Complexity of routes to chaos and global regularity of fractal dimensions in bimodal maps

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The dual-star composition rule of doubly superstable (DSS) sequences presents a complete renormalizable algebraic structure for studying Feigenbaum's metric universality and self-similar classification of DSS sequences in symbolic dynamics of bimodal maps of the interval. Here an important feature is that the complete combinations of up- and down-star products create all the generalized Feigenbaum's routes of transitions to chaos. These routes can be classified into two types: one consists of countably infinitely many regular routes which preserve Feigenbaum's metric universality; another consists of uncountably infinitely many universal nonscaling routes described by the irregularly mixed dual-star products, which break Feigenbaum's asymptotically convergent metric universality although they are structurally universal. The combinatorial complexity of dual-star products may increase the grammatical complexity of languages of symbolic dynamics. Moreover, it is found that there exists a global regularity between the fractal dimensions d and the scaling factors $\{\alpha_C, \alpha_D\}$ for Feigenbaum-type attractors: $d(Z)\log_{|Z|}|\alpha_C(Z)\alpha_D(Z)|=\beta^{(2)}$, where $\beta^{(2)}$ is independent of the concrete DSS sequences Z. [S1063-651X(99)07508-X]

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I. INTRODUCTION

The Derrida-Gervois-Pomeau (DGP) star composition [1] of symbolic sequences is a powerful and valuable tool for studying metric universalities [2–4] in symbolic dynamics of unimodal maps [5–10]. It presents a complete algebraic structure for renormalization. When maps change from unimodal to bimodal (e.g., two-parameter cubic families), physical systems become more complicated. Thus, one should establish an algebraic composition rule for bimodal maps. The physical motivation for a study of bimodal maps comes from the fact that it can help to understand the dynamics of trimodal or multimodal maps [11], degree-1 circle maps [12–14], and Lorenz maps [15–19]. After efforts of about 20 years, a rigorous generalization of the DGP star composition rule, namely, the normal *dual-star* composition rule for bimodal maps [20,21], has been found.

During the course of solving this problem, there have been many significant studies. MacKay and Tresser [13,14] presented a description of symbolic dynamics for the kneading plane and conducted a fundamental research on the period-doubling bifurcation (PDB) for bimodal maps. They gave the boundary of topological chaos and the complete set of monotone equivalent classes of bimodal maps for the sequences with periods 2^n. Mumbrú [22] and Llibre and Mumbrú [23] made an extension of the star product for bimodal maps which is restricted to four kinds of special sequences, namely, $C1$, $C2$, $C1C2$, and $C2C1$ sequences; they also presented the mother operator for bimodal maps [24], which is an effective tool for the study of renormalization [25,26] and periodic structure. Ringland and co-worker [27,28] generated a genealogy of finite kneading sequences by using the hierarchical transformations for the α seed, ψ seeds, and χ seed. They presented all monotone equivalence classes and an important zero entropy class. Brucks *et al.* [29] discussed a generalization of the star product transformation to multimodal maps by introducing linear graphs of permutations, which is based on an investigation of the factorization of permutations into products of permutations. In this generalization, however, the star product cannot be described explicitly in terms of the symbols of its factors. Recently, in the order topological space Σ_3 of three letters, we presented explicit algebraic composition rules of dual-star products which can form all the equal topological entropy classes [30–34] in which all the Feigenbaum's universalities [2–4] are contained. On the basis of dual-star products, we can deepen and promote the understanding of knowledge in the following aspects:

(i) *Generalization of Feigenbaum's metric universality*. In unimodal maps, an arbitrary period-p-tupling bifurcation (PpTB) can be described by the DGP star product $(WC)^{*n}$ of basic period p, with the metrically universal convergent rate $\delta(WC)$ and scaling factor $\alpha(WC)$, which returns to a PDB when WC is replaced by period-2 superstable sequence RC. Similarly in bimodal maps, for each doubly superstable (DSS) periodic sequence $XDYC$, the dual-star products provide us with two bifurcation modes, up-bifurcation and down-bifurcation, described by $(XDYC)^{*n}$, $*\in\{\overline{*},\underline{*}\}$; and universal constants δ and α of unimodal maps are generalized to a pair of universal convergent rates $\{\overline{\delta},\underline{\delta}\}$ and two pairs of universal scaling factors $\{\overline{\alpha}_C,\overline{\alpha}_D\}$ and $\{\underline{\alpha}_C,\underline{\alpha}_D\}$ (notations corresponding to $\overline{*},\underline{*}$) with dual symmetry, which are all contained in the equal topological entropy class of $XDYC$. Here the dual symmetry of metric universal constants may lead to the duality of renormalization group equations which may be a system of equations.

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(ii) *Complexity and diversity of the generalized Feigenbaum's routes of transitions to chaos*. The two bifurcation modes of dual-star products complicate the routes to chaos in bimodal maps much more than in unimodal maps. In unimodal maps, the routes to chaos are PDB's [including the main $(RC)^{*n}$, and the associated $AC*(RC)^{*n}$] and arbitrary PpTBs [$(WC)^{*n}$ and $AC*(WC)^{*n}$], their formal languages belong to the type-3 (finite n) and type-1 ($n\to\infty$) languages of the Chomsky hierarchy [35]; however, the type-2 languages have not yet been found [36]. In bimodal maps, the two bifurcation modes result in an infinite number of combinatorial patterns of dual-star products. The regularly mixed dual-star products create a countably infinite number of regular routes to chaos which preserve both the structural and the metric universalities. While the irregularly mixed dual-star products form an uncountably infinite number of irregular routes to chaos, which break Feigenbaum's (asymptotically convergent) metric universality although they are structurally universal. So the complexity of routes to chaos comes from the combinatorial complexity of patterns of dual-star products $\{\overline{*},\underline{*}\}$. Such complexity does not exist in the order topological space Σ_2 of two letters of unimodal maps. On the one hand, all the patterns of $\{\overline{*},\underline{*}\}$ correspond to practical bifurcations and form a complete combinatorial set, which correspond to the admissible real maps. On the other hand, the binary expressions of all the patterns cover all the real numbers on the interval [0,1]. These may enrich the languages of symbolic dynamics. The grammatical complexity of the languages of such patterns may be beyond that of unimodal maps. It provides a new direction for the study of complexity of dynamics. This would be rather interesting.

(iii) *Global regularity in the period-doubling and p-tupling bifurcations.* It is well-known that all the quantitative universalities, such as Feigenbaum's metric universal constants (convergent rates and scaling factors), fractal dimensions or singularity spectra, depend rigorously on the sequences of symbolic dynamics in the topological space Σ_2 or Σ_3, equivalently on the parameter values of systems. So the exploration of global regularities independent of sequences or parameters is very important for a thermodynamic formalism of chaotic dynamics in the whole topological space. Such global regularities in unimodal maps were found as the global relationship of fractal dimensions for Feigenbaum-type attractors [37] and the devil's staircase of topological entropy [30]. Since the algebraic composition rule of dual-star products successfully solves the geometric construction of Feigenbaum-type attractors and the structure of equal topological entropy classes in bimodal maps, we can further discuss such global regularities for bimodal maps. We find that in bimodal maps the global regularity of fractal dimensions has a generalization of similar form in comparison with unimodal case, while the entropy devil's staircase in unimodal maps is generalized to the devil's carpet [38].

The paper is organized as follows. In Sec. II, preliminaries of symbolic dynamics of bimodal maps are presented, the dual-star products are reviewed, and the self-similar classification of DSS sequences in the kneading plane is also shown. In Sec. III, we study the metric universality of dual-star products. In Sec. IV, the complexity of routes of transitions to chaos are discussed. Section V presents a global regularity of fractal dimensions independent of DSS sequences. Finally in Sec. VI, we give a short discussion.

II. PRELIMINARIES

A. Symbolic dynamics of three letters for bimodal maps

A *bimodal* map of the interval I is a piecewise monotone continuous map f from I to itself with two turning points c_1 and c_2. In this paper we consider the case of $+-+$ bimodal maps of the interval $I=[c_0,c_3]$ $(c_0<c_3;c_0,c_3\in\mathbb{R})$, i.e., maps which are increasing on $[c_0,c_1]$ and $[c_2,c_3]$, and decreasing on $[c_1,c_2]$. Let L, M, and R be assigned as *addresses* to the points belonging to the three intervals of monotonicity of f (for left, middle, and right, respectively), and C and D to the two turning points. The *itinerary* of a point $x\in I$, $A(x)=a_0a_1\ldots a_n\ldots$, is defined to be the sequence of addresses $a_n\in\{L,C,M,D,R\}$, such that $f^n(x)\in a_n$. The *kneading sequences* $K_i(f)=k_0^ik_1^i\ldots k_n^i\ldots$ are defined to be the itineraries of the extremal points $f(c_i)$, $i=1,2$; and the *kneading invariant* of the map f to be the 2-tuple $(K_1(f),K_2(f))$ which determine some important properties of the map.

Let the symbolic order $\prec$ be the Metropolis-Stein-Stein (MSS) order [5] or equivalently, the *lexicographical order* [6], which is complete [7]. Obviously, the ordering on the addresses is a natural order $L\prec C\prec M\prec D\prec R$. To induce the ordering of sequences, we define the *parity* of a sequence W as *even* if it contains an even number of M's, and *odd* otherwise, and a *parity operator* $\tau(W)$ by

$$\tau(W)=\begin{cases}+1 & \text{if } W \text{ is even}\\ -1 & \text{if } W \text{ is odd.}\end{cases}$$

Then, if two distinct sequences U and V are written as $U=Gu_k\ldots$ and $V=Gv_k\ldots$, with a common leading string G and $u_k\prec v_k$:

$$U\prec V \quad \text{if } \tau(G)=+1, \quad V\prec U \quad \text{if } \tau(G)=-1.$$

It is useful to define the *shift* operator φ, which deletes the first symbol of the sequence to which it is applied; one thus has $\varphi^k(W)=w_kw_{k+1}\ldots$ for the sequence $W=w_0w_1\ldots w_k\ldots$. For any two sequences U and V, if $\varphi^k(U)\preccurlyeq U$, and $V\preccurlyeq\varphi^k(V)$, for all $k\in\mathbb{Z}_+$ (where $\mathbb{Z}_+$ is the set of positive integers), then U is called *maximal*, V *minimal*, and $S:=(U,V)$ is an *extremal pair*. A pair S is called *compatible* if $\varphi^k(V)\preccurlyeq U$ and $V\preccurlyeq\varphi^k(U)$ for all $k\in\mathbb{Z}_+$. If the compatible pair S further satisfies the condition

$$\begin{aligned} V\prec\varphi^k(U)\preccurlyeq U, &\quad k\in\mathbb{Z}_+,\\ V\preccurlyeq\varphi^{k'}(V)\prec U, &\quad k'\in\mathbb{Z}_+, \end{aligned} \tag{2.1}$$

then S is called *admissible*. All the admissible pairs form an *admissible set* $\mathcal{K}$, they fill up the whole kneading parameter plane.

To obtain the sets $\{U\}$ and $\{V\}$, we can repeatedly operate the superorder left-handed multiplication $(L\prec M\prec R)\otimes$ [39] on the natural order $L\prec C\prec M\prec D\prec R$. For instance, $(L\prec M\prec R)\otimes(L\prec C\prec M\prec D\prec R)$ generates the following ordered sequences:

$$LL \prec LC \prec LM \prec LD \prec LR$$
$$\prec MR \prec MD \prec MM \prec MC \prec ML$$
$$\prec RL \prec RC \prec RM \prec RD \prec RR.$$

Along with the natural order $L \prec C \prec M \prec D \prec R$, we have the sequences

$$LL \prec LC \prec LM \prec LD \prec LR \prec C$$
$$\prec MR \prec MD \prec MM \prec MC \prec ML \prec D$$
$$\prec RL \prec RC \prec RM \prec RD \prec RR.$$

So the operation $(L \prec M \prec R)^{\otimes n} \otimes (L \prec C \prec M \prec D \prec R)$ will produce all the sequences $\{U\}$ and $\{V\}$ when $n \to \infty$,

$$L^\infty \prec \cdots \prec M^\infty \prec \cdots \prec R^\infty.$$

Obviously, $L^\infty \prec \cdots \prec M^\infty$ are the sequences of $\{V\}$, and $M^\infty \prec \cdots \prec R^\infty$ that of $\{U\}$. The *order topological space* $\Sigma_3 :=\{\prec, A\}$ is defined as the product of sets $\{U\}$ and $\{V\}$, where the pairs $A \equiv (U,V) \in \mathcal{K}$; and the order $\prec$ is used in the sense of the following meaning: for any two admissible sequence pairs $A_1=(U_1,V_1)$ and $A_2=(U_2,V_2)$, $A_1 \prec A_2$ if $U_1 \prec U_2$, or if $U_1=U_2$ and $V_2 \prec V_1$.

