-optic feedback. Concluding remarks are drawn in Sec. IV.

II. THE MODEL

The model we use for the CO_2 laser, based on a four-level scheme, can be stated in the following way (after a suitable normalization) [13]:

$$
\begin{aligned}
\dot{x}_1 &= k_0 x_1[x_2 - 1 - k_1 \sin^2(x_6)],\\
\dot{x}_2 &= -\Gamma_1 x_2 - 2k_0 x_2 x_1 + \gamma x_3 + x_4 + P_0,\\
\dot{x}_3 &= -\Gamma_1 x_3 - x_5 + \gamma x_2 + P_0,\\
\dot{x}_4 &= -\Gamma_2 x_4 - \gamma x_5 + z x_2 + z P_0,\\
\dot{x}_5 &= -\Gamma_2 x_5 - z x_3 + \gamma x_4 + z P_0,\\
\dot{x}_6 &= -\beta x_6 + \beta B_0 - \beta f(x_1),
\end{aligned}
\tag{1}
$$

where

TABLE I. Numerical values of the parameters of the theoretical model.

Γ_1	10.0643	γ	0.05
Γ_2	1.0643	k_0	28.5714
α	32.8767	k_1	4.5556
β	0.4286	P_0	0.016

$$f(x_1)=\frac{Rx_1}{1+\alpha x_1}.$$

In these equations the variable x_1 is the normalized photon number and thus is proportional to the laser output intensity, x_2 is proportional to the population difference N_2-N_1 between the two resonant levels, x_3 to the sum N_2+N_1, x_4 and x_5 are proportional to the difference and sum of the populations of the rotational manifolds M_2 and M_1, respectively. Each manifold contains $z=10$ sublevels. The variable x_6, proportional to the feedback voltage, affects the cavity loss parameter through the expression $k_0[1+k_1\sin^2(x_6)]$. The time has been rescaled according to $\tau=t^*7\times10^5\ \mathrm{sec}^{-1}$. The control parameters of the system are B_0 and R, proportional to the bias voltage and to the gain of the feedback loop, respectively. Γ_1, Γ_2, γ, and β represent decay rates, α is a saturation factor, P_0 is the pump parameter. The numerical parameter values are reported in Table I.

In order to implement a suitable control, it is important to identify the fixed-point solutions of Eqs. (1) in the parameter space, and to study their stability. For simplicity, we consider the stationary values of x_1 as a function of the control parameter $B_0\in[0,0.222]$, for different values of the other control parameter $R\in[50,230]$ (see Fig. 1). Obviously, whatever values are assigned to B_0 and R, it is always present a solution corresponding to the not-lasing state, that is $x_1=0$ [and $x_6=B_0$, as can be easily verified from the last of Eqs. (1)]. This solution is unstable for $B_0\leqslant B_{0C}$ and stable for $B_0\geqslant B_{0C}$. $B_{0C}=0.163$ is the value of B_0 for which the losses $1+k_1\sin^2(x_6)$ balance the equilibrium value of x_2, representing the unsaturated gain provided by the excited active medium. Thus, $B_0\leqslant B_{0C}$ and $B_0\geqslant B_{0C}$ correspond to laser above and below threshold, respectively.

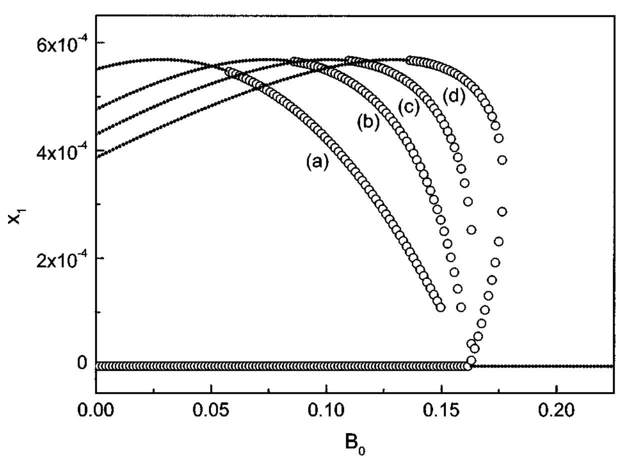

FIG. 1. Stationary values of the rescaled laser intensity x_1 as a function of the control parameter B_0 for different values of R: (a) $R=50$; (b) $R=133$; (c) $R=180$; (d) $R=230$. Dots and open circles represent stable and unstable fixed points, respectively.

