Spatiotemporal irregularity in an excitable medium with shear flow

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We consider an excitable medium moving with relative shear, subjected to a localized disturbance that in a stationary medium would produce a pair of spiral waves. The spiral waves so created are distorted and then broken by the motion of the medium. Such breaks generate new spiral waves, and so a ''chain reaction'' of spiral wave births and deaths is observed. This leads to a complicated spatiotemporal pattern, the ''frazzle gas'' [term suggested by Markus *et al.*, Nature (London) **371**, 402 (1994)], which eventually fills the whole medium. In this paper, we display and interpret the main features of the pattern. [S1063-651X(99)09407-6]

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I. INTRODUCTION

Excitable medium models, in the form of partial differential equations of the reaction-diffusion type, have been used to account for nonlinear wave phenomena in many areas of biology, physical chemistry, and physics [1]. An excitable system responds to a small subthreshold perturbation by a graded, decremental response, and to a suprathreshold perturbation by a large amplitude pulse or pulse train. This threshold property is characteristic of a cubic nonlinearity, as in the FitzHugh-Nagumo equations for an excitation process E and a recovery process g. In a spatially extended system, the suprathreshold response is a nondecremental traveling wave or a wave train. When such a cubic nonlinearity is included in a reaction-diffusion equation,

$$\frac{\partial E}{\partial t}=c_1E(E-a)(1-E)-g+D\nabla^2E,$$

$$\frac{\partial g}{\partial t}=\epsilon(c_2E-g)+\delta D\nabla^2 g, \qquad (1)$$

in a two-dimensional medium appropriate initial conditions can lead to a spiral wave. Such spiral waves (or scroll waves in three dimensions) have been observed in many biological excitable media, and a spiral source acts to organize the surrounding medium.

When the excitable medium, such as a fluid or an elastic solid, is itself undergoing spatial strain, the otherwise stable spiral pattern is deformed and possibly broken. The effects of the motion of the medium on excitation-wave dynamics in the Belousov-Zhabotinsky system has been studied experimentally for thermoconvective motion in [2], and experimentally and theoretically for small deformations in [3].

We have shown that an arbitrarily small, linear shear flow can break repetitive wave trains [4]. In a medium subject to a shear flow, the wavelength of the train changes with time. This change depends on the mutual orientation of the flow and wave train. In excitable media, there is a shortest possible wavelength, below which the waves cannot propagate. When the flow deforms the wave train so that wavelength is less than this critical value, the propagation is blocked. If the wave train and/or the flow is not strictly periodic, the blocking is localized and the waves that extend across a ''blocked'' and ''unblocked'' region break. The minimum time for the first wave break to occur has been estimated in [4] as

$$t_*\approx\alpha^{-1}(k_*-1/k_*), \qquad (2)$$

where α is the shear (i.e., the gradient of the flow velocity) and k_* is the critical deformation, i.e., the ratio of the initial wavelength of the train and the minimum wavelength.

Here we consider the effects of simple shear flows on spiral wave behavior in excitable media and show that spiral wave activity is broken down by *arbitrarily small shear flows* into spatiotemporal irregularity (an autowave turbulence, or ''frazzle gas'' similar to one described by Markus *et al.* [5]).

II. THE NUMERICAL MODEL

For simplicity the numerical illustrations were performed using a FitzHugh-Nagumo system with cubic nonlinearity and added shear flow. We expect other excitable systems to display qualitatively similar behavior. The equations considered were

$$\frac{\partial E}{\partial t}=c_1E(E-a)(1-E)-g+v(y)\frac{\partial E}{\partial x}+D\nabla^2E,$$

$$\frac{\partial g}{\partial t}=\epsilon(kE-g)+v(y)\frac{\partial g}{\partial x}+\delta D\nabla^2 g, \qquad (3)$$

