Semiclassical interaction of moving two-level atoms with a cavity field: From integrability to Hamiltonian chaos

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The dynamics of an ensemble of two-level atoms moving through a single-mode lossless cavity is investigated in the semiclassical and rotating-wave approximations. The dynamical system for the expectation values of the atomic and field observables is considered as a perturbation to one of the following integrable versions: (i) a model with atoms moving through *a spatially inhomogeneous resonant* field, and (ii) a model with atoms interacting with *a nonresonant* eigenmode which is assumed to be *homogeneous* on the cavity size. We find the general exact solutions for both the models and show that they contain special solutions describing a coherent effect of population and radiation trapping. Using the Melnikov method, we prove analytically transverse intersections of stable and unstable manifolds of a hyperbolic fixed point under a small modulation of the vacuum Rabi frequency. These intersections are believed to provide the Smale horseshoe mechanism of Hamiltonian chaos. The analytical results are accompanied with direct computation of topographical maps of maximal Lyapunov exponents that give a representative image of regularity and chaos in the atom-field system in different ranges of its control parameters—the frequency detuning, the number, and the velocity of atoms. [S1063-651X(99)09605-1]

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I. INTRODUCTION

The basic simple model of interaction between matter and radiation comprises two-level atoms interacting with a single-mode electromagnetic field in a lossless cavity. In spite of its simplicity, the model is intrinsically nonlinear with the atom-field coupling being the coefficient of this nonlinearity. In recent decades, there has arisen a considerable interest in extremely nonlinear dynamics in the atom-field interaction that is connected, mainly, with the fundamental problem of correspondence of classical and quantum dynamics. The concept of dynamical chaos is at the heart of this problem. Despite the large amount of effort devoted to the problem of how chaos in the classical domain manifests itself in the evolution of corresponding systems in the quantum domain, a number of important questions still remained unanswered [1].

The semiclassical picture, when one treats atoms quantum mechanically and the electromagnetic field as classical degrees of freedom, may be considered as a kind of a palliative. In the semiclassical approximation, the powerful methods of nonlinear dynamics and ergodic theory are applied to a hybridized system with quantum and classical degrees of freedom while leaving subtle questions of the effect of quantum correlations and fluctuations beyond the framework of the description. Obviously, the profound analysis of the complicated behavior of a semiclassical system can serve as an important step towards an understanding of the respective properties of the corresponding fully quantized system.

In recent years it has become clear that the semiclassical hybridized systems can demonstrate truly chaotic motion with positive maximal Lyapunov exponents, i.e., they can show extremal sensitivity to initial conditions. Such systems were first treated in the context of laser physics. Hamiltonian chaos in the interaction of two-level atoms with their own radiation field has been found by Belobrov, Zaslavskii, and Tartakovskii [2] and later numerically and analytically studied in great detail by other authors [3–9]. Similar models, which treat two-level objects interacting with a classical oscillator, arise quite naturally in other fields of physics, mainly in molecular and solid-state physics (see, for example, the small-polaron [10], the spin-boson [11,12], the nonlinear dimer [13], and the cavity-polariton [14] systems).

Analytical and numerical studies of the semiclassical atom-field systems prove that the counter-rotating terms neglected in the rotating-wave approximation (RWA) can lead to chaotic behavior. Including the non-RWA terms breaks the regular evolution of the RWA model. The latter one can be shown to be equivalent to an unforced nonlinear oscillator [2]. Systematic corrections to the RWA provide a periodic modulation of the near-separatrix motion of the oscillator [4] that is known to be a generic mechanism for Hamiltonian chaos. The physical mechanism for chaos may therefore be tied to virtual transitions in the atom-field system [4] that are, of course, hardly probable under usual conditions.

The idea to use an additional external field, which is injected into a cavity, in order to cause Hamiltonian chaos *even in the RWA* was proposed in [15]. Recently, it was shown numerically by two of the present authors [16] that the RWA chaos in the semiclassical matter-radiation model may arise with moving two-level atoms *even without any additional external field*. Recent developments in cavity quantum electrodynamics (for a review on this subject, see [17]) give rise to many situations where one has to deal with two-level atoms moving through a high-Q cavity (for instance, a micromaser, a microlaser, an atom laser, atoms in traps, and so on).

In this paper, we present the analytical and numerical treatment of the problem of Hamiltonian chaos with two-

level atoms moving through an ideal single-mode cavity that was numerically found in [16]. In Sec. II the model is specified in detail. In the semiclassical limit and with the approximations adopted, we derive the dynamical atom-field system from the operator Heisenberg equations by replacing all operators with their expectation values and consistently dropping all the quantum-correlator terms. This procedure may be considered as an alternative to the Maxwell-Bloch equations. The Heisenberg picture is, however, more preferable because it enables us to take into account, in a natural way, some effects of the quantum correlations [18,19]. It is shown that the semiclassical dynamics is governed by a complex Duffing oscillator with a parametric excitation being caused by a spatial inhomogeneity of the cavity mode that modulates the vacuum Rabi frequency of moving atoms.

In Sec. III we obtain general exact solutions of the semiclassical atom-field system in two limits. It is shown in the first subsection that, if the frequency of the cavity mode coincides exactly with the atomic transition frequency, the atom-field system is exactly integrable *even in the case of an arbitrary spatial structure of the selected mode*. In Sec. III B we show that the model with an arbitrary detuning is integrable if atoms move in a direction along which the cavity mode may be assumed to be homogeneous. In both cases, a homoclinic structure consisting of two homoclinic tori contracting to a hyperbolic equilibrium point, which corresponds to the equilibrium semiclassical state with fully inverted atoms and vacuum field, is found.

In Sec. IV we use the general exact solutions for the nonautonomous resonant system and the autonomous nonresonant system to describe an interesting effect of locking of the oscillations of the atomic inversion and of the average number of photons that may occur, under appropriate conditions, with moving atoms if the atoms were prepared in the same superposition state with an arbitrary phase, and the cavity mode was prepared in a coherent state with the atomic phase.

In Sec. V we show that spatial inhomogeneity of the nonresonant cavity field breaks the regularity and produces Hamiltonian chaos with moving atoms. Using the homoclinic tori found in Sec. III as a framework, we calculate a Melnikov function whose simple zeros imply transverse intersections of the stable and unstable manifolds of a hyperbolic fixed point. The breakup of these homoclinic orbits is believed to be a source of Smale horseshoe chaos in our atom-field system. Poincaré sections calculated show the breakup of the unperturbed separatrix and a homoclinic tangle. For an investigation of the chaotic oscillations in the system in different ranges of its control parameters, the number of atoms, the detuning, and the velocity of atoms, we calculate topographical maps of the maximal Lyapunov exponents.

II. SPATIAL INHOMOGENEITY OF THE CAVITY FIELD MODULATES THE VACUUM RABI FREQUENCY

Consider ''a droplet'' with N two-level atoms that moves through a single-mode cavity along the axis $\mathbf{r}$. The cavity field is supposed to be inhomogeneous along this axis with the function $\mathbf{f}(\mathbf{r})$ describing its spatial structure. The volume of the droplet is supposed to be much smaller than λ_f^3, where λ_f is the wavelength of the field mode. We are working in the strong-coupling regime, $\Omega_0\sqrt{N}\gg\omega_f/Q$, where Ω_0 is the amplitude of the vacuum Rabi frequency and ω_f and Q are the frequency and the quality factor of the cavity, respectively. In this regime, N atoms exchange excitation with a cavity field with a period $2\pi/\Omega_0\sqrt{N}$ that is much shorter than the atomic T_a and cavity T_f relaxation times. Since the recoil energy of atoms accompanying emission of photons, $R=(\hbar\omega_f)^2/2mc^2$, is very small (it is of the order of 10^{-18} eV in the microwave range), the change in kinetic energy of the atoms can be neglected (the Raman-Nath approximation).