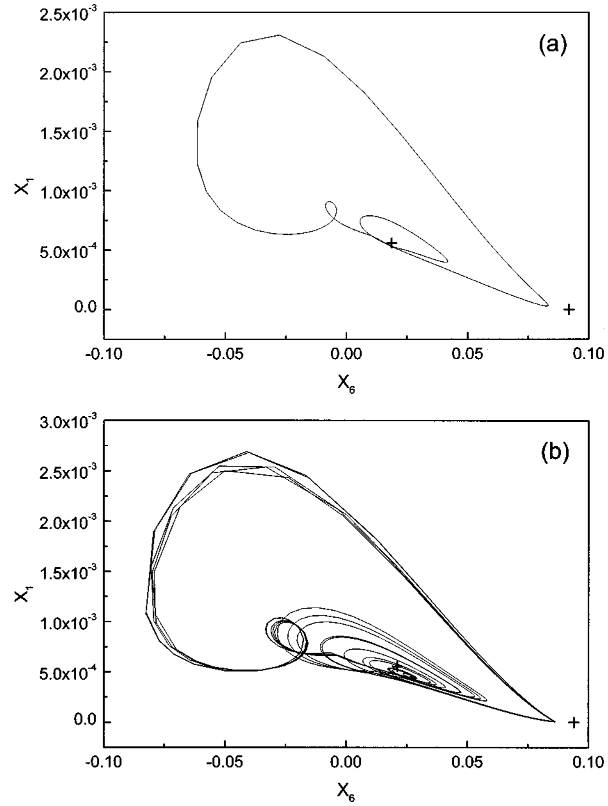

FIG. 2. Projections of the attractors in the phase plane (x_1,x_6). (a) $B_0=0.092$, $R=133$. (b) $B_0=0.094$, $R=133$. Crosses represent the unstable fixed points.

For $B_0\leqslant B_{0C}$ and small values of R, the system presents a second fixed point with $x_1\neq0$. These stationary solutions, lying on a single valued curve, are initially stable (CW laser output), become unstable through an Hopf bifurcation when B_0 is increased (self-pulsing and chaos), and finally converge to the solution $x_1=0$. If the gain is large [case (d)] the curve of the solutions is no more single valued, and a third unstable fixed point appears in the region $B_0\geqslant B_{0C}$. Anyway, beyond the critical value B_{0C}, the stability of the zero intensity solution implies that the trajectory in the phase space can only visit the vicinity of the other unstable fixed points during a transient before approaching $x_1=0$. For this reason no dynamical behaviors except transients are observable for $B_0\geqslant B_{0C}$.

We will present in the following a summary of the typical dynamics, influenced by the competition between the two unstable fixed points. In Fig. 2 we show projections of the attractor in the phase plane (x_1,x_6) corresponding to two different values of B_0, at the same R value. Figure 2(a) shows a stable limit cycle visiting the regions close to the two unstable steady state solutions. It is important to observe that in this condition it exists a small region around the unstable fixed point never visited by the trajectory. In Fig. 2(b) we report the chaotic evolution, again showing the competition between the two unstable states. For lower values of B_0 we observe a stable limit cycle around the nonzero stationary solution.

The implementation of a method to control the unstable

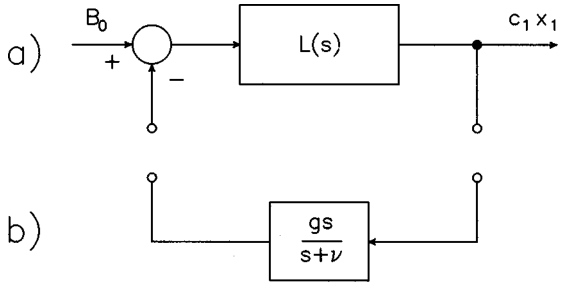

FIG. 3. Logical diagram of (a) the linearized laser equations and (b) the control loop implemented by a washout high-pass filter.

steady states requires to fulfill two general conditions: (i) the maintenance of the position of the stationary state, and (ii) control robustness with respect to large variations of the two system parameters B_0 and R. Moreover, in our case it is also necessary that, in the phase space, the region of attraction created by the control is large enough to intersect the region occupied by the unperturbed attractor.

