Fractal behavior in quantum statistical physics

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The properties of an ideal gas of spinless particles are investigated by using the path integral formalism. It is shown that the quantum paths exhibit a fractal character which remains unchanged in the relativistic domain provided the creation of new particles is avoided, and the Brownian motion remains the stochastic process associated with the quantum paths. These results are obtained by using a special representation of the Klein-Gordon wave equation. On the quantum paths the relation between velocity and momentum is not the usual one. The mean square value of the velocity depends on the time needed to define the velocity and its value shows the interplay between pure quantum effects and thermodynamics. The fractal character is also investigated starting from wave equations by analyzing the evolution of a Gaussian wave packet via the Hausdorff dimension. Both approaches give the same fractal character in the same limit. It is shown that the time that appears in the path integral behaves like an ordinary time, and the key quantity is the time interval needed for the thermostat to give to the particles a thermal action equal to the quantum of action. Thus, the partition function calculated via the path integral formalism also describes the dynamics of the system for short time intervals. For low temperatures, it is shown that a time-energy uncertainty relation is verified at the end of the calculations. The energy involved in this relation has not a thermodynamic meaning but results from the fact that the particles do not follow the equations of motion along the paths. The results suggest that the density matrix obtained by quantification of the classical canonical distribution function via the path integral formalism should not be totally identical to that obtained via the usual route. [S1063-651X(99)13608-0]

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I. INTRODUCTION

In recent years fractal geometry has been used in a vast area of knowledge [1]. The concept of fractals has proven to be very useful in simulating irregular structures that we can assume to be exactly, or at least statistically, self-similar. In mathematics, fractals are associated with the existence of curves that are nowhere differentiable. In this paper we would like to show that the concept of fractals is unavoidable in quantum statistical physics provided we use the path integral formalism; then the fractal character of the paths appears as a consequence of the Planck constant. The seminal work in this field was published more than 30 years ago by Feynman and Hibbs [2]. They have shown that the quantum paths exhibit a fractal character, although the concept of fractal was not introduced at that time. The main goal of this paper is to investigate some properties of the paths which appear in the calculation of the partition function Z. As a consequence, we shall see that the simplest system that we can investigate, i.e., the ideal gas, may lead to a nontraditional point of view in statistical physics. Moreover, in this case all the calculations can be performed analytically, and accordingly, we can consider all the results as exact.

The paper is organized as follows. First, in Sec. II we characterize the paths in nonrelativistic quantum mechanics by considering the statistical average of the velocity and its exact relation to the mean value of the momenta. Then, when focusing on short time intervals, we will see that we have to extend our approach to the relativistic domain; this will be done in Sec. III. In what follows, we only investigate the introduction of relativistic dynamics staying in the one-particle formalism. To have a consistent approach in this domain we have to use a particular representation of the Klein-Gordon wave equation. This will lead us to redefine the partition function in this representation. Then the paths are analyzed as in the nonrelativistic case. The physical meaning of these results is discussed in Sec. IV. In particular, (i) we compare our results with those obtained in the canonical approach via the Hausdorff dimension and (ii) the time-energy uncertainty relation is investigated. In the last section the main conclusions are presented.

II. STATISTICAL MECHANICS IN THE NONRELATIVISTIC DOMAIN

In the path integral formalism and in nonrelativistic quantum mechanics the partition function, Z, can be written [3]

$$Z=\frac{1}{(2\pi\hbar)^d}\int D\mathbf{x}(t)\int D\mathbf{p}(t)\exp\left[-\frac{1}{\hbar}A\{\mathbf{x}(t),\mathbf{p}(t)\}\right], \tag{1}$$

where d is the dimension of space, $\mathbf{x}(t)$ and $\mathbf{p}(t)$ are the position and momentum vectors at time t on a given trajectory, and the symbols $D\mathbf{x}(t)$ and $D\mathbf{p}(t)$ mean that we have to perform a functional integration. In Eq. (1), $A\{\mathbf{x}(t),\mathbf{p}(t)\}$ is the Euclidean action [3],

$$A\{\mathbf{x}(t),\mathbf{p}(t)\}=\int_0^{\beta\hbar}dt\left[-i\mathbf{p}(t)\dot{\mathbf{x}}(t)+\frac{\mathbf{p}^2(t)}{2m}+V(\mathbf{x}(t))\right], \tag{2}$$

in which m is the mass and β is the reverse of the absolute temperature $T,\beta=1/(k_BT)$. In the calculation of Z we only have to consider the cyclic paths for which $\mathbf{x}(0)=\mathbf{x}(\beta\hbar)$; this has to be associated with the fact that the partition function in standard quantum physics is only determined by the trace of the density matrix. In principle, in Eq. (1) there is an

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extra term related to the fact that we consider a statistics involving bosons. In what follows, we have dropped this term, which is irrelevant in this work.

