Kinematic dynamos surrounded by a stationary conductor

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We investigate kinematic dynamos in cylinders and spheres surrounded by an insulator. The flow volume is divided into an inner region, in which the conducting fluid is in motion, and an outer region enclosing the inner one, in which the conductor is at rest. The dependence of the critical magnetic Reynolds number on the thickness of the outer conducting shell is discussed as well as implications for the design of experimental dynamos. [S1063-651X(99)04309-3]

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Most numerical simulations of planetary dynamos assume that a liquid conductor is moving inside a spherical volume surrounded by vacuum. However, in all real circumstances, the domain filled with moving fluid borders a region of finite conductivity, which itself is embedded in insulating space. This paper investigates kinematic dynamos in volumes surrounded by a layer of stationary conductor, which has the same conductivity as the fluid responsible for dynamo action. There are two complementary problems. First, one may consider the volume in which fluid is moving as fixed and one imagines conductor being added to a surrounding shell. Some studies of this problem have appeared before [1–3] providing examples of flows whose critical magnetic Reynolds number (defined with the characteristic dimension of the volume of fluid in motion) decreases if such a blanket is added, but none where it increases. According to Refs. [3, 4], a current sheet forms at the conductor vacuum interface. If the conducting volume is increased, the current (or alternatively the toroidal component of the magnetic field) escapes the domain to which it was initially confined and varies on a larger length scale than before, hence causing less ohmic dissipation and yielding a lower critical magnetic Reynolds number. It is plausible that a larger volume of conductor allows the magnetic field to better adjust to the velocity field and thus leads to a more efficient dynamo. Here, we present the first examples in which the addition of stationary conductor is on the contrary detrimental to dynamo action. The effect is small but important because it dispels the notion that more freedom for the magnetic field necessarily implies a lower critical magnetic Reynolds number.

The second problem leads to more sizeable effects and is relevant for the design of experimental dynamos. Several groups presently plan to build laboratory dynamos with liquid sodium (for a review, see Ref. [5]). The experiments are constrained by the maximum volume of sodium they can handle or which fits into the dynamo cell. There is thus a need to find flows with the lowest possible critical magnetic Reynolds number, where this number is now defined with the characteristic length scale of the entire conducting volume. Considering the examples cited above, it may turn out to be advantageous to leave some of the available sodium volume at rest. It will be shown below that for flows under consideration for experimental realization, an optimum is indeed achieved when only part of the sodium is set into motion.

The kinematic dynamo problem is formally posed by asking whether the nondimensional induction equation

$$\frac{\partial}{\partial t}\mathbf{B}+R_m\boldsymbol{\nabla}\times(\mathbf{B}\times\mathbf{v})=\nabla^2\mathbf{B},\quad \boldsymbol{\nabla}\cdot\mathbf{B}=0, \tag{1}$$

admits solutions for the magnetic field $\mathbf{B}$, which grow in time for a given velocity field $\mathbf{v}$ at a magnetic Reynolds number R_m. The numerical effort implied by spherical geometry prohibits a systematic study and we therefore start with cylindrical dynamos for which Eq. (1) can be reduced to a small set of algebraic equations. Consider a helical velocity field given in cylindrical polar coordinates (s,φ,z) by $\mathbf{v}=s\hat{\boldsymbol{\varphi}}-\hat{\mathbf{z}}$ for $0<s\leqslant 1$. A stagnant conductor is contained in the layer $1<s\leqslant s_1$. The region $s>s_1$ is assumed insulating so that $\boldsymbol{\nabla}\times\mathbf{B}=0$ in this region. Ponomarenko's dynamo model [6] is recovered in the limit $s_1\to\infty$. The induction equation is solved by the ansatz $\mathbf{B}(r,t)=\mathbf{b}(s)e^{i(kz+m\varphi)}e^{\sigma t}$. With $b_\pm=b_s\pm ib_\varphi$, the solution may be represented by