Nonlinear approaches, such as the feedback linearization or sliding control methods [14], are not applicable because not all the state variables are accessible in our experiment. For this reason we first study the control problem for the system linearized around the unstable fixed point. Then, we verify *a posteriori* that it exists a condition which ensures a large domain of attraction from which there is convergence to the fixed point. After linearization around the unstable stationary solution, the local dynamics can be obtained in terms of the Jacobian matrix J evaluated at the fixed point. The linearized system can be represented by the scheme shown in Fig. 3(a) with transfer function $L(s)$ given by

$$L(s)=\tilde{C}(sI-J)^{-1}\tilde{B},$$

where

$$\tilde{B}=\begin{pmatrix}0\\0\\0\\0\\0\\-\beta\end{pmatrix},\quad \tilde{C}=(c_1,0,0,0,0,0),\quad c_1=\frac{-\beta R}{(1+\alpha x_{1e})},$$

$x_{1e}\neq 0$ being the equilibrium value of the variable x_1.

The input to the linear block is given by B_0, while the output is proportional to x_1. Since both the quantities are experimentally accessible, the control can be implemented by a function of x_1 fed back and added with B_0. The requirement of a control which maintains the position of the stationary point excludes a conventional feedback proportional to x_1, but suggests to use a feedback proportional to the derivative of the output signal. The simplest realization of this is represented by a washout filter, which is a stable high-pass filter commonly used in control of aircrafts [15], with transfer function

$$C(s)=gs/(s+\nu). \tag{2}$$

The corresponding block diagram is shown in Fig. 3(b). The first step to realize the filter consists in selecting suitable values for the parameters g and ν. A possible choice for the

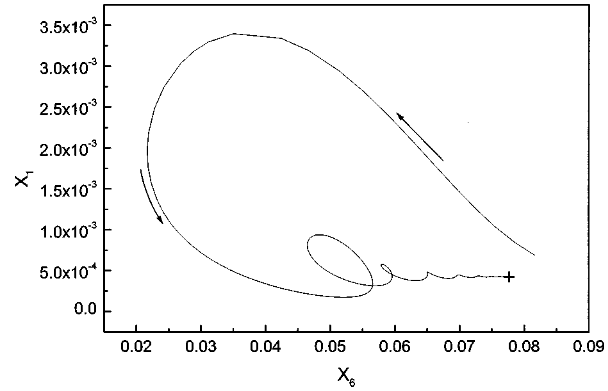

FIG. 4. Typical transient evolution in the phase plane (x_1,x_6) toward the stabilized fixed point (cross) for $B_0=0.133$ and $R=133$.

cut-off frequency ν could be a value corresponding to β, which represents the bandwidth of the feedback loop. Although in principle it is always possible to find a suitable g value providing stabilization when $\nu=\beta$, only a small set of initial conditions, contained in a narrow region around the fixed point, can be driven to the stabilized steady state. This implies that the control may fail in a dynamical condition such that of Fig. 2(a). In order to obtain a largely attractive fixed point, ν can be selected so that the filter presents a flat amplitude response over the frequency range corresponding to the bandwidth of $L(s)$. This feature assures feedback, and consequently rejection, of all the undesired frequency components of the uncontrolled system. Once ν is fixed, we have to optimize the gain g of the filter. To do this, it is useful to determine the poles of the closed loop transfer function, that is, to study the root locus of the complex function $1+F(s)$, where $F(s)=L(s)\ C(s)$ is the open loop transfer function. For small values of g, all the roots are real and negative, except a pair of complex conjugate ones having positive real part. By increasing g, the two complex roots move towards the negative half plane, while the other roots move (more slowly) in the opposite direction. After a critical value g^*, all the roots have negative real parts and the system becomes stable. Nevertheless, the time required to approach the steady state can be very long since two eigenvalues are only weakly negative. For this reason it is convenient