By using a time slicing procedure [3] the integral in equation Eq. (2) is transformed into a Riemann sum and the integrals in Eq. (1) acquire a meaning in the limit $t_{n+1}-t_n=\Delta t\rightarrow 0$, where t_n and t_{n+1} are two successive values of t such as $0<t_n<t_{n+1}<\beta\hbar$. In addition to the position and momentum variables, a third quantity $\dot{\mathbf{x}}(t)$ appears in Eq. (2), which is defined according to $\dot{\mathbf{x}}(t)=[\mathbf{x}(t_{n+1})-\mathbf{x}(t_n)]/\Delta t$. This quantity looks like a velocity, but it is important to note that $\dot{\mathbf{x}}(t_n)$ is totally disconnected from the set of values $\{\mathbf{p}(t_i)\}$ taken by the momentum. Accordingly, on the quantum paths, no simple relation is expected between velocity and momentum. If we only consider some properties related to the momentum, we can perform the integration over the positions $\mathbf{x}(t)$. Then the functional integral leads to a Riemann sum and we get the well known form of the partition function,

$$Z=\frac{V}{(2\pi\hbar)^d}\int d\mathbf{p}\exp\left[-\beta\frac{\mathbf{p}^2}{2m}\right], \tag{3}$$

where V is the volume of the sample. According to Eq. (3), any function of the momentum will have the same value as in the classical case, in particular $\langle\mathbf{p}^2\rangle/2m=d/(2\beta)$. However, the path integral formalism via Eq. (1) leads to a more detailed description of the quantum behavior than just the introduction of the volume $(2\pi\hbar)^d$ in phase space, which appears in Eq. (3). This can be illustrated by considering the velocity on the path.

We define the velocity as the change of position $\delta\mathbf{x}$ corresponding to a given finite time interval δt provided $\delta t\leqslant\beta\hbar$. For a free particle $\langle\delta\mathbf{x}/\delta t\rangle$ vanishes due to the symmetry of space, and hereafter we focus on $\langle(\delta\mathbf{x}/\delta t)^2\rangle$. This quantity can be calculated by using the propagator $K(\mathbf{x}_b-\mathbf{x}_a;t_b-t_a)$ connecting two points $\mathbf{x}_a$ and $\mathbf{x}_b$, which are associated with two different times t_a and t_b. In real space, we have [3]

$$K(\mathbf{x}_b-\mathbf{x}_a;t_b-t_a)=\left(\frac{m}{2\pi\hbar(t_b-t_a)}\right)^{d/2}\exp\left[-\frac{m(\mathbf{x}_b-\mathbf{x}_a)^2}{2\hbar(t_b-t_a)}\right], \tag{4}$$

or in momentum representation

$$K(\mathbf{x}_b-\mathbf{x}_a;t_b-t_a)=\frac{1}{(2\pi\hbar)^d}\int d\mathbf{p}\exp\left[-\frac{\mathbf{p}^2(t_b-t_a)}{2mh}+\frac{i}{\hbar}\mathbf{p}(\mathbf{x}_b-\mathbf{x}_a)\right]. \tag{5}$$

This propagator corresponds to the transition amplitude of a free particle for an imaginary time; it verifies the composition law

$$K(\mathbf{x}_b-\mathbf{x}_a;t_b-t_a)=\int K(\mathbf{x}_b-\mathbf{x}_c;t_b-t_c)\times K(\mathbf{x}_c-\mathbf{x}_a;t_c-t_a)d\mathbf{x}_c, \tag{6}$$

and Z is given by $Z=VK(0;\beta\hbar)$. Now we can calculate

$$\left\langle\left(\frac{\delta\mathbf{x}}{\delta t}\right)^2\right\rangle=\frac{1}{Z}\int d\mathbf{x}_a d\mathbf{x}_b d\,\delta x K(\mathbf{x}_b-\mathbf{x}_a;t_b-0)\,K(\delta\mathbf{x};\delta t)\times\left(\frac{\delta\mathbf{x}}{\delta t}\right)^2K(\mathbf{x}_b-\mathbf{x}_a+\delta\mathbf{x};\beta\hbar-t_b+\delta t), \tag{7}$$

from which we can easily derive the following result:

$$\left\langle\left(\frac{\delta\mathbf{x}}{\delta t}\right)^2\right\rangle=\left[\frac{\beta\hbar}{\delta t}-1\right]\frac{\langle\mathbf{p}^2\rangle}{m^2}=\left[\frac{\beta\hbar}{\delta t}-1\right]\frac{d}{\beta m}, \tag{8}$$

which shows the relation between $\langle(\delta\mathbf{x}/\delta t)^2\rangle$ and $\langle\mathbf{p}^2\rangle$.