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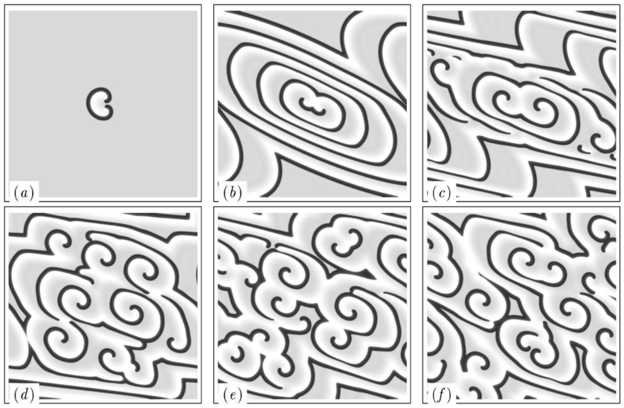

FIG. 1. Development of a ''frazzle gas'' of spiral waves in linear shear flow (4). Shown are snapshots of the E field at successive 100 t.u., in a 400×400-s.u. medium, with a flow velocity gradient $\alpha=0.02$ t.u.$^{-1}$.

with parameters $c_1=10$, $a=0.02$, $\epsilon=0.1$, $k=5$, $\delta=1$, and $D=1$. This system was solved using an explicit Euler scheme, with space step $h_s=0.5$ s.u. (space units) and time step $h_t=0.0025$ t.u. (time units), in a rectangular medium $(x,y)\in[0,L]\times[-M/2,M/2]$. The sizes $L\times M$ were varied in different experiments. We used two flow velocity profiles, a linear profile

$$v(y)=\alpha y, \tag{4}$$

with no-flux boundary conditions at $y=\pm M/2$, and a sine profile

$$v(y)=v_{\max}\sin(2\pi y/M), \tag{5}$$

with periodic boundary conditions at $y=\pm M/2$. In all cases, we used periodic boundary conditions at $x=0,L$. The properties of the stationary $(\alpha=0)$ medium were as follows: the minimum wavelength of a periodic train $\lambda_{\min}\approx19.0$ s.u., the asymptotic wavelength of the spiral wave $\lambda_{\rm sw}\approx41.0$ s.u., and the asymptotic velocity of the spiral wave $c_{\rm sw}=1.80$ s.u./t.u.. The initial condition for this system was a short excitation wavelet, just wide enough to give birth to a pair of spiral waves (horseshoe pattern) as shown in Fig. 1(a).

III. DEVELOPMENT OF THE FRAZZLE GAS

The phenomenon of conduction blocking of periodic wave trains has macroscopic consequences for the properties of large-scale two-dimensional excitable media with shear flow. Since this conduction block is dependent on the orientation of the waves, it leads to breaking of the waves when there is a complicated autowave pattern. Moreover, in an excitable medium, each wave break typically leads to the generation of a new pair of spiral waves, which are sources of periodic wave trains. This leads to a ''chain reaction'' of spiral wave births, as shown in Fig. 1.

Here a local finite initial perturbation has led to the transition of the whole medium into a turbulencelike state, the ''frazzle gas.'' To characterize quantitatively the complexity of the frazzle-gas solution, we counted the number of the free ends, defined as intersections of the isolines $E=0.2$ and $g=0.48$. Some typical dependencies of this number on time, for different values of the velocity gradient α, are shown in Fig. 2. It appears that for any α, a statistically steady value is reached after an initial period of development.

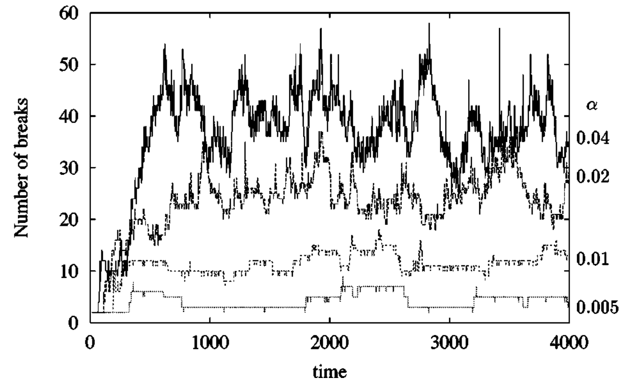

FIG. 2. Number of free ends as a function of time, in t.u., for different velocity gradients (values of α shown in t.u.$^{-1}$), in a 300×300-s.u. medium.