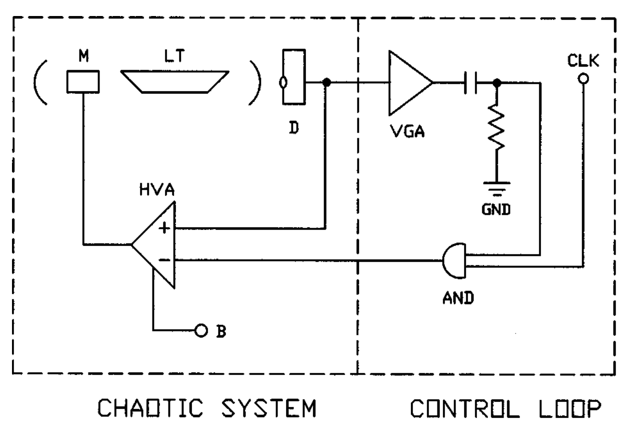

FIG. 5. Scheme of the experimental setup: LT, laser tube; *M*, electro-optic modulator; *D*, fast HgCdTe detector; HVA, high voltage amplifier; VGA, variable gain amplifier; CLK, TTL clock port.

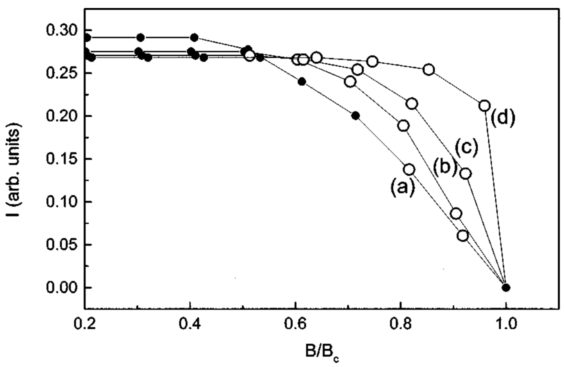

FIG. 6. Stationary values of the laser intensity I as a function of the rescaled control parameter B/B_C for different values of the high voltage amplifier gain A: (a) $A=390$; (b) $A=1200$; (c) $A=1400$; (d) $A=2200$. Dots represent originally stable fixed points; open circles denote originally unstable fixed points stabilized by the control.

to further increase g up to a value for which the real part of the two complex roots equals the value of the larger real root.

In order to insure a robust control with respect to changes of B_0 and R in the previously specified ranges, we can use for ν and g the minimum and the average values, respectively, among those calculated for each pair (B_0,R). Following this strategy, the transfer function of this tracking filter was found to be

$$C(s)=\frac{0.86s}{s+0.015}.$$

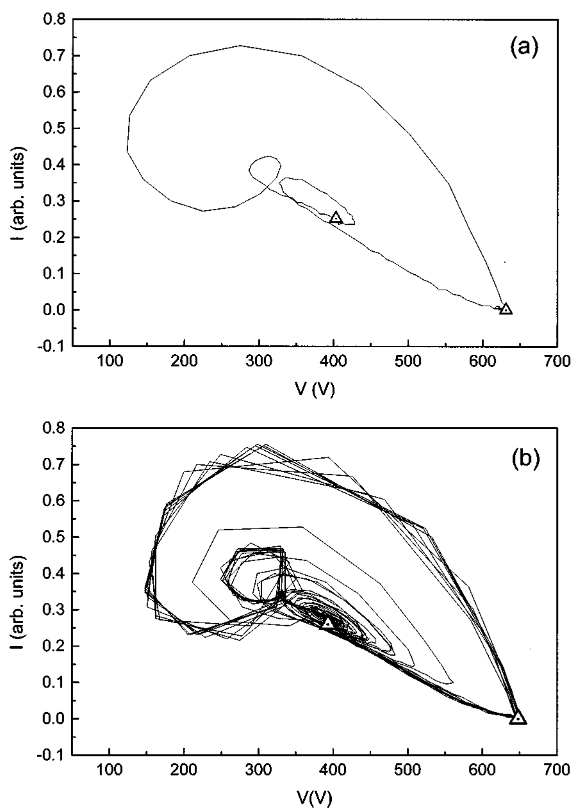

FIG. 7. Projections of the attractors in the phase plane (I,V). (a) $B=631$ V, $A=1900$. (b) $B=648$ V, $A=1900$. Triangles represent the unstable fixed points.