We can also characterize the path by studying the thermal average of the change of velocity at a given point $(\mathbf{x}_2,t_2)$ of the path. In order to do that, we consider the quantity $\langle[(\mathbf{x}_2-\mathbf{x}_1)/\delta t][(\mathbf{x}_3-\mathbf{x}_2)/\delta t]\rangle$ in which the positions $\mathbf{x}_1,\mathbf{x}_2$, and $\mathbf{x}_3$ are taken at times t_1,t_2, and t_3 respectively, and we must have $0<t_1<t_2<t_3<\beta\hbar$. We can write the average $\langle[(\mathbf{x}_2-\mathbf{x}_1)/\delta t][(\mathbf{x}_3-\mathbf{x}_2)/\delta t]\rangle$ in a form similar to Eq. (7) and by using Eq. (5) we derive the following exact result:

$$\left\langle\left[\frac{\mathbf{x}_2-\mathbf{x}_1}{\delta t}\right]\left[\frac{\mathbf{x}_3-\mathbf{x}_2}{\delta t}\right]\right\rangle=-\frac{\langle\mathbf{p}^2\rangle}{m^2}=-\frac{d}{\beta m}, \tag{9}$$

which shows that the path is not differentiable at any time, whatever the value of δt, provided $\delta t<\beta\hbar$. The physical content of Eqs. (8) and (9) will be discussed after their generalization to the relativistic case. However, we can note that there are two kinds of properties on the path; some of them may depend on δt as $\langle(\delta\mathbf{x}/\delta t)^2\rangle$ while others, such as $\langle[(\mathbf{x}_2-\mathbf{x}_1)/\delta t][(\mathbf{x}_3-\mathbf{x}_2)/\delta t]\rangle$, are independent of δt.

The relation (8) shows that $\langle(\delta\mathbf{x}/\delta t)^2\rangle$ grows indefinitely when δt goes to zero, but from the theory of relativity we expect that $[\langle(\delta\mathbf{x}/\delta t)^2\rangle]^{1/2}$ must be smaller than the velocity of light, c. Thus, we have to restart our calculations in the scheme of special relativity.

III. STATISTICAL MECHANICS IN THE RELATIVISTIC DOMAIN

The special relativity introduces two modifications to the previous approach, We have to (i) change the Hamiltonian by introducing the relativistic dynamics and (ii) take into account that new particles can be created. In what follows we will stay in the one-particle formalism and only the first modification will be considered. Among the recent works devoted to the path integral formalism for relativistic particles, to our knowledge there is no paper in which the thermodynamics is included [4,5].

For spinless particles, the Klein-Gordon (KG) wave equation is an acceptable starting point for a relativistic approach but the definition of operators in the one-particle formalism is a nontrivial task (see, for instance, [6–8]). Moreover, as a consequence of the restriction on the states of positive energy, it is no longer possible to use a δ function in order to represent localized states; these states have a spatial extension determined by the Compton wave length $\lambda=\hbar/mc$, where m is the rest mass. In order to define a meaningful position operator, first we put the KG equation in a Schrödinger form. The price that we have to pay in such a transformation is that the wave function ψ is now a two-component vector for which the scalar product is defined via a Pauli matrix by [7,8]

$$\langle \psi | \psi' \rangle = \int d\mathbf{x} \psi^\dagger \tau_3 \psi', \quad \text{where} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (10)$$

The superscript † means the Hermitian conjugate. Second, we consider the Feshbach-Villars (FV) representation [7], which transforms ψ into φ according to $\varphi = \hat{U}\psi$, where $\hat{U}$ is not a 2×2 unitary matrix in the usual sense since we have $\hat{U}^{-1} = \tau_3 \hat{U}^\dagger \tau_3$, where $\hat{U}^\dagger$ is the Hermitian conjugate of $\hat{U}$. In this representation, the position operator is defined as usual and its eigenstates of positive and negative energy are given by

$$\varphi_x^+(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{d/2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar} \mathbf{p}\cdot\mathbf{x} \right),$$

$$\varphi_x^-(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{d/2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\left(\frac{i}{\hbar} \mathbf{p}\cdot\mathbf{x} \right), \quad (11)$$

In this φ representation we can define the so-called even operators for which there is no mixing between states of positive and negative energy. The scalar product defined in Eq. (10) keeps the same form as the φ representation. More generally we define the expectation $\langle A \rangle$ of an operator A according to [7,8]

$$\langle A \rangle = \int d\mathbf{x} \psi^\dagger \tau_3 A \psi = \int d\mathbf{x} \varphi^\dagger \tau_3 A_\varphi \varphi, \quad (12)$$

in which $A_\varphi = \hat{U} A \hat{U}^{-1}$. The second relation takes into account the fact that any operator associated with a real dynamical variable must be only pseudo-Hermitian, since we must have $A = \tau_3 A^\dagger \tau_3$ [8].