If $U=K_1$, $V=K_2$, the kneading pair (K_1,K_2) obviously satisfies the admissibility condition (2.1), namely,

$$K_2 \prec \varphi^k(K_1) \preccurlyeq K_1, \quad k \in \mathbb{Z}_+,$$
$$K_2 \preccurlyeq \varphi^{k'}(K_2) \prec K_1, \quad k' \in \mathbb{Z}_+. \tag{2.2}$$

In particular, if the kneading sequences K_1 and K_2 are periodic, and they contain both turning points C and D, i.e., $K_1 = XDYC \equiv K$, and $K_2 = YCXD \equiv \tilde{K}$, then the pair $(K,\tilde{K})$ is called the *DSS kneading pair*, and its admissibility condition (2.1) is reduced to

$$YC \prec \varphi^k(XD) \preccurlyeq XD, \quad \text{for } k=0,1,\ldots,|XD|-1,$$
$$YC \preccurlyeq \varphi^{k'}(YC) \prec XD, \quad \text{for } k'=0,1,\ldots,|YC|-1, \tag{2.3}$$

where $|XD|$ is the length of XD and $|YC|$ that of YC. Denoting by $\mathcal{K}_0$ the set of all DSS kneading pairs, then one obviously has $\mathcal{K}_0 \subset \mathcal{K}$.

The DSS kneading pairs are the typical representatives of all admissible pairs; they correspond to the *joints* of the skeleton in the kneading plane, from which the bones (singly superstable kneading sequences) spanning the kneading plane grow [13]. Therefore, to study the properties of the DSS kneading pairs is crucial in analyzing the structure of the kneading plane. In this paper we will mainly concentrate on the DSS kneading pairs $(K,\tilde{K})$.

For a periodic sequence W, we can introduce a *permutation* operation $\sim$ between its maximal (W_M) and minimal (W_m) representations as $\tilde{W}_M = W_m$, $\tilde{W}_m = W_M$. Obviously, any DSS kneading pair can be expressed by $(K,\tilde{K})$, with $K = XDYC$ and $\tilde{K} = YCXD$. When no confusion arises, we will simply call a DSS kneading pair $(K,\tilde{K})$ a DSS sequence K, or denote it as $XDYC$ or (XD,YC).

B. Composition rule of dual-star products

Let us first review the algebraic composition rule of the DGP star product [1] in symbolic dynamics of unimodal maps. Let $QC = q_1 q_2 \ldots q_m C$ and $SC = s_1 s_2 \ldots s_n C$ denote two superstable sequences, with $q_j, s_k \in \{L,R\}$, then their DGP star product $QC * SC$ is defined by symbol multiplication $\bullet$ and parity operation τ,

$$QC*SC = Q(C\bullet s_1)^{\tau(Q)} Q(C\bullet s_2)^{\tau(Q)} \ldots Q(C\bullet s_n)^{\tau(Q)} Q(C\bullet C)^{\tau(Q)},$$

with

$$C\bullet L = C^{-1}, \quad C\bullet R = C^{+1}, \quad C\bullet C = C^0 \equiv C.$$

Here in the unimodal case, the parity operator $\tau(Q)$ is defined by $\tau(Q)=+1$ if Q contains an even number of R's, and $\tau(Q)=-1$ otherwise, with $C^{-1}=L$, $C^0=C$, and $C^{+1}=R$. We can see that the DGP star product concerns the regular disturbance (multiplication and parity operation) of turning point C. The DGP star product is a standard or normal star product, which has many good algebraic properties: for instance, (i) admissibility; (ii) order preservation; (iii) period-doubling and p-tupling transformations; (iv) entropy preservation, namely, the first and the second topological conjugate transformations [40], etc. These properties should be considered in the generalization of star product.

In symbolic dynamics of bimodal maps, the generalization of the DGP star product should seek the regular disturbances of two turning points C and D. In the following, we will frequently be concerned with the replacement in a sequence of C by L or M, and of D by M or R. We define $C^{-1}=L$, $C^0=C$, $C^{+1}=M$; $D^{-1}=M$, $D^0=D$, and $D^{+1}=R$. Obviously, $C^{-1} \prec C \prec C^{+1}$ and $D^{-1} \prec D \prec D^{+1}$. This exponential notation will frequently be used in conjunction with the parity operation. It is easy to verify that $YC^{-\tau(Y)} \prec YC \prec YC^{+\tau(Y)}$ and $XD^{-\tau(X)} \prec XD \prec XD^{+\tau(X)}$.

Now we present the definition of the dual-star products for the $+-+$ bimodal maps [20,21]. Let $Z = XDYC$ and $W = UDVC \equiv w_1 w_2 \ldots w_{k+l+2}$ be two DSS kneading sequences, where $X = x_1 \ldots x_m$, $Y = y_1 \ldots y_n$, $U = u_1 \ldots u_k \equiv w_1 \ldots w_k$, $V = v_1 \ldots v_l \equiv w_{k+2} \ldots w_{k+l+1}$, $x_\zeta, y_\eta, u_\xi, v_\rho \in \{L,M,R\}$, $\zeta=1,\ldots,m$, $\eta=1,\ldots,n$, $\xi=1,\ldots,k$, and $\rho=1,\ldots,l$; obviously, $w_{k+1}=D$, $w_{k+l+2}=C$. There are two kinds of star products, i.e., $Z \overline{*} W$ and $\tilde{Z} \underline{*} \tilde{W}$, $\forall\ Z,W \in \mathcal{K}_0$. The *up-star* product $\overline{*}$ is defined as

$$Z\overline{*}W = (XDYC)\overline{*}(UDVC)$$
$$= (XDYC)\overline{*}u_1 \ldots (XDYC)\overline{*}u_k (XDYC)\overline{*}D$$
$$(XDYC)\overline{*}v_1 \ldots (XDYC)\overline{*}v_l (XDYC)\overline{*}C,$$

where the up-star product $\overline{*}$ consists of up-multiplication $\overline{\cdot}$ and parity operation τ,

$$(XDYC)\overline{*}a = X(D\overline{\bullet}a)^{\tau(X)} Y(C\overline{\bullet}a)^{\tau(Y)},$$

TABLE I. Multiplication table of up- and down-star products for the case $(+-+)$.

a	$D\overline{\bullet}a$	$C\overline{\bullet}a$	$C\underline{\bullet}a$	$D\underline{\bullet}a$
L	D^{-1}	C^{+1}	C^{-1}	D^{+1}
C	D^{-1}	C^{0}	C^{0}	D^{+1}
M	D^{-1}	C^{-1}	C^{+1}	D^{+1}
D	D^{0}	C^{-1}	C^{+1}	D^{0}
R	D^{+1}	C^{-1}	C^{+1}	D^{-1}

$$a\in\{L,C,M,D,R\}; \tag{2.4}$$

and the *down-star* product $\underline{*}$ as

$$\begin{aligned}\tilde{Z}\underline{*}\tilde{W}&=(YCXD)\underline{*}(VCUD)\\&=(YCXD)\underline{*}v_1\ldots(YCXD)\underline{*}v_l(YCXD)\underline{*}C\\&\quad(YCXD)\underline{*}u_1\ldots(YCXD)\underline{*}u_k(YCXD)\underline{*}D,\end{aligned}$$

where the down-star product $\underline{*}$ consists of down-multiplication $\underline{\cdot}$ and parity operation τ,

$$(YCXD)\underline{*}a=Y(C\underline{\bullet}a)^{\tau(Y)}X(D\underline{\bullet}a)^{\tau(X)},$$

$$a\in\{L,C,M,D,R\}. \tag{2.5}$$

Table I is the dual-star multiplication table, which lists the results of $D(C)\overline{\bullet}a$ and $C(D)\underline{\bullet}a$. For an arbitrary DSS sequence $XDYC$, its period-doubling transformations are just the products $(XDYC)\overline{*}(DC)$ and $(YCXD)\underline{*}(CD)$ which coincide with the operations l and r defined by MacKay and Tresser [14], and its period-p-tupling transformations are the products $(XDYC)\overline{*}(UDVC)$ and $(YCXD)\underline{*}(VCUD)$, where basic period $p=|UD|+|VC|$. For example, one can easily obtain the four 3×2 DSS sequences

$$DLC\overline{*}DC=DLLMLC,\quad RDC\overline{*}DC=RDLRMC,$$

$$LCD\underline{*}CD=LCRLMD,\quad CRD\underline{*}CD=CRRMRD,$$

and four 2×3 DSS sequences

$$DC\overline{*}DLC=DLMMMC,\quad DC\overline{*}RDC=RLDLMC,$$

$$CD\underline{*}LCD=LRCRMD,\quad CD\underline{*}CRD=CRMMMD.$$

Just as the DGP star product, the up- and down-star products $Z\overline{*}W$ and $\tilde{Z}\underline{*}\tilde{W}$ are admissible *compound* DSS kneading sequences [20], and have the following good algebraic properties, $\forall\ Z,W,S\in\mathcal{K}_0$ and $Z\neq W$.

(i) Noncommutativity: $Z\overline{*}W\neq W\overline{*}Z$, $\tilde{Z}\underline{*}\tilde{W}\neq\tilde{W}\underline{*}\tilde{Z}$.

(ii) Associativity: $Z\overline{*}(W\overline{*}S)=(Z\overline{*}W)\overline{*}S$, $\tilde{Z}\underline{*}(\tilde{W}\underline{*}\tilde{S})=(\tilde{Z}\underline{*}\tilde{W})\underline{*}\tilde{S}$.

(iii) Order preservation: $W\prec S\Rightarrow Z\overline{*}W\prec Z\overline{*}S$, $\widetilde{\tilde{Z}\underline{*}\tilde{W}}\prec\widetilde{\tilde{Z}\underline{*}\tilde{S}}$.

(iv) Kneading admissibility preservation: $Z,W\in\mathcal{K}_0\Rightarrow Z\overline{*}W,\tilde{Z}\underline{*}\tilde{W}\in\mathcal{K}_0$.

(v) Duality: $Z\overline{*}W=[(Z)^T\underline{*}(W)^T]^T$, $\tilde{Z}\underline{*}\tilde{W}=[(\tilde{Z})^T\overline{*}(\tilde{W})^T]^T$ where the *parity preservation transformation T* is defined as

$$R\rightleftharpoons L,\quad D\rightleftharpoons C,\quad M\rightleftharpoons M.$$

Therefore, these two star products possess dual symmetry under the parity preservation transformation T, and are thus called *dual-star* products.

According to the stipulation at the end of Sec. II A, the definitions in Eqs. (2.4) and (2.5) really represent the compound DSS kneading pairs. We will concisely denote them as $Z\overline{*}W\equiv(Z\overline{*}W,\widetilde{Z\overline{*}W})$, and $Z\underline{*}W\equiv(\widetilde{\tilde{Z}\underline{*}\tilde{W}},\tilde{Z}\underline{*}\tilde{W})$.

C. Word-lifting technique: Parametrization of DSS sequence

There exists a correspondence between a DSS sequence and a point in the kneading parameter plane. The word-lifting technique [9] provides a method to determine the *loci* of DSS sequences in the parameter plane.

Consider an arbitrary DSS sequence $W=UDVC$, where $U=u_1u_2\ldots u_k$, $V=v_1v_2\ldots v_l$, and $u_i,v_j\in\{L,M,R\}$. Let $x_C=c_1$ and $x_D=c_2$ be the coordinates of the turning points C and D of the bimodal map $y=f_{\lambda,\mu}(x)$; then a system of equations can be obtained:

$$\begin{aligned}f(x_C)&=f_{u_1}^{-1}\circ f_{u_2}^{-1}\circ\cdots\circ f_{u_k}^{-1}(x_D),\\f(x_D)&=f_{v_1}^{-1}\circ f_{v_2}^{-1}\circ\cdots\circ f_{v_l}^{-1}(x_C).\end{aligned} \tag{2.6}$$

Equation (2.6) determines an isolated point (i.e., joint) in the kneading parameter plane.