IV. DENSITY OF THE FRAZZLE GAS

As can be seen in Fig. 1, the dynamics of the generation of new wave breaks in this particular experimental setup is determined, in the first instance, by two different processes: the growth of the ''horseshoe'' pattern, due to the revolution of the spiral waves, and the deformation of that pattern. Subsequently, the development of secondary breaks further increases the density of the free ends, until the pattern reaches a state of statistical equilibrium, when the average number of the new free ends is balanced by the average rate of their annihilation, which happens if two opposite free ends come too close to each other. The resulting pattern and fluctuations in the number of free ends depends on the value of the velocity gradient, as illustrated in Fig. 3.

The simple criterion for the wave break introduced in [4] can be used for a rough analytical estimate of the equilibrium density of spiral waves. First, let us estimate the typical distance between the spiral waves as being of the same order of magnitude as the distance from the spiral center to the point at which the first break in a spiral wave occurs. This is made up of a minimum distance, of the order of the spiral core, or spiral wavelength $\lambda_{\rm sw}$, plus the distance traveled by the spiral wave in the time before the breakup, which is $t_*\approx\alpha^{-1}(k_*-1/k_*)\propto\alpha^{-1}$, since the critical deformation $k_*=\lambda_{\min}/\lambda_{\rm sw}\approx2.16$. The typical distance between the spiral waves in the frazzle gas can thus be expected to be

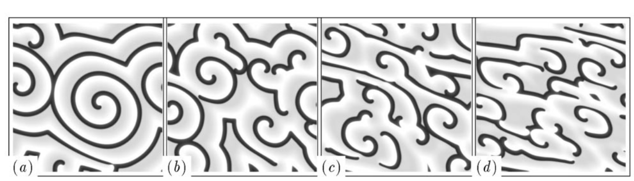

FIG. 3. Structure of the dynamically equilibrated ''frazzle gas'' of spiral waves (snapshots of the E field) at different velocity gradients: (a) 0.005 t.u.$^{-1}$, (b) 0.01 t.u.$^{-1}$, (c) 0.02 t.u.$^{-1}$, and (d) 0.04 t.u.$^{-1}$. Size of the medium is 300×300 s.u.

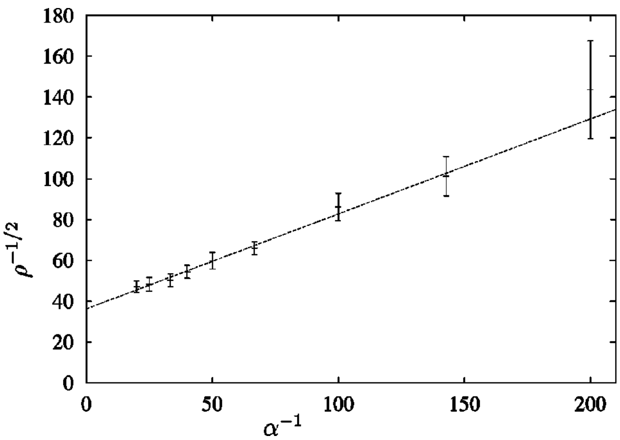

FIG. 4. Time-average density of free ends ρ measured in s.u.$^{-2}$, for established ''frazzle-gas'' state, as a function of the velocity gradient α measured in t.u.$^{-1}$, in coordinates $\rho^{-1/2}$ vs α^{-1}. Points with errorbars show values obtained from simulation; the line shows the best fit to the theoretical dependence (7).