$$b_z=C_{z,1}I_m(\kappa_1 s)\quad b_\pm=C_{\pm,1}I_{m\pm1}(\kappa_1 s)\quad \text{for } 0<s\leqslant 1,$$

$$b_z=C_{z,2}I_m(\kappa_2 s)+D_{z,2}K_m(\kappa_2 s),$$

$$b_\pm=C_{\pm,2}I_{m\pm1}(\kappa_2 s)+D_{\pm,2}K_{m\pm1}(\kappa_2 s)\quad \text{for } 1<s\leqslant s_1,$$

$$b_z=D_{z,3}K_m(ks)\quad b_\pm=iD_{z,3}K_{m\pm1}(ks)\quad \text{for } s>s_1. \tag{2}$$

These expressions guarantee that b is regular at $s=0$ and decays to zero for $s\to\infty$ if $k>0$. κ_1 and κ_2 are given by the square root with positive real part of $\kappa_1^2=k^2+\sigma+\mathrm{Im}\,R_m$ and $\kappa_2^2=k^2+\sigma$. The ten coefficients in Eq. (2) must be chosen such that the following matching conditions are satisfied: $\mathbf{B}$, $\partial B_s/\partial s$, and $\hat{\mathbf{s}}\times(\boldsymbol{\nabla}\times\mathbf{B}-R_m\mathbf{v}\times\mathbf{B})$ are continuous at $s=1$ and $\mathbf{B}$ and $\partial B_s/\partial s$ are continuous at $s=s_1$. The resulting ten equations specify an eigenvalue problem for σ, which is easily solved numerically. See Refs. [6, 7] for more detailed descriptions of related problems. Gailitis and Freiberg [8] treat a more general situation and allow for different conductivities in the inner and outer regions.

In the following, we will only be interested in the onset of dynamo action, i.e., in combinations of ''critical'' parameters such that $\mathrm{Re}\{\sigma\}=0$. Figures 1 and 2 show the critical k and $\mathrm{Im}\{\sigma\}$ as a function of s_1 for various R_m and $m=1$. One

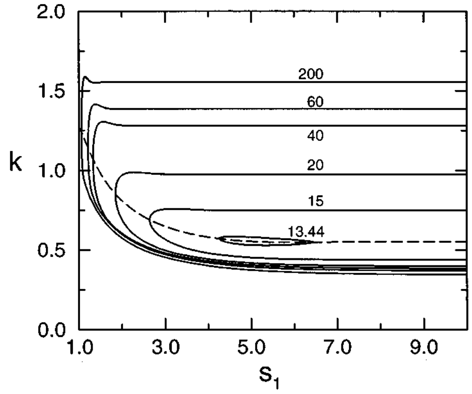

FIG. 1. The k for which $\mathrm{Re}\{\sigma\}=0$ is shown for $m=1$ as a function of s_1 for different choices of R_m. The labels next to every curve give the corresponding R_m. The values of k for which the smallest R_m is found at fixed s_1 are given by the dashed line.

can of course also interpret these figures as giving the critical R_m and $\mathrm{Im}\{\sigma\}$ for fixed m, s_1, and k. For every value of s_1, there is a mode for which the critical R_m is minimum. The corresponding k and $\mathrm{Im}\{\sigma\}$ are also indicated in Figs. 1 and 2. As R_m is increased from zero in a cylinder of infinite length, this particular mode will be the first to grow. The minimal value for R_m, which needs to be exceeded before growth occurs, is about 13.40 and is realized in a cylinder with $s_1=4.93$ for a wave number $k=0.556$. Increasing s_1 (i.e., adding stationary conductor) beyond $s_1=4.93$ deteriorates the dynamo in the sense that the critical R_m increases as can be deduced from the closed isoline in Fig. 1. The critical R_m for $s_1\to\infty$ is 13.457. Modes with $m=2$ are first excited at $R_m=59.44$, $s_1=2.13$, $k=1.78$, and $\mathrm{Im}\{\sigma\}=7.29$.