In this paper, we will mainly employ the two-parameter cubic map

$$f_{r,s}(x)=r+s(4x^3-3x) \tag{2.7}$$

as the actual metric model of bimodal maps $f_{\lambda,\mu}$; its turning points are $x_C=-\frac{1}{2}$ and $x_D=\frac{1}{2}$. In $f_{r,s}(x)$, $T_3(x)=4x^3-3x=\cos(3\arccos x)$ is just the Chebyshev polynomial, so the inverse functions of Eq. (2.7) can be easily found as follows:

$$\begin{aligned}f_L^{-1}(y)&=\cos\left(\frac{1}{3}\arccos\frac{y-r}{s}+\frac{2\pi}{3}\right),\\f_M^{-1}(y)&=\cos\left(\frac{1}{3}\arccos\frac{y-r}{s}-\frac{2\pi}{3}\right),\\f_R^{-1}(y)&=\cos\left(\frac{1}{3}\arccos\frac{y-r}{s}\right).\end{aligned} \tag{2.8}$$

By use of Eq. (2.8), from Eq. (2.6) the values of parameters (r,s) of a DSS sequence for map (2.7) can be solved. If one considers the standard cubic map

$$f_{a,b}(x)=a+(1-b)x-ax^2+bx^3, \tag{2.9}$$

the case will be similar. Because there is only a linear transformation between Eqs. (2.7) and (2.9), one can easily obtain parameters (a,b) from (r,s).

D. Periodic window, window band, and equal topological entropy class

For an arbitrary DSS kneading pair $Z=(XDYC,YCXD)$, the kneading sequence $K_C=XDYC$ has its upper and lower sequences denoted as

$$K_C^+=XD^{+\tau(X)}YC,\quad K_C^-=XD^{-\tau(X)}YC.$$

Similarly, the kneading sequence $K_D=YCXD$ has its upper and lower sequences as

$$K_D^+=YC^{+\tau(Y)}XD,\quad K_D^-=YC^{-\tau(Y)}XD.$$

These four singly superstable sequences $K_C^{\pm}$ and $K_D^{\pm}$ form the *basic* periodic windows of *Z*; they are what MacKay and Tresser called the *bones* [13]. The basic windows can be further divided into the *internal* and *external* windows. The internal window of K_C refers to

$$(K_C^+)^-=XD^{+\tau(X)}YC^{-\tau(Y)}$$

and

$$(K_C^-)^+=XD^{-\tau(X)}YC^{-\tau(Y)},$$

and the external window of K_C to

$$(K_C^+)^+=XD^{+\tau(X)}YC^{+\tau(Y)}$$

and

$$(K_C^-)^-=XD^{-\tau(X)}YC^{+\tau(Y)};$$

similarly, the internal window of K_D refers to

$$(K_D^+)^-=YC^{+\tau(Y)}XD^{+\tau(X)}$$

and

$$(K_D^-)^+=YC^{-\tau(Y)}XD^{+\tau(X)},$$

and the external window of K_D to

$$(K_D^+)^+=YC^{+\tau(Y)}XD^{-\tau(X)}$$

and

$$(K_D^-)^-=YC^{-\tau(Y)}XD^{-\tau(X)}.$$

Obviously, $(K_C^+)^-$ merges with $(K_D^-)^+$, $(K_C^-)^+$ with $(K_D^-)^-$, $(K_C^+)^+$ with $(K_D^+)^-$, and $(K_C^-)^-$ with $(K_D^+)^+$.

The *window band* of *Z* refers to the total of the windows of *Z* and all the associated PDB sequences $Z*(DC)^{*n}$ with $*\in\{\overline{*},\underline{*}\}$ and $n=1,2,\ldots$. Their windows are connected in the kneading plane, which can be deduced from the fact that Z, $Z\overline{*}(DC)$ and $Z\underline{*}(DC)$ have connected windows: the internal window sequence $XD^{-\tau(X)}YC^{-\tau(Y)}$ of *Z* merges with the external one $XD^{-\tau(X)}YC^{-\tau(Y)}XD^{-\tau(X)}YC^{-\tau(Y)}$ of $Z\overline{*}(DC)$, and the external one $XD^{+\tau(X)}YC^{+\tau(Y)}$ of *Z* with the internal one $XD^{+\tau(X)}YC^{+\tau(Y)}XD^{+\tau(X)}YC^{+\tau(Y)}$ of $Z\underline{*}(DC)$. To describe the window band of *Z*, we only choose the DSS sequences in the window band as representative for simplicity, and use the notation $Z_{\mathrm{WB}}=\cup_{*\in\{\overline{*},\underline{*}\},n\geq 0}Z*(DC)^{*n}$.

It has been proved that dual-star products have the property of preserving topological entropy, namely, for $Z\in\mathcal{K}_0$ and $A\in\mathcal{K}$ one has [20]

$$h(Z*A)=\begin{cases} h(Z) & \text{if } Z\neq(DC)^{*n},\\ \dfrac{1}{2^n}h(A) & \text{if } Z=(DC)^{*n},\end{cases}\tag{2.10a}$$

and

$$h((DC)^{*n})=0.\tag{2.10b}$$

Therefore, similar to the case of unimodal maps [30–34], an equal topological entropy class (ETEC) can be formed by dual-star products as $H_Z=Z*\mathcal{K}$, which exhibits as a *plateau* with a constant topological entropy $h(Z)$ in the space (h,λ,μ) [21]. The ETEC is a contraction formed by the dual-star map $Z*$ which compresses all phenomena occurring in the whole kneading plane to an ETEC plateau. Feigenbaum's metric universality [2–4] is confined within an ETEC plateau. There exist infinitely many plateaus with cardinal number $2^{\aleph_0}$ above the whole kneading plane. Their projections to the kneading plane construct a multifractal with a positive measure, i.e., a devil's carpet of topological entropy (a Sierpinski-like carpet). We will discuss the fractal characteristics of ETECs elsewhere [38].

Obviously, the zero topological entropy class is the set of the PDB sequences, $H_0=\cup_{*\in\{\overline{*},\underline{*}\},n\geq 1}(DC)^{*n}$, and the window band Z_{WB} is a part (subset) of the ETEC H_Z of *Z*, namely, the compression of the zero topological entropy class H_0 in H_Z. It should be indicated that besides the first topological conjugate transformation (2.10), there also exists a second topological conjugate transformation for dual-star products [41], just as the case for the DGP star product [40].

E. Self-similar structure of kneading plane

The kneading plane of bimodal maps has a perfect self-similarity that can be shown by the dual-star products. In Fig. 1(a) we present a classification for the *basic* periodic and quasiperiodic sequences in the kneading plane according to the structural similarity of the sequences. The kneading plane can be divided into two main regions: the PDB region Ω_{PDB} with zero topological entropy, and the chaotic region Ω_c with positive topological entropy. The chaotic region Ω_c can be further divided into a series of subregions Ω_m ($m=0,1,2,\ldots$), namely, $\Omega_c=\cup_{m=0}^{\infty}\Omega_m$; the set of the *basic* periodic and quasiperiodic sequences corresponding to each subregion Ω_m is denoted as $\wp_m$. Here the word ''basic'' means that we do not include the associated PDB sequences $Z*(DC)^{*n}$ ($Z\in\wp_m, n\geqslant 1$) in each subregion Ω_m.

Let $\mathcal{K}_\pi$ denote the set of all the *primitive* DSS sequences that cannot be decomposed by the composition rule of dual-star products. All these primitive sequences locate in the subregion Ω_0, namely, $\mathcal{K}_\pi\subset\mathcal{K}_0$. Besides the primitive sequences, there also exist the compound sequences of the form $Z_k*(DC)^{*n_1}*\cdots*Z_k*(DC)^{*n_k}*Z_{k+1}$ ($Z_k\in\mathcal{K}_\pi;*\in\{\overline{*},\underline{*}\},\quad k\geqslant 1,\quad n_k\in\mathbb{Z}_+$) in Ω_0, thus

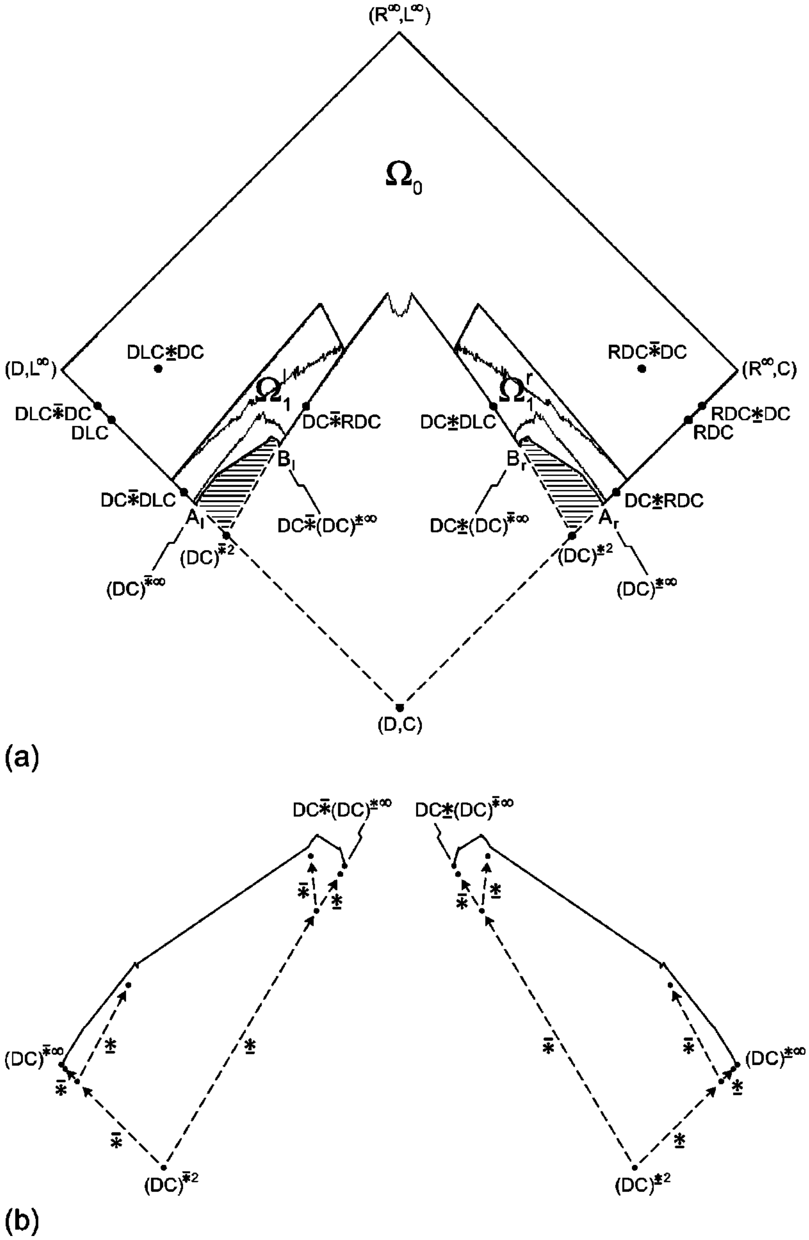

FIG. 1. (a) Schematic diagram for the self-similar classification of the basic sequences in the kneading plane. Only subregions Ω_0 and $\Omega_1=\Omega_1^l\cup\Omega_1^r$ are displayed for convenience. The black circles represent some special joints; they correspond to the period-2 DSS sequence DC, two period-3 DSS sequences (DLC and RDC), and two 2×2, four 3×2, and four 2×3 compound DSS sequences. The curves $\widetilde{A_lB_l}$ and $\widetilde{A_rB_r}$ indicate the boundary of topological chaos presented by the accumulation points of dual PDB products $(DC)*^{\infty}$. (b) The enlargement of the shaded regions in (a) that reveals the complexity of the PDB routes to chaos.

$$\wp_0=\{Z,Z_1*(DC)*^{n_1}*\cdots*Z_k*(DC)*^{n_k}*Z_{k+1}|Z,Z_k \in\mathcal{K}_\pi;k\geqslant 1,\ n_k\in\mathbb{Z}_+\}. \tag{2.11}$$

Obviously, the various possible combinations, such as $Z*^j$, $Z_1^{*j_1}*(DC)*^{n_1}*\ldots*Z_k^{*j_k}$, etc., are included in $\wp_0$. It should be indicated that there exist three types of DSS sequences in $\wp_0$; that is, we can divide $\wp_0$ into three subsets:

$$\wp_0=\wp_0^l\cup\wp_0^c\cup\wp_0^r.$$

Here $\wp_0^c$ is the set of symmetrical sequences of the form XDX^TC or Y^TDYC; these symmetrical sequences locate in the central line $K_C=K_D$ [i.e., $(D,C)\to(R^\infty,L^\infty)$] in the kneading plane. While $\wp_0^l$ and $\wp_0^r$ are two mirror-image sets symmetrical to the central line $K_C=K_D$, i.e., if $Z=XDYC\in\wp_0^l$, then $\widetilde{Z}^T=Y^TDX^TC\in\wp_0^r$.