$$l_{sw}=\beta\lambda_{sw}+\gamma c_{sw}\alpha^{-1}, \tag{6}$$

where β and γ are some dimensionless coefficients of the order of 1. The density of the spiral waves is then estimated by

$$\rho=l_{sw}^{-2}=\alpha^2/(K_1+K_0\alpha)^2, \tag{7}$$

where

$$K_0=\beta\lambda_{sw}, \quad K_1=\gamma c_{sw}. \tag{8}$$

Figure 4 shows the dependence $\rho(\alpha)$ found in numerical experiments, and the best fit to Eq. (7). This best fit is achieved with $K_0\approx36$ and $K_1\approx0.46$, which means $\beta\approx1.9$ and $\gamma\approx0.26$. Thus, the simple argument presented above correctly predicts the qualitative dependence of ρ on α, for a reasonable choice of the dimensionless coefficients. Recall that the estimates of [4] also were only valid to within an order of magnitude.

V. FRAZZLE GAS IN AN INHOMOGENEOUS FLOW

The linear shear is a highly simplified case. To check the robustness of the features of the frazzle gas of spirals, we studied its behavior in a more complicated flow, the sine shear flow (5). The results are illustrated in Fig. 5.

The sine-shear profile (see panel $t=6$ of Fig. 5) provides two regions with high velocity gradient, clockwise in the middle ($y\approx0$) and counterclockwise around the upper and lower boundary ($y\approx\pm M/2$, recall that the boundary conditions are periodic), and two regions with lower velocity gradient, around $y\approx M/4$ and $y\approx-M/4$. The horseshoe pattern is initiated in the region with high shear, and is first deformed ($t=12$) and then displays wave breaks ($t=24$). In turn, the free ends curl into new spirals, which lead to secondary breaks ($t=48$) and subsequently to the frazzle gas of spirals ($t=96$). At $t=96$, one can see that the frazzle gas is localized in the high-shear region, but some of the spirals are driven away from that region (notice the two spirals at $x\approx3L/4$, $y\approx M/4$, and at $x\approx L/4$, $y\approx-M/4$), and that the

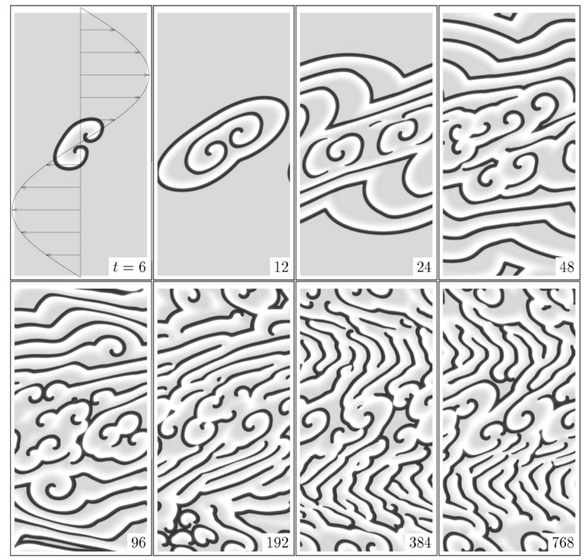

FIG. 5. Development of a ''frazzle gas'' of spiral waves in a sine-shear flow (5). Shown are snapshots of E in 300×600 s.u. with maximal flow velocity $v_{max}=1.5$ s.u.×t.u.$^{-1}$ at time moments (shown on each panel, measured in t.u.), chosen in geometric progression. $h_x=1.0$, $h_t=0.01$.

nearly plane wave trains are compressed in the other high-shear region (see near the upper and lower boundaries). This subsequently leads to the generation of the frazzle gas in the other high-shear region ($t=192$), which with time relaxes to a dynamic macroscopic equilibrium state ($t=384$). This then remains statistically constant—or at least does not qualitatively change over the following time interval ($t=768$).

The structure of the frazzle gas is inhomogeneous, and may seem counterintuitive. The free ends are seen in both high- and low-shear regions. In high-shear regions, one can see well-developed spiral waves, while in low-shear regions, where in the homogeneous case one would expect even better developed spirals [see Fig. 3(a)], there are no spirals at all, but only dislocations in quasiplane wave trains.