We verified the results of Gailitis and Freiberg [8] for the case of uniform conductivity, to the extent that this is possible from a comparison of graphs (assuming a misprint in [8], χ should be $\chi=v^I/\omega^I$). An optimal value of s_1 for which the critical magnetic Reynolds number is minimum also exists for the parameters chosen in [8] even though this is not mentioned by the authors.

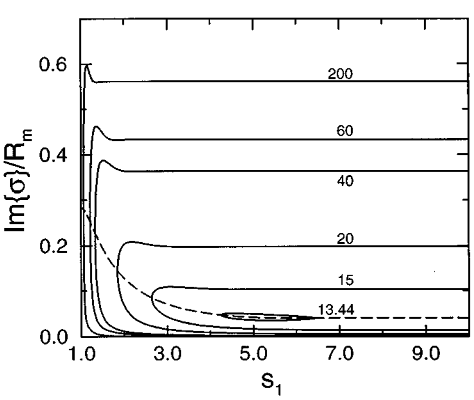

FIG. 2. The $\mathrm{Im}\{\sigma\}/R_m$ for which $\mathrm{Re}\{\sigma\}=0$ is shown for $m=1$ as a function of s_1 for different choices of R_m. The labels next to every curve give the corresponding R_m. The values of $\mathrm{Im}\{\sigma\}/R_m$ for which the smallest R_m is found at fixed s_1 are given by the dashed line. The solutions with small $\mathrm{Im}\{\sigma\}/R_m$ correspond to those with small k in Fig. 1.

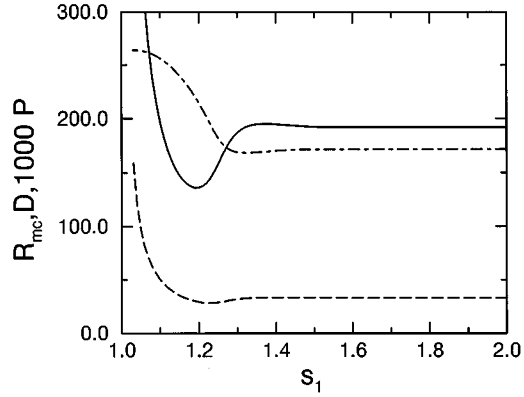

FIG. 3. R_{mc} (continuous), D (dashed), and P (dot dashed), as a function of s_1 for $k=1.55$ and $m=1$. P has been multiplied by 1000.

In a cylinder of finite length, the range of admissible k is bounded from below [9]. If k is forced to be larger than 0.556, the smallest critical R_m is reached at a finite value of s_1. This may also be deduced from Fig. 1 because for $k>0.556$, (i) smaller k yield smaller critical R_m at constant s_1 and (ii) a line $k=\mathrm{const}$ crosses isolines $R_m=\mathrm{const}$ twice due to the small excursion these lines make towards large k in the region $1<s_1<5$. Therefore, there is an optimal thickness of the outer layer of conductor in the case of a finite cylinder, too.

We are thus led to investigate why the picture developed in Ref. [3] does not apply to the present case. To this end, Fig. 3 shows for $k=1.55$ as a function of s_1 the critical R_m, R_{mc}, and two quantities, D and P, which measure dissipation and production of magnetic field, respectively, defined by

$$D=\int|\nabla\times\mathbf{B}|^2 dV\bigg/\int|\mathbf{B}|^2 dV, \tag{3}$$

$$P=\int\mathbf{B}(\mathbf{B}\cdot\nabla)\mathbf{v}dV\bigg/\int|\mathbf{B}|^2 dV, \tag{4}$$

where the integrals extend over the entire space. At onset, $R_m=R_{mc}=D/P$. The variation of all the quantities in Fig. 3 is less pronounced for smaller values of k. It is seen that D has a minimum near the value of s_1 for which R_{mc} is minimal. The minimum in D would directly account for the minimum in R_{mc} if P was independent of s_1, which it is not. But P decreases monotonically throughout the range in which the R_{mc} curve goes through a trough, so that the minimum in R_{mc} basically retraces the minimum in D.