For the subregion $\Omega_1=\Omega_1^l\cup\Omega_1^r$, we have

$$\wp_1=(DC)\bar{*}\wp_0\cup(DC)\underline{*}\wp_0, \tag{2.12}$$

where $(DC)\bar{*}\wp_0\equiv\wp_1^l$ and $(DC)\underline{*}\wp_0\equiv\wp_1^r$ are two mirror-image sets symmetrical to the central line $K_C=K_D$ in Ω_1; they locate in two small subregions $\Omega_1^l=(DC)\bar{*}\Omega_0$ and $\Omega_1^r=(DC)\underline{*}\Omega_0$, respectively. Obviously, $(DC)\bar{*}(XDYC)$ and $(DC)\underline{*}(Y^TDX^TC)$ are a pair of mirror-image sequences. It is noted that there do not exist symmetrical sequences in $\wp_1$, i.e.,

$$\wp_1=\wp_1^l\cup\wp_1^r,\quad \wp_1^c=\varnothing.$$

Similarly, for any other subregions $\Omega_m=\Omega_m^l\cup\Omega_m^r\equiv(\cup_{j=1}^{2^{m-1}}\Omega_{m,j}^l)\cup(\cup_{k=1}^{2^{m-1}}\Omega_{m,k}^r)$ consisting of 2^m small subregions with $m>1$, we have

$$\wp_m=(DC)\bar{*}\wp_{m-1}\cup(DC)\underline{*}\wp_{m-1}=\bigcup_{*\in\{\bar{*},\underline{*}\}}(DC)*^m*\wp_0 \tag{2.13}$$

and

$$\wp_m = \wp_m^l \cup \wp_m^r, \quad \wp_m^c = \varnothing.$$

Therefore, the set of all the basic DSS sequences in the chaotic region Ω_c is

$$\wp_c = \bigcup_{m=0}^{\infty} \wp_m. \tag{2.14}$$

The structure of each subregion Ω_m in the chaotic region Ω_c is completely similar. From Ω_{m-1} to Ω_m, the number of small subregions is doubled. The differences between Ω_m and Ω_{m-1} are in fact only contraction maps $(DC)\overline{*}$ and $(DC)\underline{*}$; that is, we have $\Omega_m^l = (DC)\overline{*}\Omega_{m-1}$ and $\Omega_m^r = (DC)\underline{*}\Omega_{m-1}$ corresponding to $\wp_m^l = (DC)\overline{*}\wp_{m-1}$ and $\wp_m^r = (DC)\underline{*}\wp_{m-1}$, respectively. Under the permutation operation $\sim$ and the parity preservation transformation *T*, there exists a dual symmetry between $\wp_m^l$ and $\wp_m^r$.

III. METRIC UNIVERSALITY OF DUAL-STAR PRODUCTS

A. Definitions of metric universal constants

1. Convergent rates δ

For a bimodal map $f_{\lambda,\mu}$, consider the dual-star products $(XDYC)*^n$, $* \in \{\overline{*}, \underline{*}\}$, of basic period $p = |XD| + |YC|$, and let

$$\lambda_n \in \{\overline{\lambda_n} \equiv \lambda((XDYC)\overline{*}^n), \quad \underline{\lambda_n} \equiv \lambda((XDYC)\underline{*}^n)\},$$

$$\mu_n \in \{\overline{\mu_n} \equiv \mu((XDYC)\overline{*}^n), \quad \underline{\mu_n} \equiv \mu((XDYC)\underline{*}^n)\}$$

denote their parameter values, then we can define at least three convergent rates,

$$\delta_{\lambda;n} = \frac{\lambda_{n-1} - \lambda_{n-2}}{\lambda_n - \lambda_{n-1}}, \tag{3.1a}$$

$$\delta_{\mu;n} = \frac{\mu_{n-1} - \mu_{n-2}}{\mu_n - \mu_{n-1}}, \tag{3.1b}$$

$$\delta_{\lambda,\mu;n} = \frac{d_{\lambda,\mu;(n-2,n-1)}}{d_{\lambda,\mu;(n-1,n)}}, \tag{3.1c}$$

where $d_{\lambda,\mu;(n-1,n)}$ is the Euclidean distance between points $(\lambda_{n-1}, \mu_{n-1})$ and (λ_n, μ_n) in the kneading parameter plane. $\delta_{\lambda;n} \in \{\overline{\delta}_{\lambda;n}, \underline{\delta}_{\lambda;n}\}$, $\delta_{\mu;n} \in \{\overline{\delta}_{\mu;n}, \underline{\delta}_{\mu;n}\}$, and $\delta_{\lambda,\mu;n} \in \{\overline{\delta}_{\lambda,\mu;n}, \underline{\delta}_{\lambda,\mu;n}\}$ describe the *asymptotic* processes of convergence along the λ direction, the μ direction and the convergent curve [or convergent points (λ_n, μ_n)] in the kneading parameter plane, respectively. Any one of them can describe the asymptotic process of convergence, because $\delta = \lim_{n\to\infty}\delta_{\lambda;n} = \lim_{n\to\infty}\delta_{\mu;n} = \lim_{n\to\infty}\delta_{\lambda,\mu;n}$. In fact, $\delta_{\lambda,\mu;n}$ can also be understood as the ratio of areas of the adjacent trapezoids, namely,

$$\delta_{\lambda,\mu;n} = \frac{(\lambda_{n-1}+\lambda_{n-2})(\mu_{n-1}-\mu_{n-2})}{(\lambda_n+\lambda_{n-1})(\mu_n-\mu_{n-1})} \quad \text{or} \quad \frac{(\lambda_{n-1}+\lambda_{n-2}-2\lambda_1)(\mu_{n-1}-\mu_{n-2})}{(\lambda_n+\lambda_{n-1}-2\lambda_1)(\mu_n-\mu_{n-1})}, \tag{3.1d}$$

which take $\lambda = 0$ and $\lambda = \lambda_1$ as the reference lines, respectively; or

$$\delta_{\lambda,\mu;n} = \frac{(\lambda_{n-1}-\lambda_{n-2})(\mu_{n-1}+\mu_{n-2})}{(\lambda_n-\lambda_{n-1})(\mu_n+\mu_{n-1})} \quad \text{or} \quad \frac{(\lambda_{n-1}-\lambda_{n-2})(\mu_{n-1}+\mu_{n-2}-2\mu_1)}{(\lambda_n-\lambda_{n-1})(\mu_n+\mu_{n-1}-2\mu_1)}, \tag{3.1e}$$

which take $\mu = 0$ and $\mu = \mu_1$ as the reference lines, respectively.

In practice, we employ the two-parameter cubic map $f_{r,s}(x)$ in Eq. (2.7) as the actual metric model of $f_{\lambda,\mu}$ to define and compute $\delta_{r;n}$, $\delta_{s;n}$, and $\delta_{r,s;n}$. We can also similarly define and compute $\delta_{a;n}$, $\delta_{b;n}$, and $\delta_{a,b;n}$ for the standard cubic map $f_{a,b}(x)$ in Eq. (2.9). The results show that whether the cubic maps are $f_{r,s}(x)$ or $f_{a,b}(x)$, so long as the bimodal maps considered are of cubic extrema the convergent rate for the same DSS sequence is the same. Therefore, it is a metric universality; we have given the values of δ for DSS sequences of basic period $p = 2$–4 in Ref. [21].

2. Scaling factors α

Let $f_{(n)}$ denote the map corresponding to $(XDYC)*^n$, $x_{C;n}$ and $x_{D;n}$ the coordinates of the turning points of $f_{(n)}$, and $p = |XD| + |YC|$ the basic period of $(XDYC)*^n$ and $* \in \{\overline{*}, \underline{*}\}$. In noting that $f_{(n)}^{p^{n-1}}(x_{C;n})$ is the nearest point to $x_{C;n}$, and $f_{(n)}^{p^{n-1}}(x_{D;n})$ that to $x_{D;n}$, the scaling properties of a DSS sequence in the phase space can be described by two scaling factors α_C and α_D. Here we only give the definition of α_C:

$$\alpha_{C;n} = \frac{f_{(n-1)}^{p^{n-2}}(x_{C;n-1}) - x_{C;n-1}}{f_{(n)}^{p^{n-1}}(x_{C;n}) - x_{C;n}}, \tag{3.2}$$

the definition of α_D can be similarly obtained with replacement of *C* by *D* in Eq. (3.2). Generally speaking, for each DSS sequence $Z = XDYC$, there exist two pairs of *asymptotically convergent* scaling factors: $\{\overline{\alpha}_C, \overline{\alpha}_D\}$ for up-star product $Z\overline{*}^n$, and $\{\underline{\alpha}_C, \underline{\alpha}_D\}$ for down-star product $Z\underline{*}^n$ with $n\to\infty$. The numerical results show again that both α_C and α_D are universal for either $f_{r,s}(x)$ in Eq. (2.7) or $f_{a,b}(x)$ in Eq. (2.9). The values of $\{\alpha_C, \alpha_D\}$ for DSS sequences of basic period $p = 2$–4 were computed in Ref. [21].

It should be indicated that Chang, Wortis, and Wright [11] studied the iterative properties of a trimodal quartic map early on. They found that the tricritical behavior of doubly stable 2^n cycles is characterized by the universal numbers $\delta_T = 7.28469$, $\alpha_T = -1.69030$, and $\alpha_T^2 = 2.85713$. This early result coincides with the special case of $p=2$ in bimodal maps.

B. Dual symmetry of metric universal constants

Just as Z and $\tilde{Z}^T$ have dual symmetry (i.e., they are symmetrical to the central line $K_C = K_D$ in the kneading plane), the metric universal constants also have a perfect dual symmetry:

$$\bar{\delta}(Z) = \underline{\delta}(\tilde{Z}^T), \tag{3.3}$$

$$\bar{\alpha}_C(Z) = \underline{\alpha}_D(\tilde{Z}^T), \quad \bar{\alpha}_D(Z) = \underline{\alpha}_C(\tilde{Z}^T). \tag{3.4}$$

This has been verified by numerical calculation. Thus it is enough to show only the numerical results of up-star products. For example, we have

$$\bar{\delta}(DLLC) = \underline{\delta}(RRDC) = 1275.1,$$

$$\bar{\delta}(RRDC) = \underline{\delta}(DLLC) = 32.187.$$

$$\bar{\alpha}_C(DLLC) = \underline{\alpha}_D(RRDC) = -6.1918,$$

$$\bar{\alpha}_D(DLLC) = \underline{\alpha}_C(RRDC) = 38.338,$$

$$\bar{\alpha}_C(RRDC) = \underline{\alpha}_D(DLLC) = -6.2185,$$

$$\bar{\alpha}_D(RRDC) = \underline{\alpha}_C(DLLC) = 7.1805.$$

Furthermore, for two special types of products $(DYC)^{\bar{*}n}$ and $(XDC)^{\underline{*}n}$ that locate in two straight lines $(D,C) \rightarrow (D,L^\infty)$ and $(D,C) \rightarrow (R^\infty,C)$, respectively, in the kneading plane [see Fig. 1(a)], we have in general,

$$[\bar{\alpha}_C(DYC)]^2 = \bar{\alpha}_D(DYC),$$

$$[\underline{\alpha}_D(XDC)]^2 = \underline{\alpha}_C(XDC).$$

This can be seen from the above examples.