In the relativistic domain, by analogy with Eq. (12), we suggest defining the partition function according to

$$Z = \int d\mathbf{p} d\mathbf{x} \psi_x^\dagger(\mathbf{p}) \tau_3 \exp[-\beta H_\psi(\mathbf{p})] \psi_x(\mathbf{p}) \quad (13)$$

in which H_ψ is a 2×2 matrix that contains a combination of the momentum operators and mc^2 via the Pauli matrices. In FV representation we have

$$Z = \int d\mathbf{p} d\mathbf{x} \varphi_x^\dagger(\mathbf{p}) \tau_3 \exp[-\beta H_\varphi(\mathbf{p})] \varphi_x(\mathbf{p}) \quad (14)$$

in which the Hamiltonian is a 2×2 matrix given by

$$\beta H_\varphi(\mathbf{p}) = \tau_3 \sigma \sqrt{1 + \left(\frac{\mathbf{p}}{mc} \right)^2}, \quad \sigma = \beta m c^2$$

and $\varphi_x(\mathbf{p})$ corresponds to $\varphi_x^+(\mathbf{p})$ given by Eq. (11). By using the properties of the matrix τ_3, it is easy to see that Z is given by

$$Z = \frac{V}{(2\pi\hbar)^d} \int d\mathbf{p} \exp\left[-\sigma \left(\sqrt{1 + \left(\frac{\mathbf{p}}{mc} \right)^2} \right) \right], \quad (15)$$

which is a natural extension of Eq. (3). In the FV representation we define a propagator according to

$$K(\mathbf{x}_b - \mathbf{x}_a; t_b - t_a) = \int d\mathbf{p} \varphi_{xa}^\dagger(\mathbf{p}) \tau_3 \times \exp[-(t_b - t_a) H_\varphi(\mathbf{p})/\hbar] \varphi_{xb}(\mathbf{p}). \quad (16)$$

By using the explicit expression of $\varphi_x(\mathbf{p})$ and the properties of τ_3, we can write Eq. (16) as

$$K(\mathbf{x}_b - \mathbf{x}_a; t_b - t_a) = \int \frac{d\mathbf{p}}{E_p} \frac{1}{(2\pi\hbar)^{d/2}} e^{i\mathbf{p}\cdot\mathbf{x}_b/\hbar} E_p^{1/2} \frac{1}{(2\pi\hbar)^{d/2}} \times e^{i\mathbf{p}\cdot\mathbf{x}_a/\hbar} E_p^{1/2} e^{-(t_b - t_a)E_p/\hbar}, \quad (17)$$

where $E_p = [\mathbf{p}^2 c^2 + m^2 c^4]^{1/2}$ and we have isolated the Lorentz invariant measure $[d\mathbf{p}/E_p]$. In Eq. (17), the quantity $[E_p^{1/2}/(2\pi\hbar)^{d/2}] e^{i\mathbf{p}\cdot\mathbf{x}_b/\hbar}$ is, in $\mathbf{p}$ representation, the exact form for a state localized at $\mathbf{x}_b$ for $t=0$ in the relativistic domain; this result has been derived by Newton and Wigner from first principle arguments [6]. This propagator verifies the composition rule (6) and we have $Z = VK(0;\beta\hbar)$. In terms of the dimensionless quantities $\mathbf{u} = \mathbf{p}/(mc)$; $\tau = t/(\beta\hbar)$; $\mathbf{r} = \mathbf{x}/\lambda$. $K(\mathbf{x}_b - \mathbf{x}_a; t_b - t_a)$ becomes $K(\mathbf{r}_b - \mathbf{r}_a; \tau_b - \tau_a)$ and takes the explicit form in the case $d=1$

$$K(\mathbf{r}_b - \mathbf{r}_a; \tau_b - \tau_a) = \frac{1}{\pi\lambda} \frac{\sigma(\tau_b - \tau_a)}{\sqrt{\sigma^2(\tau_b - \tau_a)^2 + (\mathbf{r}_b - \mathbf{r}_a)^2}} \times K_1[\sqrt{\sigma^2(\tau_b - \tau_a)^2 + (\mathbf{r}_b - \mathbf{r}_a)^2}], \quad (18)$$

where $K_1(z)$ is a modified Bessel function. A similar result is obtained in the case $d=3$. Starting from this propagator and after some straightforward integrations we get the following exact results:

$$\left\langle \left(\frac{\delta\mathbf{x}}{\delta t} \right)^2 \right\rangle = \left[\frac{\beta\hbar}{\delta t} - 1 \right] \left\langle \frac{\mathbf{p}^2 c^4}{\mathbf{p}^2 c^2 + m^2 c^4} \right\rangle \quad (19)$$

and

$$\left\langle \frac{\mathbf{x}_2 - \mathbf{x}_1}{\delta t} . \frac{\mathbf{x}_3 - \mathbf{x}_2}{\delta t} \right\rangle = -\left\langle \frac{\mathbf{p}^2 c^4}{\mathbf{p}^2 c^2 + m^2 c^4} \right\rangle. \quad (20)$$

Equations (19) and (20) represent the generalization of Eqs. (8) and (9) to the relativistic case. The relation (19) shows the exact relation between the mean squared value of the velocity $\langle (\delta\mathbf{x}/\delta t)^2 \rangle$ and $\langle \mathbf{p}^2 c^4 / (\mathbf{p}^2 c^2 + m^2 c^4) \rangle$, which is the average of the relativistic expression of the square of the velocity in terms of momentum in the classical domain.