The integral in Eq. (3) measures a characteristic length scale of $\mathbf{B}$. The minimum in D indicates that if the thickness of the outer conductor is increased beyond a certain value, electric currents do not extend further out radially but rather contract. This effect is shown directly in Fig. 4 where $|\nabla\times\mathbf{B}|^2 s/\int|\mathbf{B}|^2 dV$, which is proportional to the integrand of Eq. (3), is plotted against s for $s_1=1.2$ (i.e., near the minimum of D), and $s_1=2$. The current densities at $s<1$ are virtually identical, but the penetration depth for the current into the outer region is smaller for the thicker outer layer.

It will now be argued that the behavior of D may be attributed to eddy currents generated in the outer conductor

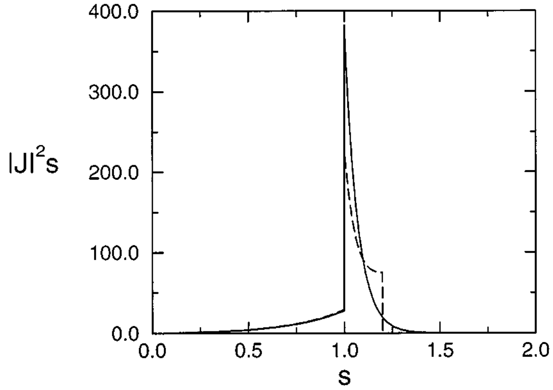

FIG. 4. The integrand of the dissipation integral (3), rewritten as $D=\int |J|^2(s) s ds$, is shown as a function of s for $s_1=2$ (continuous curve) and $s_1=1.2$ (dashed curve).

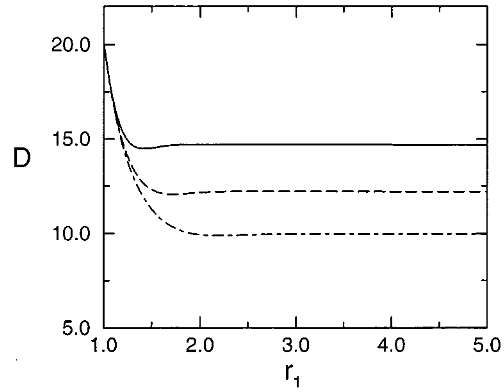

FIG. 5. D as a function of r_1 computed for the solution of problem (5) for $l=1$ and $\omega=20$ (continuous trace), $\omega=10$ (dashed), and $\omega=5$ (dot dashed).

and concomitant flux expulsion from that region. The minimum in D, and hence in R_{mc}, as a function of s_1 for fixed k and $m=1$ only occurs for $k>0.556$. For smaller k, R_{mc} decreases monotonically with increasing s_1. But the modes with a small k also have a small $\mathrm{Im}\{\sigma\}$ (see Fig. 2), so that eddy currents must necessarily be small in this parameter range. The profile of the current density in Fig. 4 is reminiscent of a skin effect, where the expulsion occurs in the outer stationary, rather than the interior mobile conductor. Flux expulsion from a cylinder is demonstrated in Ref. [10], §3.8, for a uniform stationary field applied to a cylindrical rotor in an infinite conductor. In the present dynamo, however, the magnetic field is time-dependent when observed from the stationary conductor and is, therefore, expelled from the outer region.

In order to corroborate this interpretation, we investigated a toy model in which a magnetic field is generated by electromotive forces that are prescribed in an interior region and which rotate relative to a surrounding shell. In a frame of reference in which the magnetic field is stationary, the outer conductor is rotating and the magnetic Reynolds number based on that velocity and the thickness of the outer layer determines whether diffusion dominates, and magnetic field escapes from the interior region, or whether induction dominates and the field is expelled from the exterior. For a formal discussion, consider a spherical conducting domain $r\leqslant r_1$ in which currents are driven by a prescribed emf, which is different from zero only in the region $r\leqslant 1$. The space $r>r_1$ is again insulating. This problem differs from the kinematic dynamo in that the electromotive forces are not determined by the magnetic field. But the electrical currents in the outer conductor are again governed by the competition between the same two effects as in the dynamo: If r_1 is increased, the current is less confined by a vacuum boundary, which tends to decrease dissipation, but additional eddy currents may be induced if the emf is time dependent, which leads to a higher D. The example that follows shows that the second effect can dominate the first. The spherical (as opposed to cylindrical) geometry is not expected to be essential for the phenomenon under investigation but is chosen for convenience and to make the transition to the spherical dynamos presented at the end of the paper.