In the pointlike, single-mode, lossless, Raman-Nath, and RWA approximations, the respective Hamiltonian has the form [20]

$$H=\tfrac{1}{2}\hbar\omega_a\sum_{j=1}^{N}\sigma_z^j+\hbar\omega_f(a^\dagger a+\tfrac{1}{2})+\hbar\Omega_0(\mathbf{r})\sum_{j=1}^{N}(a\sigma_+^j+a^\dagger\sigma_-^j), \tag{1}$$

where $\hbar\omega_a$ is the energy separation between two working atomic levels, σ_z, $\sigma_\pm=\sigma_x\pm i\sigma_y$ are the usual Pauli matrices, and a and $a^\dagger$ are the field destruction and creation operators, respectively. The vacuum Rabi frequency depends on the position $\mathbf{r}$ of the center of the atomic droplet inside a cavity

$$\Omega_0(\mathbf{r})=|\mathbf{d}\mathbf{f}(\mathbf{r})|\left(\frac{2\pi\omega_f}{\hbar V_c}\right)^{1/2}, \tag{2}$$

where V_c is the cavity volume and d is the value of the electric dipole moment.

The Heisenberg equations for the atomic and field operators can be derived from the Hamiltonian (1) in a straightforward manner,

$$\begin{aligned}
\frac{d}{dt}\sum\sigma_x&=-\omega_a\sum\sigma_y+i\Omega_0(\mathbf{r})(a-a^\dagger)\sum\sigma_z,\\
\frac{d}{dt}\sum\sigma_y&=\omega_a\sum\sigma_x-\Omega_0(\mathbf{r})(a+a^\dagger)\sum\sigma_z,\\
\frac{d}{dt}\sum\sigma_z&=-i\Omega_0(\mathbf{r})(a-a^\dagger)\sum\sigma_x+\Omega_0(\mathbf{r})(a+a^\dagger)\sum\sigma_y,\\
\frac{d}{dt}(a+a^\dagger)&=-i\omega_f(a-a^\dagger)-\Omega_0(\mathbf{r})\sum\sigma_y,\\
\frac{d}{dt}(a-a^\dagger)&=-i\omega_f(a+a^\dagger)-i\Omega_0(\mathbf{r})\sum\sigma_x.
\end{aligned} \tag{3}$$

The crucial point is to disentangle the operator products of the type $\langle(a\pm a^\dagger)\sigma\rangle$, when deducing from Eq. (3) the equations for the expectation values of the respective operators. The simplest factorization of expectation values of the operator products to the products of the respective expectation values [e.g., $\langle(a\pm a^\dagger)\sigma\rangle=\langle a\pm a^\dagger\rangle\langle\sigma\rangle$] is known as the semiclassical approximation. As is shown in the Appendix,

with N atoms this approximation is valid with the accuracy of the order of $\mathcal{O}(1/N)$. After taking expectation values with respect to a factorized quantum state, we obtain from Eq. (3) the set of five coupled equations

$$\begin{aligned}\dot{x}&=-y-\Omega_N f(\tau)zp,\\ \dot{y}&=x-\Omega_N f(\tau)ze,\\ \dot{z}&=\Omega_N f(\tau)(xp+ye),\\ \dot{e}&=\omega p-\Omega_N f(\tau)y,\\ \dot{p}&=-\omega e-\Omega_N f(\tau)x\end{aligned} \tag{4}$$

for the following quantities:

$$\begin{aligned}x&=\frac{1}{N}\sum_{j=1}^{N}\langle\sigma_x\rangle,\quad y=\frac{1}{N}\sum_{j=1}^{N}\langle\sigma_y\rangle,\quad z=\frac{1}{N}\sum_{j=1}^{N}\langle\sigma_z\rangle,\\ e&=\frac{1}{\sqrt{N}}\langle a+a^{\dagger}\rangle,\quad p=\frac{i}{\sqrt{N}}\langle a^{\dagger}-a\rangle.\end{aligned} \tag{5}$$

For atoms moving through a cavity with a constant velocity v_a, the vacuum Rabi frequency (2) becomes a time-dependent function, $\Omega_0(r)\rightarrow\Omega_0 f(v_a t)$. The nonlinear nonautonomous dynamical system (4) is written in the dimensionless form with the derivatives with respect to $\tau=\omega_a t$. As the control parameters, it has the dimensionless collective vacuum Rabi frequency

$$\Omega_N=\frac{\Omega_0\sqrt{N}}{\omega_a} \tag{6}$$

and the dimensionless detuning

$$\omega=\frac{\omega_f}{\omega_a}. \tag{7}$$

When specifying a spatial profile of the eigenmode in Sec. V, the atom-field system (4) will be show to have the third control parameter, the velocity of atoms. The integrals of motion

$$\mathcal{R}=x^2+y^2+z^2=1,\quad \mathcal{W}=e^2+p^2+2z \tag{8}$$

reflect the unitarity of atomic evolution and a conservation of energy, respectively.

Using the simplest factorization, we reduce the infinite-dimensional state space of the fully quantum system to the five-dimensional phase space of the semiclassical system. Due to the integrals (8), the motion is, in fact, restricted on a three-dimensional hypersurface. By introducing new complex-valued variables

$$\eta=x+iy,\quad \xi=p+ie, \tag{9}$$

the dynamical system (4) may be reduced to a single complex ODE of the second order

$$\ddot{\xi}-\left(i(\omega+1)+\frac{\dot{f}}{f}\right)\dot{\xi}+\left[\omega\left(i\frac{\dot{f}}{f}-1\right)-\frac{1}{2}(\Omega_N f)^2\mathcal{W}\right]\times\xi+\frac{1}{2}(\Omega_N f)^2\xi|\xi|^2=0 \tag{10}$$

with the initial conditions

$$\xi(0)=p(0)+ie(0),\quad \dot{\xi}(0)=i\omega\xi(0)-\Omega_N f(0)\,\eta(0). \tag{11}$$

It is a complex Duffing oscillator with a parametric excitation being caused by a spatial inhomogeneity of the cavity mode that modulates the vacuum Rabi frequency of atoms moving through the cavity.

III. INTEGRABLE LIMITS

In this section we will show that the atom-field dynamical system (4) is exactly integrable, at least in the two cases. In Sec. III A the exact solution of Eqs. (4) will be obtained in the case of the resonant interaction between moving atoms and a cavity mode with an arbitrary spatial structure along the axis of propagation of the atomic droplet. In Sec. III B the exact solution will be found for the system (4) with an arbitrary detuning but with a constant vacuum Rabi frequency, the case corresponding to the model with two-level atoms moving in a direction along which the cavity field does not vary spatially. In the Raman-Nath approximation, it is, of course, equivalent to the model with atoms at rest.

A. Nonautonomous resonant system

If the frequency of the cavity mode ω_f coincides exactly with the atomic transition frequency ω_a, i.e., if $\omega=1$, the dynamical system (4) has the additional integral of motion

$$\mathcal{J}=xe-yp \tag{12}$$

resulting from a conservation of the interaction energy between moving atoms and the resonant field that is valid in the RWA *for an arbitrary spatial structure of the cavity field* $\mathbf{f}(\mathbf{r})$.