IV. ANALYSIS OF THE RESULTS

The thermal average of quantities that are only related to the momentum can be calculated by using Eq. (3) or its relativistic extension (15). Using these partition functions at the equilibrium we will get the classical thermodynamic values associated with these quantities. However, the path integral formalism gives us more informations, in particular on the dynamics at short time intervals, i.e., for $\delta t \leqslant \beta\hbar$. This is not very surprising since, by construction, the path integral is related to the motion on paths. However, in order to conclude something about the dynamics we must consider the time on the path as ordinary time and not just as a purely mathematical quantity that has the same dimension as time. This point will be analyzed below. For quantities related to the position, Eqs. (19) and (20) show that they may or may not depend on the time interval δt.

A. Special values of δt

When $\delta t \ll \beta\hbar$ and $d=1$, we get, from Eq. (19),

$$\left\langle \left(\frac{\delta \mathbf{x}}{\delta t}\right)^2 \right\rangle \sim \frac{\hbar}{m\,\delta t}\,\frac{\int d\mathbf{u}(\mathbf{u}^2+1)^{-3/2}\exp[-\sigma(\mathbf{u}^2+1)^{1/2}]}{\int d\mathbf{u}\exp[-\sigma(\mathbf{u}^2+1)^{1/2}]}. \tag{21}$$

A similar result is obtained for $d=3$, only the ratio of integrals on the right hand side (rhs) of Eq. (21) is different. In any case, the ratio of the two integrals is smaller than 1, leading to $\langle(\delta \mathbf{x}/\delta t)^2\rangle/c^2 = \hbar/(mc^2\delta t) f(\sigma)$, where $f(\sigma)$ is approximately 1 for large values of σ. Thus, for $T=0$ we get $\langle(\delta \mathbf{x})^2\rangle = (\hbar/m)\,\delta t$ without any restriction about δt since $\beta\hbar$ is now infinite. This result shows that a Brownian motion can be associated with the quantum path. The effective diffusion coefficient corresponding to this motion, $(\hbar/m)$, is due to the existence of the Planck constant. Note that a Brownian motion is frequently associated with the Schrödinger equation [9]. However, in order to establish this association a purely formal process is invoked. We perform an analytic continuation of this equation by considering an imaginary time; accordingly, such an association although well accepted is not so clear [10]. Note that $\langle(\delta \mathbf{x})^2\rangle = (\hbar/m)\,\delta t$ is the starting point of the Newton approach of the Schrödinger equation [11]. In what follows, by investigating the Hausdorff dimension we will try to introduce some extra arguments that, although not definitive, support this association.

In the relativistic domain we have the same result provided the time interval δt is such that $\langle(\delta \mathbf{x}/\delta t)^2\rangle/c^2$ is smaller than 1; this is verified if $\delta t \geqslant \hbar/mc^2$ as expected since we are in a regime with a fixed number of particles. For smaller time intervals the uncertainty relation $\delta t\,\delta E = \hbar$ shows that we could create new particles. Thus δt must be large enough to avoid the creation of particles by quantum fluctuations. This constraint on δt implies that we must have $\beta\hbar \geqslant \delta t \geqslant \hbar/mc^2$, leading to $\sigma = \beta mc^2 \geqslant 1$, which means that no particle has to be created by thermal excitation. Thus the relation (19) is exact in the one-particle formalism. For the smallest time interval $\delta t = \hbar/(mc^2)$ we get the smallest value of $\langle(\delta \mathbf{x})^2\rangle$ that we can investigate in the one-particle formalism; it corresponds to

$$\langle(\delta \mathbf{x})^2\rangle = \lambda^2(\beta mc^2-1)\left\langle \frac{\mathbf{p}^2c^2}{\mathbf{p}^2c^2+m^2c^4}\right\rangle, \tag{22}$$

which shows the interplay between the Compton wavelength, λ, and the properties of the thermostat. At very low temperatures $\langle(\delta \mathbf{x})^2\rangle$ tends to λ^2, as expected. We can note that Eq. (19) does not lead to the pure relativistic regime for which we may expect $\langle(\delta \mathbf{x})^2\rangle \sim (c\,\delta t)^2$. For higher values of δt we can see the influence of the thermostat.