We wish to solve $(\partial/\partial t)\mathbf{B}-\nabla^2\mathbf{B}=\mathbf{f}$. Assume the forcing $\mathbf{f}$ is purely toroidal. $\mathbf{B}$ is then also purely toroidal, $\mathbf{B}=\nabla\times(\Sigma_{l,m} t_l^m(r) P_l^m(\cos\theta) e^{im\varphi} e^{-i\omega t}\hat{\mathbf{r}})$ in spherical polar coordinates (r,θ,φ). P_l^m denotes associated Legendre functions. The toroidal scalar is governed by

$$\left(i\omega+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right)t_l^m=-\frac{r^2}{l(l+1)}\{\hat{\mathbf{r}}\cdot\nabla\times\mathbf{f}\}_l^m e^{i\omega t}, \quad (5)$$

where $\{\}_l^m$ stands for the l,m component of a decomposition in spherical harmonics of the quantity inside the curly brackets. We treat in detail an example for which $\mathbf{f}$ is constant in time except for a rotation about the z axis, e.g., $\hat{\mathbf{r}}\cdot\nabla\times\mathbf{f}=-2/r^2 A_0 \alpha r j_1(\alpha r) P_1^1(\cos\theta) e^{i\varphi} e^{-i\omega t}$ for $r\leqslant 1$ with $\alpha=4.49...$ being the first root of the spherical Bessel function j_1 of order 1. t_1^1 may be written as

$$t_1^1(r)=A_1\alpha r j_1(\alpha r)+A_2 x j_1(x) \quad \text{for } \mathbf{r}\leqslant 1,$$
$$t_1^1(r)=A_3 x j_1(x)+A_4 x y_1(x) \quad \text{for } 1<r\leqslant r_1, \quad (6)$$

with $x=r\sqrt{i\omega}$. y_1 is the spherical Bessel function of the second kind of order 1. $A_1=A_0/(i\omega-\alpha^2)$ stems from a particular solution of the inhomogeneous equation and the remaining coefficients are determined by the conditions that t_1^1 and dt_1^1/dr are continuous at $r=1$ and that $t_1^1=0$ at $r=r_1$. From the numerically obtained solution, the dissipation D is computed according to Eq. (3) and plotted against r_1 in Fig. 5 for values of ω, which are in the range of values of $\mathrm{Im}\{\sigma\}$ obtained for the cylindrical dynamo above and the spherical dynamos below. One observes in Fig. 5 qualitatively the same behavior for D as for kinematic dynamos (e.g., Fig. 3) and in particular a similar value for $D_{\min}/D(r_1\to\infty)$, where $D_{\min}$ is the minimum of D. When r_1 is increased from 1, D first decreases because the current diffuses into the outer conducting shell, but D increases again for large r_1. The dissipation for $r_1\to\infty$ is increasing with ω because a faster rotation of the magnetic field induces more eddy currents. Accordingly, the minimum in D becomes shallower with decreasing ω and is certainly absent for $\omega=0$.