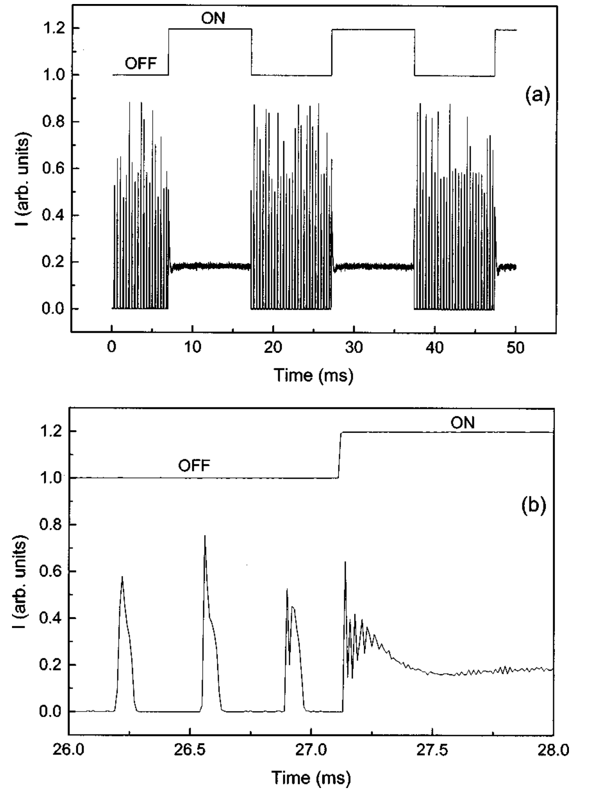

FIG. 8. Temporal evolution of the laser intensity I when the control is switched on and off synchronously with a TTL clock signal [same parameter values of Fig. 7(b)].

The use of this control provides a stable fixed point in the phase space with a large domain of attraction. An example of a transient is shown in Fig. 4.

III. THE EXPERIMENT

The experiment (Fig. 5) has been performed on a single mode CO_2 laser with an intracavity electro-optic modulator. After detection and suitable amplification, the laser intensity signal is sent to the modulator, summed with a constant voltage B acting as the control parameter. The relationship between B and B_0 is $B_0=\pi(B-V_0)/V_\lambda$, where $V_0=100$ V and $V_\lambda=4240$ V. This loop provides the extra degree of freedom necessary to observe chaotic oscillations [12]. The second feedback loop contains the tracking filter used to stabilize the unstable fixed point. It has been implemented by a variable gain amplifier in series with an RC high-pass filter, with $R=11.4\,\mathrm{k}\Omega$ and $C=9.2$ nF. This circuit presents a transfer function with the same analytical structure of Eq. (2). The cut-off frequency is 1.52 kHz, which, following the time rescaling operated in the model, gives $\nu=0.014$ in good agreement with the theoretical predictions. The control signal can be gated by a TTL square wave in order to observe several transients toward the desired steady state solution.

The stabilized fixed points, namely, the values of the CW laser intensity I $(I\propto x_1)$, are reported in Fig. 6 as a function of the control parameter B, for different values of the high voltage amplifier gain A $(A\propto R)$. The horizontal axis is rescaled to the critical value $B_C=978$ V (corresponding to B_{0C}), where the cavity losses balance the laser gain provided by the pump mechanism. The solutions corresponding to I

$=0$, unstable for $B<B_C$ and stable for $B>B_C$, are not reported. It is important to observe that the curves of Fig. 6 present the same qualitative behavior than those of Fig. 1 when the gain A is increased. Moreover, stabilization has been achieved over the whole ranges of B and A without any change in the control loop parameters. Figure 7 shows the two dimensional projections of the attractors in terms of I versus the feedback voltage V $(V\propto x_6)$, and proves the relevant role played by the unstable fixed points in the dynamics. The steady state with $I\neq 0$ has been detected by using the stabilization loop, while the other one can be easily recognized considering that for $I=0$ we have $V=B$. Finally, several transients from chaos towards the stabilized steady state are reported in Fig. 8(a), paced by the TTL clock signal; the enlargement of Fig. 8(b) shows that the typical transient times are of the order of 0.5 msec.

IV. CONCLUSIONS

In this paper we have shown that chaotic and nonchaotic dynamics of a CO_2 laser with feedback can be stabilized on the unstable steady state by using a washout filter with appropriate cut-off frequency. The proposed feedback scheme, which does not require a real time analysis of the state of the system and is able to track the steady states over a wide range of the control parameters, seems particularly suitable to stabilize the laser output intensity on fast time scales, as those accessible acting on the cavity losses. The possibility to continuously select the output laser intensity starting from its maximum value up to the laser threshold is of practical importance in applications where an high stability laser source with adjustable output is required.

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