It is noteworthy that the time interval $\delta t = \beta\hbar/2$ plays a special role. For this δt we focus on the simplest and the most symmetrical paths. They are formed by two parts and with each of them we can associate the same time interval. Then Eq. (19) shows that we recover, for the thermal average, the classical relation between the velocity and the momentum in the relativistic domain. In addition, $\langle(\delta \mathbf{x})^2\rangle$ gets its largest value, which is

$$\langle(\delta \mathbf{x})^2\rangle = \left(\frac{\beta\hbar}{2}\right)^2\left\langle \frac{\mathbf{p}^2c^4}{\mathbf{p}^2c^2+m^2c^4}\right\rangle. \tag{23}$$

For large values of σ this quantity is the square of the thermal de Broglie wavelength $\Lambda = 2\pi\beta\hbar^2/m$. Thus with the path integral formalism, at the thermal equilibrium, we explore some distances larger than λ but smaller than Λ; this is also an expected result.

In contrast with Eq. (19), the rhs of Eq. (20) does not contain δt and the result is exact provided that $\delta t \leqslant \beta\hbar$. From Eq. (20) we can see that

$$\frac{\left|\left\langle \dfrac{\mathbf{x}_2-\mathbf{x}_1}{\delta t}\,\dfrac{\mathbf{x}_3-\mathbf{x}_2}{\delta t}\right\rangle\right|}{c^2} < 1. \tag{24}$$

Equation (20) shows that the paths are not differentiable. The change of velocity at any point $(\mathbf{x},t)$ is only determined by m and σ. It the temperature goes to zero the rhs of Eq. (20) goes to zero and the velocities before and after $(\mathbf{x},t)$ are uncorrelated, in thermal average. For a given value of T, the thermostat tends to reverse the direction of the velocity at any time. This result is quite clear if we consider the particular time interval $\delta t = \beta\hbar/2$. In this case we have $\mathbf{x}_3 = \mathbf{x}_1$ and Eq. (20) can be immediately deduced from Eq. (19). In this case Eq. (20) is simply a consequence of Eq. (19) and of the cyclic character of the path which is due to the presence of the thermostat. In fact, the role of the thermostat is clearly to localize the particle in a volume which has Λ for a radius.

B. Brownian motion

The results (19) and (20) are exact; they seem rather meaningful, as well as their consequences, but, in fact, they are new and not trivial. Comparison of Eq. (19) and (20) with (8) and (9) shows that the relativity does not change the stochastic process associated with the quantum path provided we stay in the one-particle formalism. Except for some obvious modifications we recover in the relativistic case all the properties of the Brownian motion. At first glance this could appear as strange, since the KG equation contains a second derivative relative to t that does not exist in the Schrödinger equation. In [9] it is claimed that a stochastic process probably different from the Wiener process has to be associated with the quantum paths in the case of special relativity. Gaveau *et al.* [5] have considered a stochastic process connected with the telegraphers equation. The origin of the difference between our approach and the one developed by these authors is due to the fact that we have put the KG equation in its FV representation in which the operators have a simple meaning. In particular, this FV representation is needed in order to define the so-called even operators, which transform a state of positive energy into another state of positive energy.

Recently, Kleinert [3] introduced a propagator associated with the KG equation. This propagator can be put in a form similar to Eq. (17). For $d=1$ it is given by

$$K'(\mathbf{x}_b-\mathbf{x}_a;t_b-t_a)=\int\frac{d\mathbf{p}}{E_p}\frac{1}{(2\pi\hbar)^{-d/2}}\times e^{i\mathbf{p}\cdot\mathbf{x}_b/\hbar}\frac{1}{(2\pi\hbar)^{-d/2}}\times e^{i\mathbf{p}\cdot\mathbf{x}_a/\hbar}\,e^{-(t_b-t_a)E_p/\hbar}, \quad (25)$$

which we can also write as

$$K'(\mathbf{r}_b-\mathbf{r}_a;\tau_b-\tau_a)=\frac{1}{2\pi\lambda}K_0[\sqrt{\sigma^2(\tau_b-\tau_a)^2+(\mathbf{r}_b-\mathbf{r}_a)^2}], \quad (26)$$

which is clearly in disagreement with our result (18). The origin of the discrepancy is related to the fact that in Eq. (25) we assume that the localized states correspond to a δ function in real space in opposition to what has been established in [6]. The use of Eq. (26) instead of Eq. (18) leads to results which are without any physical meaning.