Finally, we investigate kinematic dynamos in a sphere with velocity fields similar to those proposed for experiments. These fields are axisymmetric and the poloidal and

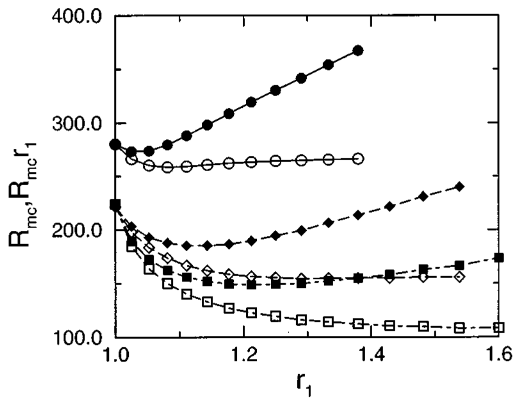

FIG. 6. R_{mc} and $R_{mc}r_1$ as a function of r_1 for the S_1T_1 flow with $\epsilon=0.25$ (circles, continuous line), $\epsilon=0.3$ (diamonds, dashed line), and $\epsilon=0.35$ (squares, dot dashed line). The full and open symbols represent $R_{mc}r_1$ and R_{mc}, respectively.

toroidal scalars S and T consist each of a single spherical harmonic of degree l_S and l_T. Following Dudley and James [11] we label the corresponding flow by $S_{l_S}T_{l_T}$. A radial function needs to be chosen, which we take as $g(r)=r(1-r)\sin(\pi r)$ for $r\leqslant 1$ and $g(r)=0$ for $r>1$. The velocity field $S_{l_S}T_{l_T}$ is given in spherical polar coordinates (r,θ,φ) by

$$\mathbf{v}=\nabla\times g(r)P_{l_T}(\cos\theta)\hat{\mathbf{r}}+\epsilon\nabla\times\nabla\times rg(r)P_{l_S}(\cos\theta)\hat{\mathbf{r}}, \tag{7}$$

where P_l denote Legendre polynomials and ϵ is another free parameter. $\mathbf{v}=0$ for $r>1$, vacuum is assumed for $r>r_1$, and Eq. (1) is solved with the same method as used in Ref. [12]. Figure 6 shows the critical R_m for the S_1T_1 flow for different ϵ. In all cases, the $m=1$ mode is preferred over higher m. The S_1T_1 flow may be regarded as the same velocity field, which was investigated above in cylindrical geometry but with an added return flow. For $\epsilon=0.25$ we observe again a minimum in R_{mc}, i.e., there is an optimal thickness for the shell of a stationary conductor. The existence of a minimum is thus not peculiar to cylindrical dynamos. The minimum in R_{mc} is found for small ϵ at which the magnetic field rotates faster (owing to a larger contribution from the toroidal component to the velocity field) than for larger ϵ, which is in line with the interpretation that the minimum occurs because of the time dependence of $\mathbf{B}$ viewed from the outer conductor.

In the experimental context, the more important issue is to minimize the magnetic Reynolds number at a constant total volume of the conductor, or equivalently, to minimize the magnetic Reynolds number on the length scale r_1, i.e., the product $R_{mc}r_1$. As is also shown in Fig. 6, this product always first decreases when r_1 is increased from 1. With the velocity profile $g(r)$ it is, therefore, better to leave part of the available conductor at rest than to force the entire fluid volume to flow according to Eq. (7). We made the same observation for other radial functions and for velocity fields containing spherical harmonics of degree $l=2$. For large enough r_1, $R_{mc}r_1$ eventually increases again because the productive region in which mechanical work is actually done on the magnetic field becomes small in comparison with the total conducting volume.

In summary, we have shown that a mechanism exists for time-dependent magnetic dynamo fields through which added stagnant conductor increases the critical magnetic Reynolds number. As the thickness of the layer of the conductor at rest is increased, the electric current may occupy a larger volume and dissipate less because it now varies on a larger length scale. But it may also occur that the current is expelled from the outer layer as in a skin effect and dissipation actually increases with increasing layer thickness. Both effects taken together lead to a finite thickness of the outer conductor for which the critical magnetic Reynolds number is minimal. Since the outer conductor may deteriorate a dynamo, one wonders whether flows exist that are unable to operate as dynamos in an infinite expanse of conductor but which lead to growing magnetic fields once the conductor is confined to a cylindrical or spherical volume. Such an example remains to be found.

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