The closed equation for the density of the atomic inversion z can be derived from the set of equations (4) with the help of the three integrals of motion (8) and (12),

$$\dot{z}=\pm\Omega_N f\sqrt{(\mathcal{W}-2z)(1-z^2)-\mathcal{J}^2}. \tag{13}$$

Its general exact solution is written in terms of the Jacobian elliptic functions

$$z(\tau)=z_1+(z_2-z_1)\mathrm{sn}^2\left(\sqrt{\tfrac{1}{2}(z_3-z_1)}\Omega_N\times\left(\int_0^{\tau}f(\tau')d\tau'-T\right);\frac{z_2-z_1}{z_3-z_1}\right), \tag{14}$$

where

$$T=\frac{1}{\Omega_N\sqrt{2}}\int_{z(0)}^{z_1}\frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)}}. \tag{15}$$

Here $-1 \leqslant z_1 < z_2 < z_3 \leqslant 1$ are the roots of the algebraic equation

$$z^3 - \tfrac{1}{2}\mathcal{W} z^2 - z + \tfrac{1}{2}(\mathcal{W} - \mathcal{J}^2) = 0. \tag{16}$$

The general exact solution for the other variables of the nonautonomous resonant atom-field system is written with the help of the solution (14) for the atomic inversion as follows:

$$\begin{aligned} x &= X \sin(\tau + \theta) - Y \cos(\tau + \theta), \\ y &= -X \cos(\tau + \theta) - Y \sin(\tau + \theta), \\ e &= Z \sin(\tau + \theta), \\ p &= Z \cos(\tau + \theta), \end{aligned} \tag{17}$$

where

$$\begin{aligned} X &= \mathcal{J} Z^{-1}, \\ Y &= \pm \sqrt{1 - z^2 - X^2}, \\ Z &= \pm \sqrt{\mathcal{W} - 2z}, \\ \theta &= 2\Omega_N \mathcal{J} \int_0^\tau f(\tau') Z^{-2} d\tau' + \theta_0, \end{aligned} \tag{18}$$

where θ_0 is a constant. When writing down the solution (17), the signs in front of the roots in Eqs. (18) should be chosen to be the same, both the upper ones, or both the lower ones.

The nonautonomous integrable version of the dynamical system (4) with $\omega = 1$ may be transformed into an autonomous one and written in a rotating frame in the form of the canonical Hamilton's equations. Following [21], we introduce new coordinates u and q and their canonically conjugated momenta v and $\mathcal{J}$ that are connected with the old variables in the following way:

$$\begin{aligned} p + ie &= u \exp[i(q + \tau)], \\ x + iy &= -\left(v + i \frac{\mathcal{J}}{u} \right) \exp[i(q + \tau)]. \end{aligned} \tag{19}$$

The integrable Hamiltonian in the new coordinates has the form

$$H_0 = \frac{1}{2}\left(v^2 + \frac{\mathcal{J}^2}{u^2} \right) + \tfrac{1}{8}(\mathcal{W} - u^2)^2. \tag{20}$$

With the canonical Poisson brackets $\{u, v\} = \{q, \mathcal{J}\} = 1$, one has Hamilton's equations

$$\begin{aligned} u' &= v, \\ v' &= \tfrac{1}{2} u(\mathcal{W} - u^2) + \frac{\mathcal{J}^2}{u^3}, \\ q' &= \frac{\mathcal{J}}{u^2}, \\ \mathcal{J}' &= 0, \end{aligned} \tag{21}$$

where a prime denotes differentiation with respect to the new "time" $\Omega_N \int_0^\tau f(t)\, dt$.

The polar coordinates $(u, q, v, \mathcal{J})$ help us to identify homoclinic orbits in the integrable version of the original system. Really, the phase portrait of Eqs. (21) for the case $\mathcal{J} = 0$ and $\mathcal{W} > 0$ contains the orbits of an unforced, undamped real Duffing oscillator with a separatrix in the plane u-v homoclinic to the hyperbolic fixed point $(u = v = 0)$. This is a fixed point $S_+ : (x_s = y_s = e_s = p_s = 0, z_s = 1)$ of the original problem that corresponds to all the atoms being in the fully inverted state and vacuum field. S_+ is always unstable at exact resonance.

In terms of the old variables, the homoclinic orbits are given explicitly by

$$\begin{aligned} x_0 &= \pm 2 \operatorname{sech}(\Omega_N \tau) \tanh(\Omega_N \tau) \cos(\tau + \theta_0), \\ y_0 &= \pm 2 \operatorname{sech}(\Omega_N \tau) \tanh(\Omega_N \tau) \sin(\tau + \theta_0), \\ z_0 &= 1 - 2 \operatorname{sech}^2(\Omega_N \tau), \\ p_0 &= \pm 2 \operatorname{sech}(\Omega_N \tau) \cos(\tau + \theta_0), \\ e_0 &= \pm 2 \operatorname{sech}(\Omega_N \tau) \sin(\tau + \theta_0). \end{aligned} \tag{22}$$

B. Autonomous nonresonant system

If detuned atoms move in a spatially homogeneous field ($f = \text{const}$), the atom-field dynamical system (4) becomes autonomous and acquires the additional integral of motion

$$\mathcal{C} = \Omega_N (xe - yp) - (\omega - 1) z \tag{23}$$

that describes a conservation of the energy of interaction between atoms and the homogeneous cavity field in the RWA *even out of resonance*. Now we can derive the equation of motion for the density of the atomic inversion

$$\dot{z} = \pm \Omega_N \left[(\mathcal{W} - 2z)(1 - z^2) - \left(\frac{\mathcal{C} + (\omega - 1) z}{\Omega_N} \right)^2 \right]^{1/2} \tag{24}$$

that is solved in terms of the elliptic Jacobian function

$$z = z_1 + (z_2 - z_1) \operatorname{sn}^2\left(\sqrt{\tfrac{1}{2}(z_3 - z_1)}\, \Omega_N (\tau - T); \frac{z_2 - z_1}{z_3 - z_1} \right), \tag{25}$$

where

$$T = \frac{1}{\Omega_N \sqrt{2}} \int_{z(0)}^{z_1} \frac{dz}{\sqrt{(z - z_1)(z - z_2)(z - z_3)}}, \tag{26}$$

and $z_1 < z_2 < z_3$ are the roots of the algebraic equation

$$\begin{aligned} z^3 - \left(\frac{\mathcal{W}}{2} + \frac{(\omega - 1)^2}{2\Omega_N^2} \right) \\ \times z^2 - \left(1 + \frac{\mathcal{C}(\omega - 1)}{2\Omega_N^2} \right) z + \left(\frac{\mathcal{W}}{2} - \frac{\mathcal{C}^2}{2\Omega_N^2} \right) = 0. \end{aligned} \tag{27}$$

Let us seek the solution of the autonomous version of Eqs. (4) for the variables x, y, e, and p in the form

$$x=\frac{S}{U}\sin(\omega\tau+\theta)-V\cos(\omega\tau+\theta),$$

$$y=-\frac{S}{U}\cos(\omega\tau+\theta)-V\sin(\omega\tau+\theta),$$

$$e=U\sin(\omega\tau+\theta), \tag{28}$$

$$p=U\cos(\omega\tau+\theta).$$

With the help of the integrals of motion (8) and (23), it can be shown that all the new variables V, U, S, and θ are the functions of the old variable z,

$$U=\pm\sqrt{\mathcal{W}-2z},$$

$$V=\pm\sqrt{1-z^2-(S/U)^2},$$

$$S=(\Omega_N)^{-1}[\mathcal{C}+(\omega-1)z], \tag{29}$$

$$\theta(\tau)=\int_0^\tau\frac{\mathcal{C}+(\omega-1)z}{\mathcal{W}-2z}d\tau'+\theta_0.$$

In polar coordinates the autonomous nonresonant version of the dynamical system (4) may be written in the form of the canonical Hamilton's equations,

$$\dot{u}=\Omega_N v,$$

$$\dot{v}=\tfrac{1}{2}\Omega_N u(\mathcal{W}-u^2)+\Omega_N\frac{\mathcal{I}^2}{u^3}-\frac{(\omega-1)^2u}{4\Omega_N}, \tag{30}$$

$$\dot{q}=\Omega_N\frac{\mathcal{I}}{u^2},$$

$$\dot{\mathcal{I}}=0$$

with the Hamiltonian

$$H_0=\tfrac{1}{2}\Omega_N\left(v^2+\frac{\mathcal{I}^2}{u^2}\right)+\frac{\Omega_N}{8}(\mathcal{W}-u^2)^2+\frac{(\omega-1)^2}{8\Omega_N}u^2. \tag{31}$$