For the concrete map $f_{r,s}(x)$, duality leads to the exact symmetry in the kneading plane, i.e.,

$$\bar{r}(Z^{\bar{*}n}) = -\underline{r}((\tilde{Z}^T)^{\underline{*}n}), \quad \bar{s}(Z^{\bar{*}n}) = \underline{s}((\tilde{Z}^T)^{\underline{*}n}).$$

For the above examples, we have

$$\bar{r}_n(DLLC) = -\underline{r}_n(RRDC), \quad \bar{s}_n(DLLC) = \underline{s}_n(RRDC);$$

$$\bar{r}_n(RRDC) = -\underline{r}_n(DLLC), \quad \bar{s}_n(RRDC) = \underline{s}_n(DLLC).$$

Thus the route of convergence of $Z^{\bar{*}n}$ is completely symmetrical to that of $(\tilde{Z}^T)^{\underline{*}n}$.

From the above, we can see that for each DSS sequence $Z = XDYC$ of period $p = |XD| + |YC|$, there exist a pair of convergent rates $\{\bar{\delta}, \underline{\delta}\}$ and two pairs of scaling factors $\{\bar{\alpha}_C, \bar{\alpha}_D\}$ and $\{\underline{\alpha}_C, \underline{\alpha}_D\}$, which describe the different convergent and scaling behaviors of the dual-star products. The up-star product $Z^{\bar{*}n}$ and down-star product $Z^{\underline{*}n}$ exhibit different bifurcation structures; they locate along different directions in the kneading plane, and are also different in the phase space. Therefore we can call them the dual PpTBs, i.e., up PpTB and down PpTB, respectively. The bifurcation diagram should be a very complicated picture in the three-dimensional space (r,s,x). The global bifurcation structure is perfectly mirror symmetrical to the central line $K_C = K_D$ in the kneading plane due to the dual symmetry of the universal constants.

C. Metric universality of regularly mixed dual-star products

Now we investigate the metric universality of *regularly* mixed dual-star products. The regularly mixed dual-star products can be formed in the following ways.

(i) Given an arbitrary primitive DSS sequence Z, we can construct infinitely many *new* compound DSS sequences of the form

$$Z_{\rm com} = Z^{\bar{*}n_1} \underline{*} Z^{\underline{*}n_2} \bar{*} Z^{\bar{*}n_3} \underline{*} Z^{\underline{*}n_4} \bar{*} \underline{*} \cdots \bar{*} Z^{\bar{*}n_{k-1}} \underline{*} Z^{\underline{*}n_k}, \tag{3.5}$$

where each n_j ($j \in \{1,2,\ldots,k\}$) is a *finite* nonnegative integer. For each $Z_{\rm com}$ with a set of fixed values $\{n_1, n_2, \ldots, n_k\}$, the products $(Z_{\rm com})^{\bar{*}n}$ and $(Z_{\rm com})^{\underline{*}n}$, with $n = 1,2,\ldots,\infty$, will lead to a pair of convergent rates $\{\bar{\delta}(Z_{\rm com}), \underline{\delta}(Z_{\rm com})\}$ and two pairs of scaling factors $\{\bar{\alpha}_C(Z_{\rm com}), \bar{\alpha}_D(Z_{\rm com})\}$ and $\{\underline{\alpha}_C(Z_{\rm com}), \underline{\alpha}_D(Z_{\rm com})\}$.

(ii) Given a series of *distinct* primitive DSS sequences $Z_1, Z_2, \ldots, Z_k$, we can also construct infinitely many compound DSS sequences of the form

$$Z_{\rm com} = Z_1^{\bar{*}n_1} \underline{*} Z_2^{\underline{*}n_2} \bar{*} Z_3^{\bar{*}n_3} \underline{*} Z_4^{\underline{*}n_4} \bar{*} \underline{*} \cdots \bar{*} Z_{k-1}^{\bar{*}n_{k-1}} \underline{*} Z_k^{\underline{*}n_k}. \tag{3.6}$$

Similarly, for each such $Z_{\rm com}$ in Eq. (3.6) with a set of fixed values $\{n_1, n_2, \ldots, n_k\}$, there also exist a pair of convergent rates $\{\bar{\delta}(Z_{\rm com}), \underline{\delta}(Z_{\rm com})\}$ and two pairs of scaling factors $\{\bar{\alpha}_C(Z_{\rm com}), \bar{\alpha}_D(Z_{\rm com})\}$ and $\{\underline{\alpha}_C(Z_{\rm com}), \underline{\alpha}_D(Z_{\rm com})\}$. In fact, Eq. (3.6) is a general form, while Eq. (3.5) is its special case by setting $Z_1 = Z_2 = \ldots = Z_k \equiv Z$.

In Table II, we list some numerical results of universal constants of compound DSS sequences, which verify the above conclusions. In this table, $(DC*DC)^{*n}$ locate in the PDB region $\Omega_{\rm PDB}$, $(DLC*DC)^{*n}$ and $(RDC*DC)^{*n}$ in the subregion Ω_0 of the chaotic region Ω_c, and $(DC*DLC)^{*n}$ and $(DC*RDC)^{*n}$ in the subregion Ω_1 of Ω_c [see also Fig. 1(a) for reference]. We can see that $\bar{\delta}(DLMC) = \bar{\delta}((DC)^{\bar{*}2}) = [\bar{\delta}(DC)]^2$ and $\underline{\delta}(RMDC) = \underline{\delta}((DC)^{\underline{*}2}) = [\underline{\delta}(DC)]^2$, it can be extended to

$$\bar{\delta}(Z^{\bar{*}k}) = [\bar{\delta}(Z)]^k, \quad \underline{\delta}(Z^{\underline{*}k}) = [\underline{\delta}(Z)]^k; \tag{3.7}$$

this is similar to $\delta((WC)^{*k}) = [\delta(WC)]^k$ for superstable periodic sequences in the unimodal case. Meanwhile, we have $\bar{\alpha}_C(DLMC) = \bar{\alpha}_C((DC)^{\bar{*}2}) = [\bar{\alpha}_C(DC)]^2$ and $\bar{\alpha}_D(DLMC)$

TABLE II. Universal constants with dual symmetry for some compound DSS sequences for the cubic map.

Z	$\overline{\delta}(Z)$	$\overline{\alpha}_C(Z)$	$\overline{\alpha}_D(Z)$	$\underline{\delta}(Z)$	$\underline{\alpha}_C(Z)$	$\underline{\alpha}_D(Z)$
$(DC)^{\overline{*}2}$	53.067	2.8571	8.1632	14.599	−4.8627	−4.8627
$(DC)^{\underline{*}2}$	14.599	−4.8627	−4.8627	53.067	8.1632	2.8571
$DLC\overline{*}DC$	465	4.94	24.4	46.0	−7.88	−12.3
$DLC\underline{*}DC$	46.0	−8.51	−13.3	46.0	17.0	6.66
$RDC\overline{*}DC$	46.0	6.66	17.0	46.0	−13.3	−8.51
$RDC\underline{*}DC$	46.0	−12.3	−7.88	465	24.4	4.94
$DC\overline{*}DLC$	465	4.94	24.4	46.0	−8.51	−13.3
$DC\underline{*}DLC$	46.0	−7.88	−12.3	46.0	17.0	6.66
$DC\overline{*}RDC$	46.0	6.66	17.0	46.0	−12.3	−7.88
$DC\underline{*}RDC$	46.0	−13.3	−8.51	465	24.4	4.94

$=\overline{\alpha}_D((DC)^{\overline{*}2})=[\overline{\alpha}_D(DC)]^2$; this leads to a general result for the compound DSS sequences:

$$\overline{\alpha}_C(Z^{\overline{*}k})=[\overline{\alpha}_C(Z)]^k,\quad \overline{\alpha}_D(Z^{\overline{*}k})=[\overline{\alpha}_D(Z)]^k,$$
$$\underline{\alpha}_D(Z^{\underline{*}k})=[\underline{\alpha}_D(Z)]^k,\quad \underline{\alpha}_C(Z^{\underline{*}k})=[\underline{\alpha}_C(Z)]^k, \tag{3.8}$$

which is also similar to $\alpha((WC)^{*k})=[\alpha(WC)]^k$ for superstable periodic sequences in the unimodal case. However, for the mixed dual-star products composed of distinct DSS sequences, say, $(Z_1*Z_2)^{*n}$, the approximate relations $\delta(Z_1*Z_2)\approx\delta(Z_1)\delta(Z_2)$ and $\alpha_{C,D}(Z_1*Z_2)\approx\alpha_{C,D}(Z_1)\alpha_{C,D}(Z_2)$ are not satisfactory in accuracy.

It should be emphasized that, in Eq. (3.6), if a certain $n_j\geqslant 3$ ($j\in\{1,2,\ldots,k\}$), then one will observe the *local* convergent behavior described by $\overline{\delta}_{n_j}(Z_j)$ or $\underline{\delta}_{n_j}(Z_j)$ depending on whether $Z_j^{*n_j}$ is up-star product $Z_j^{\overline{*}n_j}$ or down-star product $Z_j^{\underline{*}n_j}$; Similarly, if $n_j\geqslant 2$, one will see the *local* scaling behavior described by $\{\overline{\alpha}_{C;n_j}(Z_j),\overline{\alpha}_{D;n_j}(Z_j)\}$ or $\{\underline{\alpha}_{C;n_j}(Z_j),\underline{\alpha}_{D;n_j}(Z_j)\}$. However, these local convergent and scaling behaviors are *approximate* because n_j is finite, while *exact* convergent and scaling behaviors [$\delta(Z_j)$, $\alpha_C(Z_j)$, and $\alpha_D(Z_j)$] can only be approached by $Z_j^{*n_j}$ with the limit $n_j\to\infty$.

Of course, the associated PDB sequences $Z*(DC)^{\overline{*}n}$ and $Z*(DC)^{\underline{*}n}$, with $n=1,2,\ldots,\infty$, in the window band, keep the universal constants $\overline{\delta}(DC)=\underline{\delta}(DC)$, $\overline{\alpha}_C(DC)=\underline{\alpha}_D(DC)$, and $\overline{\alpha}_D(DC)=\underline{\alpha}_C(DC)$. Similarly, the associated PpTB sequences $Z_1*Z_2^{\overline{*}n}$ and $Z_1*Z_2^{\underline{*}n}$ keep the universal constants $\{\overline{\delta}(Z_2),\overline{\alpha}_C(Z_2),\overline{\alpha}_D(Z_2)\}$ and $\{\underline{\delta}(Z_2),\underline{\alpha}_C(Z_2),\underline{\alpha}_D(Z_2)\}$, respectively.

IV. COMPLEXITY OF ROUTES OF TRANSITIONS TO CHAOS

A. Combinatorial complexity of patterns of dual-star products

Before discussing the routes of transitions to chaos, we will introduce the patterns of dual-star products which can help us understand the combinatorial complexity of dual-star products. Here we will mainly focus on the self-combinations of a primitive DSS sequence.

Let the symbol 0 denote the up-star operation $\overline{*}$, and 1 the down-star operation $\underline{*}$. For an arbitrary primitive DSS sequence Z, we define the *pattern* of its dual-star product of a finite power k:

$$Z^{*k}=Z*Z*\cdots*Z,\quad *\in\{\overline{*},\underline{*}\}, \tag{4.1a}$$

where there are $k-1$ operations, to be

$$\sigma_{\{k\}}=\sigma_1\sigma_2\cdots\sigma_{k-1},\quad \sigma_j\in\{0,1\}. \tag{4.1b}$$

Thus we can transfer the study of various dual-star products to that of patterns.

In such a way, Eq. (3.5) can be described by the following pattern:

$$\sigma_{\{n_1,n_2,\ldots,n_k\}}\equiv\sigma_1\sigma_2\cdots\sigma_{n_1+n_2+\cdots+n_k-1}$$
$$=0\ldots01\ldots10\ldots10\ldots01\ldots1, \tag{4.2}$$

where there are n_1-1 zeros, n_2 ones, . . . , followed by n_{k-1} zeros and n_k ones. If $n_1=0$, we stipulate that the above pattern should be written as

$$\sigma_{\{n_2,\ldots,n_k\}}=\sigma_1\sigma_2\ldots\sigma_{n_2+\cdots+n_k-1}$$
$$=1\ldots10\ldots10\ldots01\ldots1,$$

which represents the product $Z^{\underline{*}n_2}\overline{*}\underline{*}\cdots\overline{*}Z^{\overline{*}n_{k-1}}\underline{*}Z^{\underline{*}n_k}$, and where there are n_2-1 ones, and n_{k-1} zeros, followed by n_k ones.