This paradox is easily explained. The presence of a shear flow breaks the spatial reflection symmetry of the reaction-diffusion system. As a result, the angular velocity of a spiral wave in the shear flow now depends on the direction of rotation. The general perturbation theory [6] predicts only that the angular velocity is

$$\omega=\omega_0+\chi m\alpha+O(\alpha^2), \tag{9}$$

where ω_0 is the angular velocity in a quiescent medium, m is the direction of rotation, say, $m=1$ for clockwise rotating spirals and $m=-1$ for counterclockwise, and coefficient χ depends on the particular model. In our model, the spirals rotating against the shear (counter-rotating spirals) rotate faster.

Furthermore, it is well known that in an autowave medium faster sources entrain slower sources, and if the slower source is a spiral wave, this causes its so-called ''induced drift'' [7–9]. As a result, in the high-shear regions, corotating spiral waves are entrained by counterrotating spirals and driven away to the low-shear regions. The spirals in the low-shear regions do not develop since the spiral rotation fre-

TABLE I. Structure of the frazzle gas in different regions. The ratios show the number of clockwise (cw) free ends/number of counterclockwise (ccw) free ends in the region.

y range	$[0,M/8]\cup[7M/8,M]$	$[M/8,3M/8]$	$[3M/8,5M/8]$	$[5M/8,7M/8]$
shear	high ccw	small	high cw	small
$t=384$	11/2 (cw/ccw)	10/12	3/13	7/7
$t=768$	11/3	10/10	4/12	6/5

quency there is approximately ω_0, which is lower than that in the high-shear regions where it is $\omega_0+|\chi\alpha_{\rm max}|$, so the free ends remain dislocations and cannot develop into spiral waves.

These processes lead to the following structure (see Table I). The high-shear regions are populated mainly (but not exclusively) by counter-rotating spirals, i.e., counterclockwise rotating in the middle region and clockwise rotating in the top/bottom region. However, some corotating spirals are also present, since the free ends are born in pairs, and it takes time to entrain a spiral wave. At the same time the low-shear regions show quasiplane wave trains with dislocations, which are former spiral waves expelled from the high-shear regions.

VI. DISCUSSION

In this paper, we have described the process of generation and main properties of a ''frazzle gas'' of spiral waves produced by shear flows in the medium. Such a frazzle gas occurs in a sufficiently large excitable medium when shear flow breaks a repetitive wave train. The conditions for the generation of the first wave breaks were described earlier [4] and the first break requires a space and time to develop (the weaker the shear, the larger the space and time required), whereafter new wave breaks are generated via a chain reaction, until a dynamical equilibrium is reached where the average number of newly generated wave breaks equals the average number of annihilated wave breaks.

The average density of wave breaks as a function of flow velocity gradient is described by a simple semiempirical ''Michaelis-Menten'' formula; understanding the key mechanisms of the dynamic equilibrium allowed us to relate, to within the order of magnitude, the constants in that formula to principal parameters of the medium.

An inhomogeneously sheared flow makes the rate of generation of new wave breaks space-dependent, which naturally leads to inhomogeneous distribution of the wave break. In addition, it introduces qualitatively new features, especially if the shear changes sign: the flow sorts the wave breaks by their chirality. The mechanism for this sorting is related to parity violation by the shear, which leads to a difference in frequency between oppositely rotating spiral waves and to induced drift of the slower rotating waves.

The mechanism and properties of this ''frazzle gas'' makes it different from other examples. In particular, the example in [5] is clearly different, since it occurs in a stationary medium. The experimental example [2] is more similar, since it is also about interaction of convection and excitation. However, the convection there was quite complicated, consisting of Bénard convection cells of size comparable to the wavelength of the spiral. It was, therefore, not clear whether the complexity of the resulting pattern should have been attributed to the presence of convective motion or to its complexity. The present study shows that the complexity of the flow is not necessary, as the irregular activity occurs even in a perfectly homogeneous linear shear flow.

ACKNOWLEDGMENTS

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