The map from (x,y,e,p) to $(u,v,q,\mathcal{I})$ is given by

$$x+iy=-\left(v+i\frac{\mathcal{I}}{u}-i\frac{(\omega-1)u}{2\Omega_N}\right)\exp i[q+\tfrac{1}{2}(\omega+1)\tau],$$

$$p+ie=u\exp i[q+\tfrac{1}{2}(\omega+1)\tau]. \tag{32}$$

The specific solution on the separatrix, which takes place with $\mathcal{W}=2$, is given explicitly by

$$x_0=\pm\frac{\alpha^2}{2}\operatorname{sech}\left(\frac{\alpha\Omega_N}{2}\tau\right)\tanh\left(\frac{\alpha\Omega_N}{2}\tau\right)\cos\left(\frac{\omega+1}{2}\tau+\theta_0\right)$$
$$\mp\frac{(\omega-1)\alpha}{2\Omega_N}\operatorname{sech}\left(\frac{\alpha\Omega_N}{2}\tau\right)\sin\left(\frac{\omega+1}{2}\tau+\theta_0\right),$$

$$y_0=\pm\frac{\alpha^2}{2}\operatorname{sech}\left(\frac{\alpha\Omega_N}{2}\tau\right)\tanh\left(\frac{\alpha\Omega_N}{2}\tau\right)\sin\left(\frac{\omega+1}{2}\tau+\theta_0\right)$$
$$\pm\frac{(\omega-1)\alpha}{2\Omega_N}\operatorname{sech}\left(\frac{\alpha\Omega_N}{2}\tau\right)\cos\left(\frac{\omega+1}{2}\tau+\theta_0\right),$$

$$z_0=1-\frac{\alpha^2}{2}\operatorname{sech}^2\left(\frac{\alpha\Omega_N}{2}\tau\right), \tag{33}$$

$$e_0=\pm\alpha\operatorname{sech}\left(\frac{\alpha\Omega_N}{2}\tau\right)\sin\left(\frac{\omega+1}{2}\tau+\theta_0\right),$$

$$p_0=\pm\alpha\operatorname{sech}\left(\frac{\alpha\Omega_N}{2}\tau\right)\cos\left(\frac{\omega+1}{2}\tau+\theta_0\right),$$

where $\alpha^2=4-[(\omega-1)/\Omega_N]^2$. The solution (33) describes the locus of states in which atoms radiate and reabsorb their own field in infinite time. The orbits (33) are homoclinic to the fixed point S_+. Contrary to the case of the resonant interaction where it is always stable, this point can be proven, out of resonance, to be unstable if $\Omega_N\geqslant|\omega-1|/2$ and stable otherwise [22]. Under this condition, the expression under the square root of α is non-negative.

Thus in the case $\Omega_N\geqslant|\omega-1|/2$ the autonomous nonresonant atom-field system possesses a two-sheeted homoclinic manifold (a pair of two-dimensional homoclinic tori) which is a collection of all the pairs of separatrices that connect the equilibrium point S_+ to itself. These tori are given explicitly by the solution (33) or implicitly by the following values of the integrals of motion ($\mathcal{W}=2,\mathcal{C}=0$). This unperturbed manifold provides a framework in which we will analyze in Sec. V the chaotic oscillations in the nonautonomous nonresonant atom-field interaction.

IV. HOW TO LOCK THE ATOMIC INVERSION AND THE AVERAGE NUMBER OF PHOTONS IN THE ATOM-FIELD INTERACTION

A. Nonautonomous resonant interaction

The general exact solutions (17) and (18) obtained in Sec. III A for resonant moving atoms contain a special solution that leaves the density of the atomic inversion z and the density of the average number of photons $n=(e^2+p^2)/4$ unaffected. We will seek the solution in the form

$$x_{\rm trap}=\mp\sqrt{1-\gamma^4}\cos\left(\gamma\Omega_N\int_0^\tau f(\tau')d\tau'-\tau+\phi\right),$$

$$y_{\rm trap}=\pm\sqrt{1-\gamma^4}\sin\left(\gamma\Omega_N\int_0^\tau f(\tau')d\tau'-\tau+\phi\right),$$

$$z(0)=-\gamma^2, \tag{34}$$

$$e_{\rm trap}=\beta\cos\left(\gamma\Omega_N\int_0^\tau f(\tau')d\tau'-\tau+\phi\right),$$

$$p_{\rm trap}=\beta\sin\left(\gamma\Omega_N\int_0^\tau f(\tau')d\tau'-\tau+\phi\right),$$

where ϕ is an arbitrary phase and γ^2 is an arbitrary number from the interval $0 \leqslant \gamma^2 \leqslant 1$. The expression (34) obeys the original system (4) with $\omega = 1$ if the field amplitude β is connected with the initial atomic inversion $z(0)$ by the following condition:

$$\beta = \pm \sqrt{z(0) - [z(0)]^{-1}}. \tag{35}$$

These results may be resumed as follows. If atoms at the cavity entrance are prepared in the same superposition state with an arbitrary phase ϕ as follows:

$$x(0) = \mp \sqrt{1-\gamma^4}\cos\phi, \quad y(0) = \pm \sqrt{1-\gamma^4}\sin\phi,$$

$$z(0) = -\gamma^2, \tag{36}$$

and the cavity mode is prepared initially in a coherent state with the atomic phase ϕ and the amplitude β satisfying the condition (35) as follows:

$$e(0) = \beta\cos\phi, \quad p(0) = \beta\sin\phi, \tag{37}$$

then the density of the atomic inversion z of moving atoms and the density of the photon number $n = \beta^2/4$ do not evolve *regardless of the spatial structure* $\mathbf{f}(\mathbf{r})$ *of the cavity mode along the propagation axis*. The total energy and the interaction energy in these trapping states are given by

$$\mathcal{W}_{\rm trap} = \beta^2 - 2\gamma^2, \quad \mathcal{J}_{\rm trap} = -\beta\sqrt{1-\gamma^4}. \tag{38}$$

Thus, the receipt for locking the atomic inversion of moving two-level atoms and the average number of photons in a single-mode lossless cavity is the following. Prepare the field mode in a coherent state (37) with a phase ϕ and atoms at the cavity entrance in the same superposition state (36) with the same phase ϕ and the initial inversion $z(0)$ connected with the field amplitude by the condition (35). In spite of the oscillations of both the atomic (x,y) and the field (e,p) variables [see Eq. (34)], the atomic inversion and the average number of photons remain stationary and equal to their initial values. This is a result of the synchronized oscillations of the atomic and field subsystems.

The nonpositive initial inversion, $-1 \leqslant z(0) = -\gamma^2 \leqslant 0$, can be considered as the necessary condition for population locking. It follows from Eq. (35) that in order to lock the population and the radiation starting with zero inversion $z(0) = 0$, one needs an infinitely large number of initial photons in the cavity mode, $\beta \to \infty$. The respective field state may be treated as a phase state. To lock the population in the ground state, one needs to prepare the field in the vacuum state, $\beta = 0$. It is a trivial consequence of the fact that the initial state $|x(0) = y(0) = e(0) = p(0) = 0,\ z(0) = -1\rangle$ is a simple equilibrium point of the dynamical system (4). One can lock the population in any superposition state with the initial Bloch vector to be chosen from the lower Bloch semisphere $-1 \leqslant z(0) \leqslant 0$ by preparing the cavity mode in the respective coherent state. To lock the population of N atoms with, say, the density of inversion $z(0) = -1/2$, one needs $3N/8$ initial photons.

The effect of coherent trapping with two-level atoms has been found [23] within the Jaynes-Cummings model treating a single two-level atom that interacts with a single mode of *the quantized radiation field* in a lossless cavity. In contrary to the multilevel case, where there are additional transition channels between which a coherent interference can occur, in the two-level case, the atomic dipole interferes destructively with a coherent cavity eigenmode inhibiting the transitions between the two levels. In the framework of the fully quantized Jaynes-Cummings model, the population inversion can be expressed in the form of a series in the photon population numbers that contains an interference term depending on a relative phase between the dipole and the coherent state of the eigenmode. If this phase is zero, the amplitude of the oscillations of the population amplitude has been shown numerically to be extremely small [23].