C. Hausdorff dimension

Until this point we have associated quantum paths and Brownian motion, but we can also investigate the fractal character of the quantum world starting from the time dependent Schrödinger equation or from the KG equation. In this case we are at $T=0$ and the time is the ordinary time. From the wave equations we can study how $\langle(\delta\mathbf{x})^2\rangle$ depends on a given initial spatial resolution, δ, for a particle located near the origin, $\mathbf{x}=0$, at $t=0$. In the nonrelativistic domain, we define δ as the width of a Gaussian wave packet. After a given time interval t, we get [12,13] $\langle(\delta\mathbf{x})^2\rangle=(\delta^2/2)+\frac{1}{2}(\hbar t/m)^2(1/\delta)^2$. The meaning of this result is the following. At $t=0$ the particle is localized near the origin in a sphere of radius $\sim\delta$, when the time is running, due to the existence of the Planck constant, the particle becomes more and more delocalized. This effect depends on t but also on the value of the initial spatial resolution. For time intervals, t, such as $(\hbar t/m)\gg\delta^2$ we see that $(\langle(\delta\mathbf{x})^2\rangle)^{1/2}$ behaves like $1/\delta$ leading to a Hausdorff dimension $D_H=2$, as already established in [12,13] (for a definition of D_H, see for instance, [1]).

Note that the spatial resolution δ leads to a time resolution δt. We may define δt as the shorter time interval that we need to wait in order to see something, i.e., to see a displacement of order δ. From the result given above this leads to $\frac{1}{2}(\hbar\,\delta t/m)^2(1/\delta)^2\sim\delta^2$ or $(\hbar\,\delta t/m)=\delta^2$. This result is reminiscent of the one obtained above in the path integral formalism. It can be interpreted as follows: δt is the time interval for which the particle may explore by diffusion a distance equal to the spatial resolution. When $(\hbar\,\delta t/m)=\delta^2$ is used we can see that the path has a Hausdorff dimension $D_H=2$ in terms of this time scale.

In the relativistic domain it seems natural to introduce a Gaussian wave packet at the level of the FV representation. This point has been recently investigated in the case of the Dirac oscillator ([14]) for which, as in our approach, a particular representation of the wave equation has been used. Thus, we introduce an extra factor, $\exp(-p^2\delta^2)/2\hbar^2$, in the wave functions given in Eq. (11). From Eq. (12) we get

$$\langle(\delta\mathbf{x})^2\rangle=\frac{\delta^2}{2}+(ct)^2\frac{1}{\sqrt{2\pi}}\left(\frac{\delta}{\lambda}\right)\exp\left[\frac{1}{2}\left(\frac{\delta}{\lambda}\right)^2\right]\left\{K_1\left[\frac{1}{2}\left(\frac{\delta}{\lambda}\right)^2\right]-K_0\left[\frac{1}{2}\left(\frac{\delta}{\lambda}\right)^2\right]\right\}, \quad (27)$$

in which $K_i(z)$ means a modified Bessel function. Of course, this result has meaning only if the resolution is such that $\delta\gg\lambda$. In the limit $\delta\gg\lambda$ we get $\langle(\delta\mathbf{x})^2\rangle=(\delta^2/2)+[(ct)^2/2](\lambda/\delta)^2$. This result does not contain any linear dependence in t as assumed in [15] and we obtain $D_H=2$ provided $\lambda ct\gg\delta^2\gg\lambda^2$ or $mc^2t\gg\hbar$, which is also the condition obtained above in order to observe Brownian motion in the relativistic domain when the path integral formalism is used.

The fractal character of the quantum world has been investigated from two points of view. The path integral formalism restricted to the case $T=0$ shows that we have Brownian motion, i.e., fractal paths with fractal dimension $D=2$, provided $\delta t\gg\hbar/mc^2$. The wave equations show that a Hausdorff dimension $D_H=2$ can be associated with the free evolution of a wave packet with the same restriction on the time interval. These results suggest that the two formalisms describe the same effect and that the motion on the quantum paths is a real process. It means that the time that appears in the path integral formalism is not just a mathematical quantity but an ordinary time. To support this conclusion, note that all the results are obtained above when $\delta t\ll\beta\hbar$ is meaningful. Moreover, we can note that the "time" $\beta\hbar$ has a clear meaning; it represents the time interval τ needed for the thermostat to give to the particle a "thermal action," τk_BT, which is equal to the quantum of action $\hbar$. It is not surprising that for shorter time intervals we observe an intricate mixture of pure quantum aspects and thermal effects. More fundamentally, it seems normal to consider τk_BT as a basic physical quantity and not just as a mathematical trick, since the product τT is Lorentz invariant. In the path integral formalism we focus on a particular value of this product, which is $\hbar/k_B$, i.e., the ratio of two universal constants.