Obviously, a periodic pattern, i.e., an infinitely repeating pattern of a finite string $\sigma_1\sigma_2\cdots\sigma_{k-1}\sigma_k$,

$$\overline{\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k}\equiv(\sigma_1\sigma_2\ldots\sigma_{k-1}\sigma_k)^{\infty}, \tag{4.3}$$

can describe an accumulation point of a regularly mixed dual-star product, i.e., $(Z^{*k})^{*\infty}$. The last symbol σ_k denotes the type of combinations of the repeating *basic* block Z^{*k}. For example, if $\sigma_k=0$, then $\overline{\sigma_1\sigma_2\ldots\sigma_{k-1}0}$ represents $(Z^{*k})^{\overline{*}\infty}$. It can also be easily deduced that an eventually periodic pattern $\sigma_1\sigma_2\ldots\sigma_m\overline{\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+k}}$ with a fixed finite non-negative integer m corresponds to an associated bifurcation $Z*^m*(Z*^k)*^\infty$.

If we assign a pattern σ with a binary number $0.\sigma$, then any one of the real numbers on $[0,1]$ in a binary system corresponds to a possible pattern of mixed dual-star products. Obviously, $0.\overline{0}=0.000\ldots$ corresponds to the pure up-star product $Z\overline{*}^\infty$, and $1.\overline{0}\equiv 0.\overline{1}=0.111\ldots$ to the pure down-star product $Z\underline{*}^\infty$. Further, we should indicate that a rational number on $[0,1]$ (i.e., a fractional number) corresponds to a periodic or eventually periodic pattern, and an irrational number to an infinite nonperiodic pattern which can describe an irregularly mixed dual-star product. We have known that there are uncountably infinitely many real numbers on the interval $[0,1]$ which possess the cardinal number of the continuum, so there also exist uncountably infinitely many patterns for the mixed dual-star products. From the above we can see that the combinatorial complexity of dual-star products embodies in the following aspects.

(i) For a concrete primitive DSS sequence Z, there are an infinite number of patterns; these patterns can correspond to an infinite number of regularly and irregularly mixed dual-star products of Z. Furthermore, since Z can be taken from the set $\mathcal{K}_\pi$ of all the infinitely many *primitive* DSS sequences, therefore, for a concrete pattern, there are also an infinite number of dual-star products to be corresponded, this can be attained by taking Z over the set $\mathcal{K}_\pi$ for such a pattern. Thus the patterns have a twofold combinatorial complexity.

(ii) If going beyond the self-combinations of a primitive sequence, then for an arbitrary pattern $\sigma=\sigma_1\sigma_2\ldots\sigma_k\ldots$, the corresponding sequences (say Z_j and Z_{j+1}) lying before and after the symbol σ_j in the pattern can be different, this complicates the combinatorial types of dual-star products much more than the case (i), because each Z_j can take over the set $\mathcal{K}_\pi, Z_j\in\mathcal{K}_\pi,\ j\in\mathbb{Z}_+$.

Now we have known that patterns are closely related to dual-star products: each pattern σ corresponds to a binary number $0.\sigma$, and it can also correspond to the infinitely many dual-star products from either case (i) or case (ii). Therefore, the infinitely many patterns make the combinations of dual-star products extremely complicated.

B. Regular universal scaling routes: Preservation of Feigenbaum's metric universality

In this subsection, we discuss the *regular* routes of transitions to chaos in bimodal maps. The PDB and PpTB are two types of bifurcation routes. Using the concept of pattern defined above, we can show the characteristics of these regular routes easily.

The PDB's are described by the dual-star products $(DC)*^n$, $*\in\{\overline{*},\underline{*}\}$, $n=1,2,\ldots,\infty$. A pattern σ related to the PDB routes must be infinite associated with taking Z as the period-2 DSS sequence DC. If this infinite pattern σ is periodic, that is, it has an infinitely repeating byte as in Eq. (4.3), we refer to this as a regular PDB route, which corresponds to a rational number on $[0,1]$. Obviously, $\sigma=\overline{0}$ and $\sigma=\overline{1}$ correspond to the *pure up-star* product $(DC)\overline{*}^\infty$ and *pure down-star* product $(DC)\underline{*}^\infty$, they provide two pure and regular routes of transitions to chaos, which are structurally universal and have metric universal convergent rates $\overline{\delta}(DC)=\underline{\delta}(DC)$, and scaling factors $\overline{\alpha}_C(DC)=\underline{\alpha}_D(DC)$ and $\overline{\alpha}_D(DC)=\underline{\alpha}_C(DC)$.

In general, according to Eqs. (4.2) and (4.3), an arbitrary regular route of transition to chaos can be described by the periodic pattern

$$\overline{\sigma_1\sigma_2\ldots\sigma_{n_1+n_2+\cdots+n_k-1}\sigma_{n_1+n_2+\cdots+n_k}}, \tag{4.4}$$

where each n_j ($j\in\{1,2,\ldots,k\}$) is a *finite* nonnegative integer. This is equivalent to the following construction. First, for each regular combination (i.e., a set of n_j values or a set of $\sigma_1\sigma_2\cdots\sigma_{n_1+n_2+\cdots+n_k-1}$), there exists a repeating byte that can be constructed by the compound PDB sequences in the form of Eq. (3.5), namely,

$$Z_{\{DC\}}=(DC)\overline{*}^{n_1}\underline{*}(DC)\underline{*}^{n_2}\overline{*}(DC)\overline{*}^{n_3}\underline{*}(DC)\underline{*}^{n_4},\quad \overline{*}\underline{*}\cdots\overline{*}(DC)\overline{*}^{n_{k-1}}\underline{*}(DC)\underline{*}^{n_k}. \tag{4.5}$$

Second, the last symbol $\sigma_{n_1+n_2+\cdots+n_k}$ in Eq. (4.4) indicates the combinatorial type of the repeating *basic* block $Z_{\{DC\}}$ in $(Z_{\{DC\}})*^\infty$. Therefore, pattern (4.4) or $(Z_{\{DC\}})*^\infty(*:\overline{*}$ or $\underline{*})$ presents a regular route of transition to chaos, which is structurally universal and preserves Feigenbaum's metric universality, i.e., there exist the metric universal constants $\delta(Z_{\{DC\}})$, $\alpha_C(Z_{\{DC\}})$, and $\alpha_D(Z_{\{DC\}})$ of the compound PDB sequence $Z_{\{DC\}}$ [see, e.g., Table II for $((DC)\underline{*}(DC))*^n=(RMDC)*^n$]. Here, we omit the discussion for another kind of regular routes, namely, the eventually periodic pattern $\sigma_1\sigma_2\ldots\sigma_m\overline{\sigma_{m+1}\sigma_{m+2}\ldots\sigma_{m+n_1+n_2+\cdots+n_k}}$ with a fixed finite non-negative integer m, since it corresponds to the associated bifurcation $(DC)*^m*(Z_{\{DC\}})*^\infty$ with the same metric universal constants as Eq. (4.5). All the regular routes are (countably) infinitely many.

It should be indicated that these infinitely many *regular universal scaling routes* of transitions to chaos are extremely similar to that in the Feigenbaum scenario of the unimodal maps; they are of zero topological entropy from Eq. (2.10b), and have metric universal constants. Each of them is connected because they belong to the window band sequences.

By using the same pattern (4.4) while replacing DC by $XDYC$ of period $p=|XD|+|YC|\geqslant 3$ in the above constructions in Eq. (4.5), each period-p DSS sequence $XDYC$ will lead to (countably) infinitely many regular PpTB routes in the chaotic region Ω_c described by $(Z_{\{XDYC\}})*^\infty$. These PpTB routes preserve the topological entropy of $XDYC$, i.e., $h((Z_{\{XDYC\}})*^n)=h(Z_{\{XDYC\}})=h(XDYC)$. In addition, the window of $(XDYC)*^n$ and that of $(XDYC)*^{(n+1)}$ [and of course, that of $(Z_{\{XDYC\}})*^n$ and that of $(Z_{\{XDYC\}})*^{(n+1)}$] are disconnected, there exist both periodic and chaotic behaviors between their windows.

C. Irregular universal nonscaling routes: Breaking of Feigenbaum's metric universality

Besides the regular routes discussed above, we know from Sec. IV A that there also exist infinitely many *irregular*

TABLE III. A numerical example of the *irregularly* mixed period-tripling dual star product $(DLC)^{*n}$ for the map $f_{r,s}(x)$.

n	Sequence $(DLC)^{*n}$	$\alpha_{C;n}$	$\alpha_{D;n}$	$\delta_{r,s;n}$	$\langle\delta\rangle_n$
1	(DLC)				
2	$(DLC)^{*(n-1)}\underline{*}(DLC)$	6.6686	−25.412		
3	$(DLC)^{*(n-1)}\underline{*}(DLC)$	5.3078	−3.8819	16.188	16.188
4	$(DLC)^{*(n-1)}\overline{*}(DLC)$	−4.3592	1.9098	23.726	19.957
5	$(DLC)^{*(n-1)}\underline{*}(DLC)$	3.5814	−19.377	6.0743	15.329
6	$(DLC)^{*(n-1)}\underline{*}(DLC)$	5.0531	−4.0157	16.219	15.552
7	$(DLC)^{*(n-1)}\overline{*}(DLC)$	−4.3044	1.9085	24.494	17.340
8	$(DLC)^{*(n-1)}\overline{*}(DLC)$	−3.2655	9.7158	95.831	30.422
9	$(DLC)^{*(n-1)}\overline{*}(DLC)$	−3.1489	9.8350	83.129	37.952

universal nonscaling routes of transitions to chaos, and each of them is structurally universal but not metrically universal. An irregular route must have an infinite nonperiodic pattern that corresponds to an irrational number on [0,1]. Such a pattern can be called an infinite random pattern which implies that each symbol σ_j in the pattern is randomly taken from {0,1}; that is, $*$ is randomly taken from $\{\overline{*},\underline{*}\}$, and the corresponding dual-star products are *irregularly* mixed. In this case, the irregular PDB routes can generally be described by

$$(DC)^{\overline{*}^{n_1}}\underline{*}(DC)^{\underline{*}^{n_2}}\overline{*}(DC)^{\overline{*}^{n_3}}\underline{*}(DC)^{\underline{*}^{n_4}}\overline{*}\underline{*}\ldots, \tag{4.6}$$

where n_k $(k=1,2,\ldots)$ can take *arbitrary* non-negative integers to combine $n_1+n_2+n_3+n_4+\cdots\to\infty$, and the sequence of numbers $\{n_1,n_2,\ldots,n_k,\ldots\}$ should be irregular. Obviously, there will be (uncountably) infinitely many combinatorial types or patterns with cardinal number $2^{\aleph_0}$. These infinitely many irregularly mixed dual PDB's are of zero entropy from Eq. (2.10b), and they are mutually connected according to Sec. II D. In the topological sense of monotone equivalence class of various maps, we know that an arbitrary DSS sequence possesses structural universality; that is, a symbolic sequence can describe the same topological behavior for different bimodal maps. Thus the mixed dual-star products (either regular or irregular) have the structural universality. However, when the combinations $(DC)^{*n}=(DC)^{*(n-1)}*(DC)$ of up- and down-star products are random, taking the limit $n\to\infty$, we can see explicitly from the numerical verifications that δ_n, $\alpha_{C;n}$, and $\alpha_{D;n}$ of irregularly mixed dual PDB's do not exhibit *definite* asymptotically convergent behavior although there may exist some approximate *local* convergent behaviors; that is, Feigenbaum's metric universality is broken [21].

Still, by replacing DC by $XDYC$ of period $p\geqslant 3$ in the above constructions in Eq. (4.6), we have (uncountably) infinitely many irregular PpTB routes in the chaotic region Ω_c, with nonzero topological entropy and disconnected windows. These irregularly mixed PpTB routes also do not exhibit the definite asymptotically universal convergent rates and scaling factors (cf. Table III).

From the above we can see that all the infinitely many generalized Feigenbaum's routes of transitions to chaos are created by combinations of up- and down-star products which are all structurally universal. However, these routes belong to two distinctive types: the regular ones possess Feigenbaum's asymptotic metric universality, and the irregular ones do not. The regular routes are related to an infinite number of periodic or eventually periodic patterns which correspond to countably infinitely many rational numbers on [0,1], and the irregular routes are related to an infinite number of nonperiodic random patterns which correspond to uncountably infinitely many irrational numbers on [0,1]. This presents a picture of complexity: the irregular routes would be complicatedly mingled with the regular ones in the present *dual bifurcation scenario*. The set of infinitely many accumulation points of the regular (universal scaling) and irregular (universal nonscaling) dual PDB's forms a fractal curve, which is just what MacKay and Tresser called the boundary of topological chaos [14] (cf. Fig. 1). From the computability theory of numbers, we know that the noncomputable random numbers in a binary system are almost full in the real line of [0,1]. Therefore, the *structurally universal nonscaling routes* of transitions to chaos which possess a positive measure deserve much attention in the practical dynamical systems with more than one parameter.