We demonstrate here that the effect of population and radiation trapping can occur with moving two-level atoms which are at exact resonance with a spatially inhomogeneous cavity eigenmode. It means that, under appropriate conditions, the effect of trapping can persist in maser-type experiments with atomic beams. In the semiclassical limit, it is even possible to find in the explicit form [see Eqs. (35)–(37)] a class of the initial conditions for the atoms and the field under which the atomic population inversion and the average number of cavity photons can be locked.

B. Autonomous nonresonant interaction

In this section we describe briefly the effect of trapping with atoms moving through a spatially homogeneous single-mode field whose frequency is not at exact resonance with the atomic frequency. It is easy to check that the special solutions

$$x_{\rm trap} = \pm \sqrt{1-[z(0)]^2}\cos\left(\frac{\omega+1}{2}\tau + \varphi\right),$$

$$y_{\rm trap} = \pm \sqrt{1-[z(0)]^2}\sin\left(\frac{\omega+1}{2}\tau + \varphi\right),$$

$$z(0) = \left(\frac{\omega-1}{2\Omega_N}\right)^2, \tag{39}$$

$$e_{\rm trap} = \mp \frac{2\Omega_N}{\omega-1}\sqrt{1-[z(0)]^2}\cos\left(\frac{\omega+1}{2}\tau + \varphi\right),$$

$$p_{\rm trap} = \pm \frac{2\Omega_N}{\omega-1}\sqrt{1-[z(0)]^2}\sin\left(\frac{\omega+1}{2}\tau + \varphi\right)$$

of the general solutions (28) and (29) leave the density of the atomic inversion z and the density of the average number of photons n unaffected. Assuming $\tau = 0$ in Eq. (39) one can find the corresponding initial states of the atoms and the cavity field that provide trapping. As in the case of the nonautonomous resonant system, trapping will occur if the atoms are prepared in a superposition state and the field in a coherent state with the same phase. Contrary to the resonant case, the respective values of the density of the initial atomic inversion should be chosen from the upper Bloch semisphere, $0 \leqslant z(0) \leqslant 1$. One limit case, $z(0) = 0$, is realized at exact resonance and considered in the preceding section. The

other limit case, $z(0)=1$, corresponds to the fixed point S_+ that is an equilibrium state in the semiclassical approximation.

V. HAMILTONIAN CHAOS

A. Melnikov analysis

Now we return to the original nonintegrable problem, nonautonomous nonresonant atom-field system (4) that takes into account both a spatial inhomogeneity of the cavity mode and a detuning between atoms and the field.

Homoclinic motion of a Hamiltonian system is a motion that is asymptotic to a periodic motion at $t\rightarrow\pm\infty$. As was shown by Poincaré [24], the existence of isolated homoclinic orbits results in a complicated behavior of trajectories in Hamiltonian systems. There exists a general method that can prove the presence of chaotic motion by detecting transverse intersections between perturbed stable and unstable homoclinic manifolds (a homoclinic structure) and by calculating the width of the respective stochastic layer. This method was developed by Melnikov [25] and generalized in [26] (for a review see, e.g., [27]).

In the absence of spatial modulation of the vacuum Rabi frequency ($f=$const) the evolution of the atom-field system is periodic and is governed by the exact solutions to be obtained in Sec. III B. The main effect of the modulation, $\Omega_0(t)=\Omega_0 f(t)$, is to produce, out of resonance ($\omega\neq 1$), a homoclinic structure in the vicinity of the separatrix of the unperturbed autonomous system.

In this section we use the Melnikov method [25] to prove the existence of chaotic trajectories by detecting transverse intersections between perturbed stable and unstable manifolds. This is a standard method suited for the case of small periodic perturbations of integrable systems. In order to apply the method, we introduce vector notations and rewrite the original atom-field dynamical system (4) in the form

$$\dot{\mathbf{s}}=\mathbf{F}(\mathbf{s})+\epsilon\mathbf{G}(\mathbf{s},\tau), \tag{40}$$

where each vector has five components, namely

$$\mathbf{s}=(x,y,z,e,p)^T,$$

$$\mathbf{F}(\mathbf{s})=(-y-\Omega_N zp, x-\Omega_N ze, \Omega_N(xp+ye), \omega p-\Omega_N y, -\omega e-\Omega_N x)^T, \tag{41}$$

$$\mathbf{G}(\mathbf{s},\tau)=(-g(\tau)zp, -g(\tau)ze, g(\tau)(xp+ye), -g(\tau)y, -g(\tau)x)^T.$$

When writing down the original system (4) in the vector form (40), we have represented the modulation $f(\tau)$ in Eq. (4) in the following form:

$$f(\tau)=1+\frac{\epsilon}{\Omega_N}g(\tau). \tag{42}$$

The basic idea of the Melnikov analysis is to make use of exact solutions of the unperturbed integrable system ($\epsilon=0$) in the computation of a perturbed system of the form (40). We referred to the autonomous nonresonant system treated in Sec. III B as the unperturbed system. The unperturbed system possesses a two-sheeted homoclinic manifold consisting of a collection of all the pairs of separatrices homoclinic to a fixed point S_+ and connecting this hyperbolic point to itself. The manifold is parametrized by τ and is given explicitly by the solutions (33) with $\theta_0=0$ and $\theta_0=\pi$. With the help of the integrals of motion one can also represent it implicitly by the equations

$$\mathcal{R}=1, \quad \mathcal{W}=2, \quad \mathcal{C}=1-\omega. \tag{43}$$

The homoclinic manifold possesses three normal vectors

$$\mathbf{n}_1=\boldsymbol{\nabla}\mathcal{R}, \quad \mathbf{n}_2=\boldsymbol{\nabla}\mathcal{W}, \quad \mathbf{n}_3=\boldsymbol{\nabla}\mathcal{C}, \tag{44}$$

where

$$\boldsymbol{\nabla}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},\frac{\partial}{\partial e},\frac{\partial}{\partial p}\right). \tag{45}$$

The Melnikov function $M(\tau_0)$ measuring the signed distance between the stable and unstable manifolds of the equilibrium point S_+ at τ_0 along the normal $\mathbf{n}$ to the unperturbed homoclinic manifold is proportional to $\epsilon M(\tau_0)+\mathcal{O}(\epsilon^2)$. It is given by [25]

$$M(\tau_0)=\int_{-\infty}^{\infty}\mathbf{n}\cdot\mathbf{G}\,d\tau, \tag{46}$$

where $\mathbf{G}$ is the perturbation part of the vector field which is given in our case by the last equation in Eqs. (41). This integral is evaluated along the separatrix (33). Since $\mathbf{n}_1(\mathbf{s}_0)\cdot\mathbf{G}(\mathbf{s}_0)=\mathbf{n}_2(\mathbf{s}_0)\cdot\mathbf{G}(\mathbf{s}_0)=\mathbf{0}$ we need to measure the Melnikov distance in only one direction $\mathbf{n}_3(\mathbf{s}_0)=(\Omega_N e_0, -\Omega_N p_0, 1-\omega, \Omega_N x_0, -\Omega_N y_0)$.

It should be noted that our analysis is applicable to the physical situation with atoms moving through a cavity in a direction along which the depth of modulation of their vacuum Rabi frequency may be considered to be small as compared with the amplitude value, i.e., $\epsilon\ll\Omega_N$ [see Eq. (42)]. We suppose the harmonic modulation $g(\tau)=\sin\omega_m\tau$, with ω_m being a dimensionless modulation frequency. After calculating the scalar product

$$\mathbf{n}_3(\mathbf{s}_0)\cdot G(\mathbf{s}_0)=\frac{1-\omega}{\Omega_N}\sin[\omega_m(\tau-\tau_0)]\frac{dz_0}{d\tau} \tag{47}$$

and substituting Eq. (47) into Eq. (46), we can carry out the integration by parts with the result

$$M(\tau_0)=\frac{2\pi(1-\omega)\omega_m^2}{\Omega_N^3 sh(\omega_m\pi/\alpha\Omega_N)}\cos(\omega_m\tau_0). \tag{48}$$

It is evident from Eq. (48) that out of resonance, $\omega\neq 1$, the Melnikov integral has simple zeros as a function of τ_0. If $M(\tau_0)$ has simple zeros, then the stable and unstable manifolds of the hyperbolic point intersect transversally, resulting in Smale horseshoe chaos [27].