D. The time-energy uncertainty relation and its consequences

In the results derived above we have considered that the time interval is only restricted by a relativistic constraint $\delta t\gg\hbar/mc^2$. This is not a trivial result. From texbooks in statistical physics [16] we know that starting from the microcanonical form of the density matrix ρ we can put ρ in its canonical form. After that, by using the Lie-Trotter formula [3] we may derive (1). Obviously, the validity of this derivation is primarily based on the constraints associated with the existence of the microcanonical expression for ρ. In order to define ρ we must avoid large quantum fluctuations of energy $\widetilde{\delta E}$ and from the time-energy uncertainty relation we must restrict our investigations to time intervals $\widetilde{\delta t}$ that are large enough. In Eq. (1) no such restriction appears explicitly; for instance, in the nonrelativistic case we performed an integration from $t=0$ to $t=\beta\hbar$ and the results are meaningful whatever the value of δt. However, we can note that a like time-energy uncertainty relation appears at the end of the calculation since we can rewrite Eq. (19) according to

$$\delta t \frac{m}{2}\left\langle \left(\frac{\delta \mathbf{x}}{\delta t}\right)^2 \right\rangle = \frac{m}{2}(\beta\hbar - \delta t)\left\langle \frac{\mathbf{p}^2 c^4}{\mathbf{p}^2 c^2 + m^2 c^4}\right\rangle, \tag{28}$$

in which the rhs tends to $\hbar$ if $\delta t \ll \beta\hbar$ and $\sigma \gg 1$. It is interesting to note that $(m/2)\langle(\delta \mathbf{x}/\delta t)^2\rangle$ is not the kinetic energy, U, in the thermodynamic sense. This quantity is defined according to $U=-(\partial \ln Z/\partial\beta)$ and its value is $d/(2\beta)$, in the nonrelativistic case. The quantity $(m/2)\langle(\delta \mathbf{x}/\delta t)^2\rangle$ represents a fluctuating energy associated with the fact that, on the path integral formalism, the particles do not follow the equation of motion and then the energy is not constant along the quantum path as noted in [17]. Our results lead one to consider that Eq. (1) is not strictly equivalent to the canonical formalism derived by the usual routes but represents a result that is little bit more general as suggested by Feynman [2]. Note also that the existence of this time-uncertainty relation on the quantum paths also suggests that the time on the quantum path is not purely formal.

V. CONCLUSIONS

By using the path integral formalism we have investigated the behavior of a gas of spinless particles without interaction. This ideal gas of bosons is in contact with a thermostat. If we focus on the thermodynamics or on properties related to the momentum we recover the standard results, including their extension to the relativistic domain. The temperature that appears in the partition function fixes the pressure or the kinetic energy but not all of the quantities in the system. In this work we get three main conclusions, which appear at three different levels of the theory.

The first conclusion relates to the path integral formalism itself. We have shown that the Brownian motion is the stochastic process associated with the path even when the relativistic dynamics is introduced. All the results are exact provided we stay in the one-particle formalism. This requires working on time intervals larger than $\hbar/mc^2$. On the path the properties may or may not depend on the time interval under consideration.

The second conclusion concerns the physical meaning of our results. It seems natural to consider that the ‘‘time’’ that appears in the expression of the partition function behaves as ordinary time. This is based on several results: (1) the partition function is determined by a time $\beta\hbar$ that has a clear physical meaning and reveals a quantity which is Lorentz invariant; (2) when the temperature goes to zero we find that the particles exhibit Brownian motion determined by an effective diffusion constant $\hbar/m$, a result frequently invoked in the literature; (3) during their Brownian motion the particles explore some distances localized between the Compton wavelength and the thermal de Broglie wavelength as expected; (4) the fractal character of the quantum path also appears from the Schrödinger or the Klein-Gordon equation when we investigate the evolution of a Gaussian wave packet; (5) the Brownian motion and the evolution of the wave packet show the same fractal character for the same limit; and finally (6) on the paths, we may associate a time-energy uncertainty relation which has its usual meaning. According to these elements we may conclude that the partition function that is calculated from the path integral formalism gives us the standard thermodynamics of the system but also the dynamics of the system for short time intervals, such as $\delta t \ll \beta\hbar$. During this time interval, the thermal average of the square of the velocity is not related to the momentum by the standard relation. The exact relation shows that the square of the velocity results from the interplay between thermodynamics and pure quantum effects. It is interesting to note that $\beta\hbar$ is about 10 femtoseconds at room temperature, and we may investigate our results from an experimental point of view.

The third conclusion relates to the expression for the partition function. We have accepted that the partition function may give us some information without any restrictions on the time interval in the nonrelativistic domain or provided we focus on a time interval larger than $\hbar/mc^2$ in the relativistic domain. This is not in agreement with the usual derivation of the quantum statistical physics for which we have to consider a time interval large enough to avoid large energy fluctuations associated with the time-energy uncertainty relation. No such restriction appears in the path integral formalism, but we have shown that such an uncertainty relation appears at the end of the calculation, which means that it is implicitly involved from the very beginning in the quantification via the path integral formalism.

This paper shows that the fractal character of the quantum world is unavoidable provided we use the path integral formalism. Here, the fractal character of the microscopic world appears as fundamental; it is intrinsically related to the existence of the Planck constant.

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