D. Exploration of characterization of the irregular routes

We now make an exploration for describing the characterization of convergence of the irregular routes to chaos. Though the irregular routes break Feigenbaum's asymptotically universal scaling, these routes are convergent. We have noted that for the map $f_{\lambda,\mu}(x)$ [$f_{r,s}(x)$ in Eq. (2.7) or $f_{a,b}(x)$ in Eq. (2.9)], the parameter values $\{\lambda_n,\mu_n\}$ of the irregularly mixed Z^{*n} ($Z=DC$ or $XDYC$) form the convergent series. So the sum of distances of every two adjacent points,

$$U_{\lambda,\mu;n}=\sum_{j=2}^{n} d_{\lambda,\mu;(j-1,j)}, \tag{4.7}$$

can clearly reflect the convergence of the irregular routes (Table IV), because there is a Cesàro summability for these routes. However, $U_{\lambda,\mu;n}$ is not a ratio; it depends on the map $f_{\lambda,\mu}(x)$. If noting that in the product sequence, where there are n Z's,

$$Z^{*n}=Z*Z*\ldots*Z*Z*Z*\ldots*Z,$$
$$\updownarrow\ \updownarrow$$
$$\sigma_a\ \sigma_b$$

every three DSS sequences and every two multiplication symbols contribute a *local* convergent rate $\delta^{\sigma_a\sigma_b}$, and there are four types of $\delta^{\sigma_a\sigma_b}$, i.e., δ^{00}, δ^{01}, δ^{10}, and δ^{11}, then we can use an average convergent rate $\langle\delta\rangle_n$ to describe an irregular route (irregularly mixed product Z^{*n}), namely,

$$\langle\delta\rangle_n=\frac{1}{n-2}\sum_{j=3}^{n}\delta_j^{\sigma_{j-2}\sigma_{j-1}}. \tag{4.8a}$$

This $\langle\delta\rangle_n$ can also be thought of as a weighted average value for four types of convergence

TABLE IV. A numerical example of the *irregularly* mixed period-tripling dual-star product $(DLC)^{*n}$ for the maps $f_{r,s}(x)$ and $f_{a,b}(x)$. Here we do not list the parameter values of (a_n, b_n).

n	Sequence $(DLC)^{*n}$	(r_n, s_n)	$U_{r,s;n}$	$U_{a,b;n}$
1	(DLC)	$(-0.213\,267\,757\,584,\ 0.713\,267\,757\,584)$		
2	$(DLC)^{*(n-1)}\underline{*}(DLC)$	$(-0.175\,493\,551\,200,\ 0.768\,859\,311\,538)$	0.067 210 948 059 4	0.177 809 314 350
3	$(DLC)^{*(n-1)}\underline{*}(DLC)$	$(-0.173\,435\,246\,026,\ 0.772\,465\,195\,875)$	0.071 362 938 189 1	0.189 147 505 908
4	$(DLC)^{*(n-1)}\overline{*}(DLC)$	$(-0.173\,384\,657\,188,\ 0.772\,632\,719\,206)$	0.071 537 933 323 4	0.189 658 575 010
5	$(DLC)^{*(n-1)}\underline{*}(DLC)$	$(-0.173\,371\,726\,413,\ 0.772\,658\,463\,428)$	0.071 566 742 521 2	0.189 738 793 064
6	$(DLC)^{*(n-1)}\underline{*}(DLC)$	$(-0.173\,370\,930\,075,\ 0.772\,660\,051\,181)$	0.071 568 518 785 3	0.189 743 739 991
7	$(DLC)^{*(n-1)}\overline{*}(DLC)$	$(-0.173\,370\,898\,152,\ 0.772\,660\,116\,294)$	0.071 568 591 302 6	0.189 743 942 578
8	$(DLC)^{*(n-1)}\overline{*}(DLC)$	$(-0.173\,370\,897\,819,\ 0.772\,660\,116\,973)$	0.071 568 592 059 3	0.189 743 944 693
9	$(DLC)^{*(n-1)}\overline{*}(DLC)$	$(-0.173\,370\,897\,815,\ 0.772\,660\,116\,981)$	0.071 568 592 068 4	0.189 743 944 718

$$\langle\delta\rangle_n = \sum_{\sigma_a=0}^{1}\sum_{\sigma_b=0}^{1} p_n^{\sigma_a\sigma_b}\langle\delta^{\sigma_a\sigma_b}\rangle_n,$$
$$p_n^{\sigma_a\sigma_b} = \frac{N_n^{\sigma_a\sigma_b}}{N_n^{00}+N_n^{01}+N_n^{10}+N_n^{11}}, \tag{4.8b}$$

where $N_n^{\sigma_a\sigma_b}$ denotes the number of the $\sigma_a\sigma_b$-type convergent rates, and $\langle\delta^{\sigma_a\sigma_b}\rangle_n$ the average value of the $\sigma_a\sigma_b$-type convergent rates in $n-2$ local convergent rates of Z^{*n}. Thus the limit values $\langle\delta\rangle_\infty \equiv \lim_{n\to\infty}\langle\delta\rangle_n$ can reflect the irregular routes. It is obvious that for a pure up-star (or pure down-star) product, $\langle\delta\rangle_\infty$ returns to the value of $\overline{\delta}$ (or $\underline{\delta}$); this can be verified according to Cesàro summability, because $\delta_n^{00} \to \overline{\delta}$ (or $\delta_n^{11}\to\underline{\delta}$) hold for sufficiently large n values.

It should be indicated that $\langle\delta\rangle_n$ cannot distinguish two different routes with the same weights $p_n^{\sigma_a\sigma_b}$, $\sigma_a, \sigma_b \in \{0,1\}$. $\langle\delta\rangle_n$ is a characteristic quantity for the irregular routes, but it is not very satisfactory at present.

E. Grammatical complexity of patterns of dual-star products

Recently, the studies of grammatical complexity of symbolic dynamical systems of two letters in unimodal maps have received increasing attention [36,10]. Now for dynamical systems of three letters in bimodal maps, the grammatical complexity of their formal languages will greatly increase due to the existence of infinitely many patterns of dual-star products.

In symbolic dynamics of two letters of unimodal maps, the general periodic sequences (including primitive sequences and compound sequences generated by a finite number of DGP star compositions) with nonpositive Lyapunov exponents and the eventually periodic sequences with positive Lyapunov exponents belong to the simplest *regular language*, i.e., the type-3 language $\mathcal{L}_3$ (the lowest level) of the *Chomsky hierarchy* [35,36,42–44]. The Feigenbaum-type limit attractors are described by the infinite DGP star products $(WC)^{*\infty}$ (quasiperiodic sequences), whose Lyapunov exponents can be regarded as zero because they are the critical values from negative to positive ones; it is proven that their language is not the *context-free language* (CFL) or the type-2 language $\mathcal{L}_2$ of the Chomsky hierarchy; it is in fact an *extended tabled zero-sided Lindenmayer* (ET0L) language [36,45–47], and, consequently, a *context-sensitive language* (CSL) or the type-1 language $\mathcal{L}_1$ of the Chomsky hierarchy, which belongs to the proper subclass of the *indexed* (IND) languages, namely,

$$\mathcal{L}(\mathrm{ET0L}) \subsetneq \mathcal{L}(\mathrm{IND}) \subsetneq \mathcal{L}(\mathrm{CS}),$$

where $\mathcal{L}(\mathrm{CS})$ represents the language class of all the CSL's, and the meanings of $\mathcal{L}(\mathrm{ET0L})$ and $\mathcal{L}(\mathrm{IND})$ are similar. These results may be correct for the dynamical systems of three letters of bimodal maps, all the regular routes with metric universal scalings correspond to the type-3 and type-1 languages ($\mathcal{L}_3$ and $\mathcal{L}_1$).

However, the CFL (type-2 language $\mathcal{L}_2$) has not yet been found in admissible sequences of symbolic dynamics of unimodal maps [36], although the other three classes of languages of the Chomsky hierarchy, even the languages with a noncomputable complexity beyond the Chomsky hierarchy (such as the Bernoulli-Chaitin-Ford infinite sequences [48,49,31]), exist. It is conjectured that the type-2 language of the Chomsky hierarchy does not exist in the formal languages of unimodal maps [36].

In symbolic dynamics of three letters of bimodal maps, due to the appearances of up- and down-star products, the constructions of admissible sequences become enormously rich. The $\{0,1\}$ patterns of dual-star products $\{\overline{*},\underline{*}\}$ form a complete set, which cover all the infinitely many combinatorial types. Binary numbers corresponding to patterns also cover the real numbers on the interval $[0,1]$ which possess the cardinal number of the continuum. Thus we can construct or introduce a new class of languages for symbolic patterns of dual-star products. This class of abstract languages of symbolic patterns would cover all four classes of the Chomsky hierarchy, and perhaps even go beyond these four classes to reach the language of noncomputable random patterns. Among them, the abstract Dyck language can be easily produced. For instance, take the symbolic patterns σ of $\{0,1\}$, and let $\wp_\sigma$ be the complete set of all patterns; then for the pattern $\sigma\in\wp_\sigma$, one can construct the grammars $\sigma\to 0\sigma 0$ or $1\sigma 1$, $\sigma\to\sigma\sigma$, ... Such resulting languages are a class of Dyck languages of patterns. Therefore, a new way to link the abstract languages of patterns of dual-star products with formal languages $\mathcal{L}(\mathrm{KS})$ of admissible kneading sequences is possible. Is there a Dyck language for admissible sequences? This interesting problem is worth further discussion in the future.

V. GLOBAL REGULARITY OF FRACTAL DIMENSIONS FOR FEIGENBAUM-TYPE ATTRACTORS IN BIMODAL MAPS

In this section, we show a global regularity of fractal dimensions on critical points (accumulation points) of transitions to chaos for bimodal systems, which is a generalization of a global regularity of Feigenbaum-type attractors in unimodal maps found ten years ago [37]. We now give a brief review.

For unimodal maps, an orbit can be characterized by a U sequence (i.e., a MSS sequence) WC which is a superstable sequence of period $p=|WC|$. A Feigenbaum-type attractor refers to a period-p-tupling attractor formed on the accumulation point corresponding to the infinite DGP star product $(WC)^{*\infty}$. One knows that the dimension $d(WC)$ reflects the result of the self-similar orbital limit when WC becomes infinite $(WC)^{*\infty}$, and the scaling factor $\alpha(WC)$ describes the self-similarity of orbits of $(WC)^{*\infty}$. A global relationship between these two characteristic quantities with a high precision was found in Ref. [37] as

$$d(WC)\log_{|WC|}|\alpha(WC)|=\beta^{(1)}, \tag{5.1}$$

where $\beta^{(1)}$ is universal for all the infinitely many MSS sequences WC (or for all Feigenbaum-type attractors). For the quadratic map, its value is $\beta_c^{(1)}=0.71749$ for the capacity dimension d_c and $\beta_i^{(1)}=0.68436$ for the information dimension d_i, with the standard deviations $\sigma_{\beta_c^{(1)}}=0.00401$ and $\sigma_{\beta_i^{(1)}}=0.00227$ by using the least-squares method. It should be emphasized that Feigenbaum's universalities [such as the scaling factor $\alpha(WC)$ and the convergent rate $\delta(WC)$] are strongly dependent on the MSS sequences WC; while Eq. (5.1) is independent of the MSS sequences WC, it is a global superuniversality on the accumulation points in the one-dimensional unimodal Feigenbaum scenario.