B. Poincaré sections and Lyapunov exponents

We have performed some computer simulations on the nonautonomous nonresonant atom-field system (4) assuming the modulation of the vacuum Rabi frequency of moving

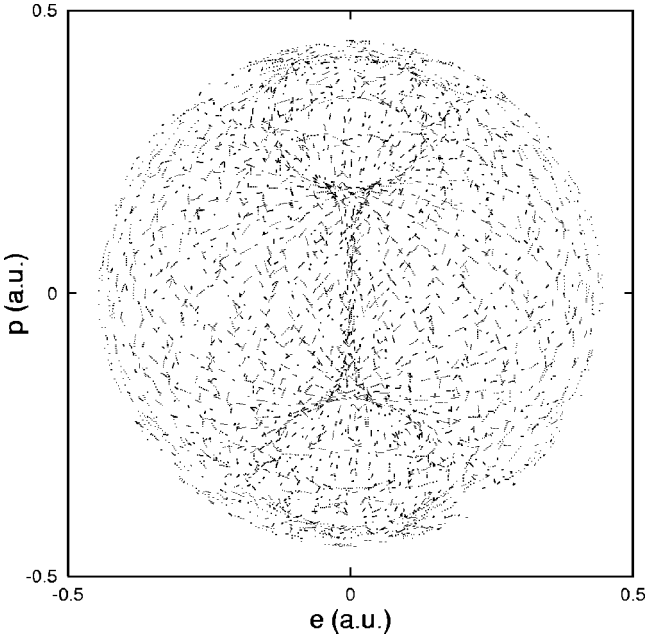

FIG. 1. Poincaré section on the field plane e-p with $\Omega_N=0.2$, $\omega=0.9$, $b=0.1$, and the energy $\mathcal{W}=-1.8$.

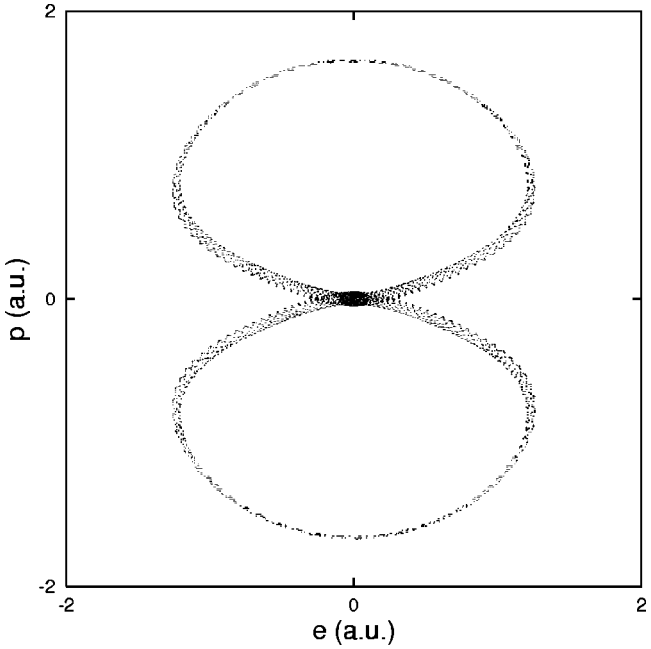

FIG. 2. Same as Fig. 1 with $\mathcal{W}=1$.

atoms to be large. Suppose the simplest spatial variation of the cavity mode $f(r)=\sin(k\pi r/L_c)$, which corresponds to a TE_k mode in a rectangular cavity with L_c being the cavity length and $k+1$ being the number of nodes in the cavity. For moving atoms, it becomes the time-periodic function that has the following form in the dimensionless time $\tau=\omega_a t$:

$$f(\tau)=\sin(b\omega\tau), \tag{49}$$

where $b\equiv v_a/c$ is the ratio of the velocity of atoms to the velocity of light.

To verify the breakup of the separatrix (33) when $\omega\neq 1$, we calculate Poincaré sections of the coupled atom-field flow generated on the direct product space of the surface of the atomic Bloch sphere and the field oscillator plane. The nonautonomous nonresonant atom-field system (4) has two integrals of motion (8). Therefore, a standard two-dimensional Poincaré surface of section is defined by fixing one variable out of the three independent ones. We define a Poincaré section by $x=0$ with $\dot{x}>0$, which is realized on the phase plane of the field variables e-p with the following fixed values of the control parameters: the collective vacuum Rabi frequency $\Omega_N=0.2$, the detuning $\omega=0.9$, and the velocity of atoms $b=0.1$. Figures 1–4 demonstrate these sections at four different values of the initial total energy $\mathcal{W}=e^2+p^2+2z$. In each figure we plot the successive sections of six trajectories started at the six different initial conditions $e_j(0)=-p_j(0)=0.1j$, $x(0)=0$, $z_j(0)=\frac{1}{2}\mathcal{W}+[e_j(0)]^2$, $[y_j(0)]^2=1-[z_j(0)]^2$, $j=0,2,4,6,8,10$.

A rather regular structure is visible in Fig. 1, which shows the Poincaré section for atoms entering a cavity, practically, in the ground state ($\mathcal{W}=-1.8$). A separatrix becomes visible on the plane e-p when the total energy reaches $\mathcal{W}=1$ (Fig. 2). With the total energy increased, a stochastic layer near the separatrix begins to show up (Fig. 3 with $\mathcal{W}=1.4$). The Poincaré section that is generic for developed chaos is shown in Fig. 4 with the energy $\mathcal{W}=2$ corresponding to the initial state with practically fully inverted atoms, $z_0\simeq 1$.

In closed dynamical system, chaos has its origin in extremal sensitivity to initial conditions, which is characterized by the Lyapunov exponents

$$\lambda_i=\lim_{\tau\to\infty}\lambda_i(\tau), \quad \lambda_i(\tau)=\lim_{\Delta_i(0)\to 0}\frac{1}{\tau}\ln\frac{\Delta_i(\tau)}{\Delta_i(0)}, \tag{50}$$

where $\Delta(\tau)$ is the distance between two initially adjacent trajectories at time τ, which may be specified as the Euclidean distance between two phase-space points. For an m-dimensional dynamical system there exist m Lyapunov

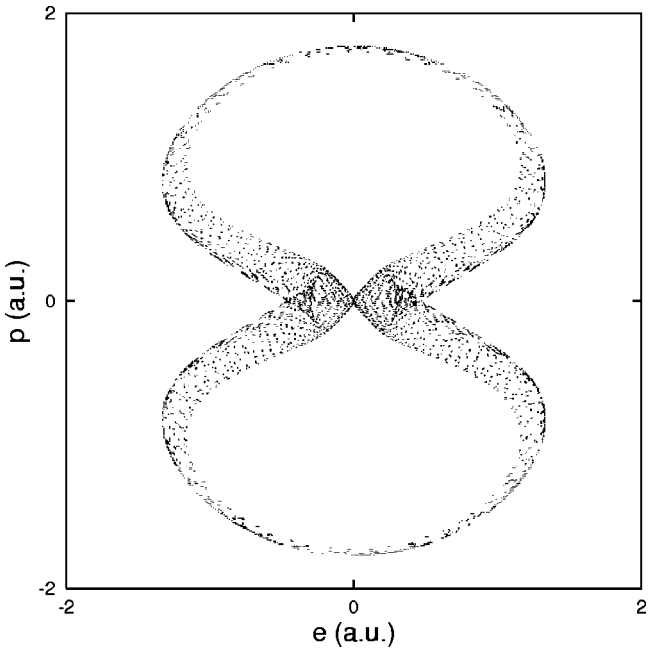

FIG. 3. Same as Fig. 1 with $\mathcal{W}=1.4$.