For bimodal maps $f_{\lambda,\mu}(x)$, for a DSS sequence $Z=XDYC$ of period p, we have two accumulation points due to the existence of two bifurcation modes (the up-star $Z^{\overline{*}\infty}$ and down-star $Z^{\underline{*}\infty}$). Therefore, a DSS sequence can lead to two Feigenbaum-type (period-p-tupling) attractors formed on two accumulation points $(\overline{\lambda}_\infty,\overline{\mu}_\infty)$ and $(\underline{\lambda}_\infty,\underline{\mu}_\infty)$, respectively. To explore the global regularity of fractal dimensions, we should compute the capacity dimensions $d_c\in\{\overline{d}_c,\underline{d}_c\}$ and the information dimensions $d_i\in\{\overline{d}_i,\underline{d}_i\}$ of the period-p-tupling attractors. It is known that a period-p-tupling attractor forms a multiscale Cantor set that is not exactly self-similar. In the following, we shall briefly describe the geometric construction of the period-p-tupling attractor and the method for computing the fractal dimensions [50].

Consider a bimodal map f_∞ obtained as the bifurcation limit of p^n cycles. For the concrete cubic map $f_{r,s}(x)$ in Eq. (2.7), this period-p-tupling limit map is given by

$$x_{n+1}=f_\infty(x_n)\equiv r_\infty+s_\infty(4x^3-3x). \tag{5.2}$$

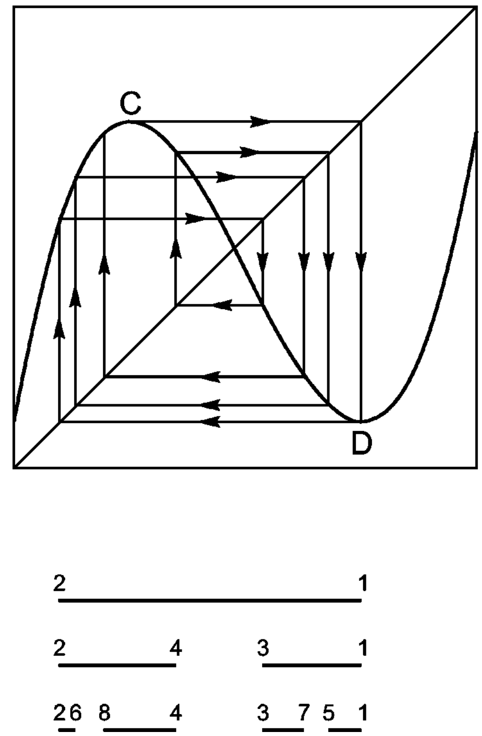

FIG. 2. The PDB limit map $f_\infty(x)$ of up-star product $(DC)^{\overline{*}\infty}$. The iterative images of the peak C exhibit the geometric construction of a period-doubling attractor.

For example, the parameter values of the PDB limit map corresponding to the pure up-star product $(DC)^{\overline{*}\infty}$ are $(\overline{r}_\infty,\overline{s}_\infty)=(-0.1504619259273784, 0.6504619259273784)$ (see Table V). For x lying in the dynamical invariant subinterval, $I_U=[f_\infty(x_D),f_\infty(x_C)]$, $f_\infty(x)$ will remain in I_U. For the limit map (5.2), $x_C=-0.5$, $x_D=0.5$. By following the itinerary of the peak C at x_C, we can obtain the points of the period-p-tupling attractor on arbitrary nth level: $x_1=f_\infty(x_C)$, $x_2=f_\infty^2(x_C)$, . . . ,$x_{2p^{n-1}}=f_\infty^{2p^{n-1}}(x_C)$. These $2p^{n-1}$ points form the end points of $N_n=p^{n-1}$ subintervals (point-clusters). The attractor is constructed on the nth level by removing all the line intervals outside these N_n subintervals from I_U. Such a procedure should be carried on *ad infinitum*, which is precisely the geometric construction of a multiscale Cantor set. We take the two adjacent end points as a minimal covering of the subinterval; then the length of the jth covering on the nth level would be [37,51]

$$l_{n;j}=|x_j-x_{j+p^{n-1}}|. \tag{5.3}$$

We can also construct the attractor by following the itinerary of the valley D at x_D: $x_1'=f_\infty(x_D)$, $x_2'=f_\infty^2(x_D)$, . . . ,$x'_{2p^{n-1}}=f_\infty^{2p^{n-1}}(x_D)$. The jth covering on the nth level still has the similar form, $l'_{n;j}=|x'_j-x'_{j+p^{n-1}}|$. We can take any one of these two constructions because they are equivalent under the limit $n\rightarrow\infty$. In Fig. 2 we show the PDB limit map and the geometric construction of the period-doubling attractor corresponding to $(DC)^{\overline{*}\infty}$.

It is easy to compute the capacity dimensions d_c and the information dimensions d_i of the period-p-tupling attractors according to Refs. [50,9]. Let $L=f_\infty(x_C)-f_\infty(x_D)$ be a

TABLE V. The parameter values of accumulation points $(\overline{r_\infty},\overline{s_\infty})$ of up-star products $Z^{\overline{*}n}$ of basic period $p=2-4$ for the cubic map $f_{r,s}(x)$.

p	Sequence Z	$(\overline{r_\infty},\overline{s_\infty})$
2	DC	$(-0.150\,461\,925\,927\,378\,4,\ 0.650\,461\,925\,927\,378\,4)$
3	DLC	$(-0.226\,152\,433\,702\,457\,6,\ 0.726\,152\,433\,702\,457\,6)$
3	RDC	$(0.173\,308\,249\,343\,404\,2,\ 0.772\,725\ \ 845\,666\,879\,1)$
4	$DLLC$	$(-0.245\,654\,715\,129\,384\,2,\ 0.745\,654\,715\,129\,384\,2)$
4	$RDLC$	$(-0.001\,109\,762\,767\,872\,1,\ 0.947\,354\,406\,614\,981\,8)$
4	$DLMC$	$(-0.150\,461\,925\,927\,378\,4,\ 0.650\,461\,925\,927\,378\,4)$
4	$RMDC$	$(0.094\,799\,379\,455\,713\,3,\ 0.700\,778\,839\,715\,518\,2)$
4	$RRDC$	$(0.230\,322\,376\,337\,138\,9,\ 0.760\,918\,525\,589\,362\,0)$

renormal scale; we can convert the dynamical invariant interval I_U to the interval $[0,1]$. The lengths of the N_n subintervals on the nth level would become $L_{n;j}=l_{n;j}/L$. The capacity dimension d_c is determined by Newton's method from the sum rule

$$\sum_{j=1}^{N_n} L_{n;j}^{d_c}=1. \tag{5.4}$$

The information dimension d_i is given by

$$d_i=\frac{\sum_{j=1}^{N_n} P_{n;j}\ln P_{n;j}}{\sum_{j=1}^{N_n} P_{n;j}\ln L_{n;j}}, \tag{5.5}$$

where $P_{n;j}$ is the relative probability of the attractor in the subinterval $L_{n;j}$, and we assume that each of the N_n subintervals has the same relative probability $P_{n;j}=1/N_n$. Theoretically, to find the ''exact'' values of d_c and d_i, one should use Eqs. (5.4) and (5.5) with the limit $n\to\infty$, but this is impossible in an actual calculation. However, as indicated in Ref. [50], we can take $d_{c,i}$ as the expression $nd_{c,i}(n)-(n-1)d_{c,i}(n-1)$ which converges to $d_{c,i}(\infty)$ very rapidly in a finite n. In this way we have computed the values of d_c and d_i of period-p-tupling attractors with $p=2$–4. These values also have the dual symmetry

$$\overline{d}_c(Z)=\underline{d}_c(\tilde{Z}^T), \quad \overline{d}_i(Z)=\underline{d}_i(\tilde{Z}^T), \tag{5.6}$$

so we only list the values of $\overline{d}_c$ and $\overline{d}_i$ in Table VI.

We now can generalize the global regularity of fractal dimensions of the unimodal case to the bimodal one. By numerical calculation shown in Tables V and VI, we find that the following global relation works very well:

$$d_{c,i}(Z)\log_{|Z|}|\alpha_C(Z)\alpha_D(Z)|=\beta_{c,i}^{(2)}, \tag{5.7}$$

where $\beta_c^{(2)}=1.4339$ and $\beta_i^{(2)}=1.2945$ are universal for all DSS sequences $Z=XDYC$ (or for all period-p-tupling attractors), and $\beta_{c,i}^{(2)}$ are the same for either the up accumulation $Z^{\overline{*}\infty}$ or the down accumulation $Z^{\underline{*}\infty}$ due to the dual symmetry of Eqs. (5.6) and (3.4), i.e., $\beta_{c,i}^{(2)}=\overline{\beta}_{c,i}^{(2)}=\underline{\beta}_{c,i}^{(2)}$. In comparison with the unimodal case, here in Eq. (5.7) the contributions of both scaling factors of two turning points C and D have been included. Furthermore, if an *equivalent* scaling factor α_e for bimodal maps is defined by

$$[\alpha_e(Z)]^2=|\alpha_C(Z)\alpha_D(Z)|, \tag{5.8}$$

TABLE VI. Fractal dimensions $\overline{d}_{c,i}$ and global constants $\beta_{c,i}^{(2)}$ for period-p-tupling attractors of basic period $p=2-4$ for the cubic map. The constants $\beta_{c,i}^{(2)}$ and the standard deviations $\sigma_{\beta_{c,i}^{(2)}}$ at the bottom are found by using the least-squares method.

p	Sequence Z	$\overline{d}_c(Z)$	$\overline{d}_i(Z)$	$\overline{\alpha}_C(Z)$	$\overline{\alpha}_D(Z)$	$\beta_c^{(2)}(Z)$	$\beta_i^{(2)}(Z)$
2	DC	0.6427	0.5544	−1.6903	2.8571	1.4600	1.2595
3	DLC	0.4732	0.4031	−3.1522	9.9361	1.4834	1.2638
3	RDC	0.5186	0.4942	−3.7018	5.2711	1.4024	1.3365
4	$DLLC$	0.3768	0.3168	−6.1918	38.338	1.4868	1.2498
4	$RDLC$	0.3602	0.3331	−9.0797	27.168	1.4312	1.3235
4	$DLMC$	0.6427	0.5544	2.8571	8.1632	1.4600	1.2595
4	$RMDC$	0.6140	0.5873	−4.8627	−4.8627	1.4010	1.3402
4	$RRDC$	0.5019	0.4813	−6.2185	7.1805	1.3753	1.3189

$\beta_c^{(2)}=1.4339$ $\quad\sigma_{\beta_c^{(2)}}=0.0146$

$\beta_i^{(2)}=1.2945$ $\quad\sigma_{\beta_i^{(2)}}=0.0145$

then the global regularity (5.7) of bimodal maps can even be reduced to form (5.1) of unimodal ones, namely,

$$d_{c,i}(Z)\log_{|Z|}|\alpha_e(Z)| = \beta_{c,i}^{(e)}, \tag{5.9}$$

where the *equivalent* global constants are $\beta_c^{(e)} = 0.7169$ and $\beta_i^{(e)} = 0.6473$, respectively, and we find that they are approximately equal to $\beta_{c,i}^{(1)}$: $\beta_c^{(e)} \approx \beta_c^{(1)}$ within an accuracy of 0.08%, and $\beta_i^{(e)} \approx \beta_i^{(1)}$ within an accuracy of 5%. This result may imply that the global regularity (5.1) of unimodal systems may be a rather general form which may hold for a wide range of systems, for instance, for trimodal, multimodal, or even discontinuous systems.

VI. DISCUSSION

From the above we have seen that dual-star products play a key role in the study of universalities in symbolic dynamics of three letters. The metric universalities are related to the renormalization. So the renormalization scheme should be able to be carried out. One can expect that the dual renormalization group equations associated with operation $* \in \{\overline{*}, \underline{*}\}$ can be obtained. We have also noted that the theoretical frame of symbolic dynamics of three letters is of practical significance to other physical systems described by such maps as circle maps, Lorenz maps, etc., because such discontinuous maps can be regarded as the breaking or pruning of the continuous bimodal maps. The applications of star products to these systems will be important in physics [52].

It is worthy to indicate that the symbolic dynamics of two letters in unimodal maps is rather simple in comparison with that of three letters in bimodal maps. There is only one kind of star product (i.e., a DGP star product) for unimodal maps, but there are two kinds of dual star products for bimodal maps. Some elementary studies for trimodal maps show that the generalization of star products to trimodal or multimodal maps would be a complicated but accessible problem. In addition, there would be a rapid growth in the kinds of star products when the number of turning points (or parameters) of the map increases [53]. This increase in the kinds of star products will enrich the routes to chaos in trimodal or multimodal maps, and result in higher degrees of complexity.

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