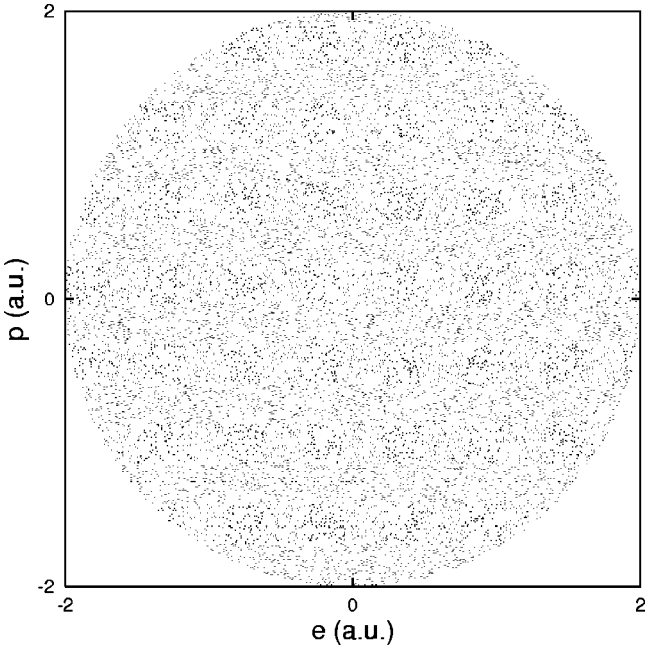

FIG. 4. Same as Fig. 1 with $\mathcal{W}=2$.

numbers (λ_i, $i=1,2,...,m$). Since the volume of a given element of phase space is always invariant for a conservative dynamical system, we have $\Sigma_{i=1}^{m}\lambda_i=0$. The values λ_i measure the rates of expansion of a volume element in the m principal directions. If $\lambda_i<0$, then the volume element shrinks in the corresponding direction. If $\lambda_i>0$, then the volume element expands exponentially in that direction, and if $\lambda_i=0$, then the growth is linear. The rates of growth of a volume element of phase space in various directions can be used to describe the trajectories that pass through such volume elements. A trajectory is chaotic if the maximal Lyapunov exponent λ is positive. Therefore, the Lyapunov exponents allow us to determine whether individual orbits are chaotic and to compare the strength of their randomness.

A dynamical system may be considered as a transformation of phase space. In other words, the volume of a given element of phase space representing different initial phase points is transformed into a deformed volume during the evolution. When this volume has smooth boundaries, then the respective flow is regular. A chaotic flow arises when the initial volume element stretches, shrinks, and folds. The local exponential divergence of trajectories produces a local stretching, but because of the global confinement in the phase space of our conservative Hamiltonian system with two degrees of freedom this stretching is accompanied by folding. Repeated stretching and folding produces very complicated motion that is known as chaotic.

When a dynamical system possesses more than a single control parameter, it is useful to compute topographical λ maps [14,16] that give a representative ''portrait'' of chaos. Such a map shows the values of the maximal Lyapunov exponent λ of the system with given initial conditions as a function of two of the control parameters at fixed values of the other parameters. We have computed the topographical λ maps with all three control parameters Ω_N, ω, and b varied for the atom-field system (4) with the following initial condition:

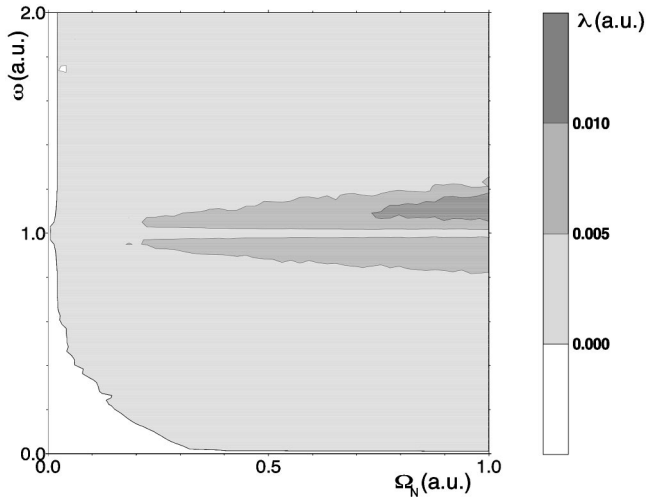

FIG. 5. Topographical λ map showing the regions of periodic and chaotic motion on the frequency plane ω-Ω_N ($b=0.01$).

$$x(0)=y(0)=0, \quad z(0)=1, \quad e(0)=p(0)=1, \quad (51)$$

which corresponds to initially fully inverted atoms entering a cavity with the initial density of photons to be equal to $n(0)=\frac{1}{4}[e(0)^2+p(0)^2]=\frac{1}{2}$. The values of λ are encoded by the color using a linear scale which is shown on the right sides of the maps. In calculating the maximal Lyapunov exponent, we employ $\lambda(\tau)$ according to Eq. (50). In taking the limit (50) we integrate the equations of motion up to $\tau=32\,000$ with a step $\delta\tau=1$ and check the results at each step with the help of the conservation laws (8).

Figures 5 and 6 show the ω-Ω_N maps calculated at the fixed values of the dimensionless velocity of atoms $b=0.01$ and 0.1, respectively. It is clear from the figures that chaos disappears when the atomic frequency is close to the field frequency. The $\log_{10} b$-Ω_N map of chaos at the fixed value of the detuning $\omega=0.9$ is given in Fig. 7. It is seen from the figure that comparatively strong chaos arises in a rather narrow range of the values of the atomic velocities from $b\simeq 0.01$ to 0.3. The $\log_{10} b$-ω map shown in Fig. 8 has a rather symmetric structure with respect to the line of exact resonance, $\omega=1$. It is calculated at the fixed value of the

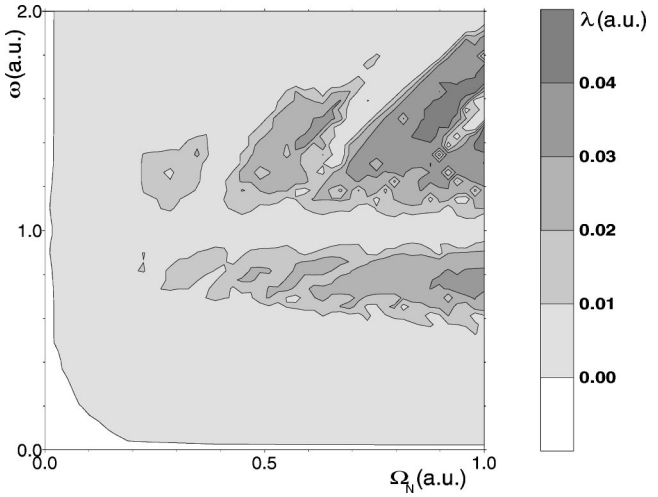

FIG. 6. Same as Fig. 5 with $b=0.1$.

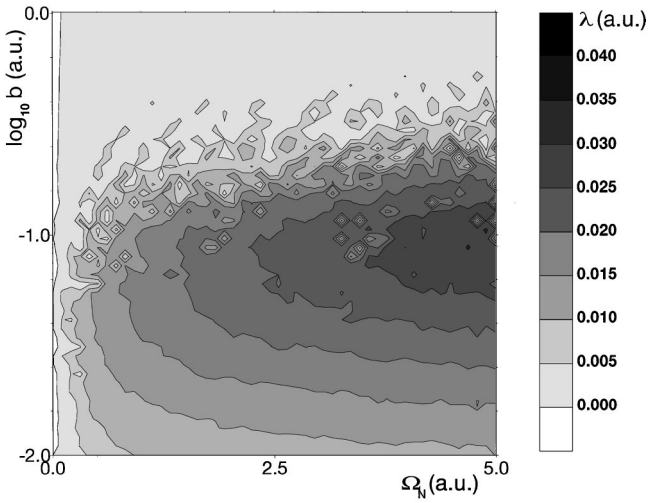

FIG. 7. Topographical λ map on the $\log_{10} b$-Ω_N plane ($\omega = 0.9$).

collective vacuum Rabi frequency, $\Omega_N = 2$. The maps b-Ω_N and ω-Ω_N give us, in fact, the values of λ as a function of the number of atoms N [see the expression (6)] which may be considered as an adjustable control parameter of the atom-field system along with b and ω.

The Hamiltonian approach we have adopted throughout the paper is valid over time intervals shorter than all the relaxation times. This approach is based on the strong-coupling limit

$$T_R \equiv \frac{2\pi}{\Omega_0 \sqrt{N}} \ll T_a, T_f, \tag{52}$$

where T_R is the period of the collective vacuum Rabi oscillations, and $T_{a,f}$ are the lifetimes of the atomic states and of the photons in a cavity, respectively. In numerical experiments that compute maximal Lyapunov exponents, Hamiltonian chaos can be diagnosed over a time interval of the order of the correlation decoupling time [28] $T_{\mathrm{cor}} = 2\pi/\omega_a \lambda$. In terms of the collective vacuum Rabi frequency Ω_N and λ, the condition for numerically observing Hamiltonian chaos can be rewritten as

$$\Omega_N \equiv \frac{\Omega_0 \sqrt{N}}{\omega_a} \gg \lambda. \tag{53}$$

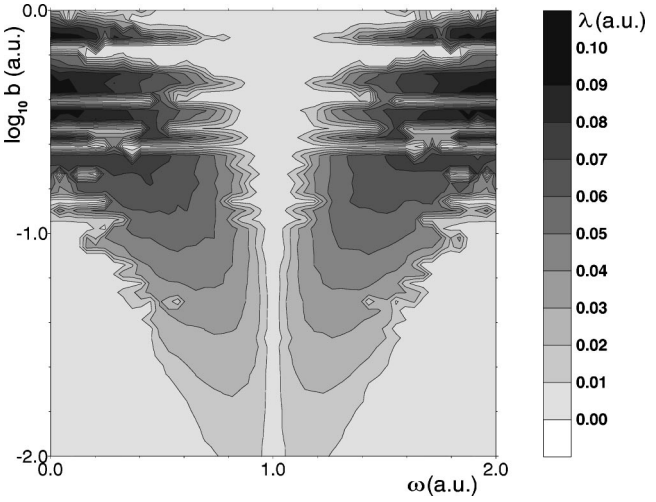

FIG. 8. Topographical λ map on the $\log_{10} b$-ω plane ($\Omega_N = 2$).

A Rydberg maser operating in the strong-coupling regime with a beam of two-level Rydberg atoms injected into a small high-Q cavity seems to be a promising device for observing semiclassical Hamiltonian chaos in real experiments. The Rydberg atoms have the transition frequency of the order of $\omega_a \simeq 10^{11} - 10^{12}$ rad/s, the electric dipole matrix element $d \simeq 10^3$ atomic units, the one-photon vacuum Rabi frequency $\Omega_0 \simeq 10^5 - 10^6$ rad/s, and the lifetime of the circular Rydberg states, $T_a \simeq 10^{-2}$ s [29]. Parameters of a microwave maser cavity are the following: $Q \simeq 10^{10}$, $L_c \simeq 1$ cm, and $T_f \simeq 10^{-1} - 10^{-2}$ s [29]. As it follows from our numerical results (Figs. 5–8), one can reach the regime of the chaotic vacuum Rabi oscillations with the strength of chaos of the order of $\lambda \simeq 0.01$ operating with a droplet consisting of $10^6 - 10^7$ atoms and flying with the velocity more than $v_a \simeq 10^8$ cm/s. It should be noted that to our knowledge micromaser experiments are performed with much slower atoms, $v_a < 10^5$ cm/s [29].

VI. CONCLUSION

We analyze the RWA nonlinear dynamics in one of the simplest models of laser and atomic physics that comprises two-level atoms moving through a high-Q cavity and interacting with a single eigenmode of the cavity. The main problem in which we are interested in this paper is a transition to Hamiltonian chaos in the case when the field cannot be assumed as homogeneous on the cavity size. The Heisenberg equations for the expectation values of a complete set of the atomic and field observables are shown to be integrable in the two limit cases: the nonautonomous resonant interaction and autonomous nonresonant interaction. The respective general exact solutions are given for both the models. We use them to reveal and describe the coherent effect of locking of the oscillations of the atomic inversion and of the radiation field that may occur under appropriate conditions. The integrable equations are shown to possess special orbits that are homoclinic to the state with fully inverted atoms and a vacuum cavity field which is an equilibrium one in the semiclassical approximation. With the help of the Melnikov method we prove analytically transverse intersections of stable and unstable manifolds of this equilibrium point under small modulation of the vacuum Rabi frequency caused by a slightly inhomogeneous field. These transverse intersections are believed to provide the Smale horseshoe mechanism of chaos. To confirm numerically the chaotic dynamics in the semiclassical RWA model, we compute Poincaré sections and maximal Lyapunov exponents under strong modulation of the vacuum Rabi frequency. The Lyapunov topographical maps showing the regions of regular and chaotic motion provide representative numerical ''portraits'' of the system's dynamics in different ranges of its control parameters.

ACKNOWLEDGMENTS

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APPENDIX: ACCURACY OF THE SEMICLASSICAL APPROXIMATION

In this appendix we introduce new atomic and field operators with commutators vanishing at the limit of a large number of atoms, $N\to\infty$, and show in an explicit form that the semiclassical approximation is valid in this limit.

Let us introduce new operators normalized to the number of atoms

$$A=\frac{a}{\sqrt{N}},\quad A^{\dagger}=\frac{a^{\dagger}}{\sqrt{N}},$$
$$S_x=\frac{1}{N}\sum_{j=1}^{N}\sigma_x^j,\quad S_y=\frac{1}{N}\sum_{j=1}^{N}\sigma_y^j,\quad S_z=\frac{1}{N}\sum_{j=1}^{N}\sigma_z^j, \tag{A1}$$

which satisfy the following commutation relations:

$$[A^{\dagger},A]=\frac{1}{N},\quad [S_i,S_j]=\frac{2i}{N}e_{ijk}S_k,\quad i,j,k=x,y,z. \tag{A2}$$

Neglecting all the quantum correlations, one can obtain from the Heisenberg equations of motion for the operators (A1) a closed set of *c*-number equations for the expectation values of the respective operators

$$\tilde{x}=\langle S_x\rangle,\quad \tilde{y}=\langle S_y\rangle,\quad \tilde{z}=\langle S_z\rangle,$$
$$\tilde{e}=\langle A^{\dagger}+A\rangle,\quad \tilde{p}=i\langle A^{\dagger}-A\rangle. \tag{A3}$$

Differentiating with respect to $\tau=\omega_0 t$, this set can be easily shown to have the same form as Eqs. (4),

$$\dot{\tilde{x}}=-\tilde{y}-\Omega_N f(\tau)\tilde{z}\tilde{p},$$
$$\dot{\tilde{y}}=\tilde{x}-\Omega_N f(\tau)\tilde{z}\tilde{e},$$
$$\dot{\tilde{z}}=\Omega_N f(\tau)(\tilde{x}\tilde{p}+\tilde{y}\tilde{e}), \tag{A4}$$
$$\dot{\tilde{e}}=\omega\tilde{p}-\Omega_N f(\tau)\tilde{y},$$
$$\dot{\tilde{p}}=-\omega\tilde{e}-\Omega_N f(\tau)\tilde{x}.$$

The commutators (A2) of the normalized operators vanish at the N infinite limit. It is obvious that the semiclassical approximation is valid at this limit with the relative error $\mathcal{O}(1/N)$. The equations of motion for these operators and their expectation values (A4) do not depend on the number of atoms N. We want to emphasize that with the help of the normalization (A1) one can incorporate in a natural way into respective equations of motion some quantum correlation terms since the dependence on the small parameter $1/N$ appears only in the quantum correlators when the commutators (A2) do not vanish identically.

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