Inverse Mermin-Wagner theorem for classical spin models on graphs

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In this paper we present the inversion of the Mermin-Wagner theorem on graphs, by proving the existence of spontaneous magnetization at finite temperature for classical spin models on transient on the average graphs, i.e., graphs where a random walker returns to its starting point with an average probability $\overline{F}<1$. This result, which is here proven for models with $O(n)$ symmetry, includes as a particular case $n=1$, providing a very general condition for spontaneous symmetry breaking on inhomogeneous structures even for the Ising model. [S1063-651X(99)12208-6]

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Geometry plays a fundamental role in phase transitions of statistical models on regular lattices. The existence itself of an ordered phase at non zero temperature only depends on large scale topology, via the Euclidean dimension d of the lattice. Indeed, a discrete symmetry is broken if and only if $d>1$, while for a continuous symmetry the corresponding condition is $d>2$. For the latter case two rigorous results, the Mermin-Wagner theorem [1,2] and the Frölich-Simon-Spencer bound [3,4], provide, respectively, the necessary and the sufficient condition for spontaneous symmetry breaking.

This simple and exhaustive picture allows us to classify a statistical system on a lattice in terms of *geometrical super-universality classes* characterized by the Euclidean dimension. Unfortunately the classification cannot be directly extended to general networks describing noncrystalline structures, where one cannot exploit the basic geometrical features of crystal lattices, such as translation invariance, the concept of Euclidean dimension, the reciprocal lattice. The proofs given in Refs. [1–4] strongly rely on these properties and more general concepts and tools are needed when dealing with a noncrystalline structure.

A basic improvement in the study of properties of geometrically disordered structures has been achieved with graph theory. A graph, i.e., a general network of sites connected pairwise by bonds, provides the most suitable mathematical tool to describe complex and irregular discrete geometries. Euclidean lattices, which are the usual model for crystalline structures, are very peculiar example of graphs characterized by complete translation invariance.

The generalization to inhomogeneous structures of the necessary condition for spontaneous breaking of continuous symmetry, the Mermin-Wagner theorem, has been the first step in this direction [5,6]. The existence of spontaneous magnetization on a graph $\mathcal{G}$ is related to the probability F_i of returning to the starting site i for a simple random walk on $\mathcal{G}$. In particular, it was proven that there is no spontaneous magnetization for recurrent on the average (ROA) graphs, i.e., when $\overline{F}=1$, where $\overline{F}$ is the average of F_i over all the points i of the graph $\mathcal{G}$ [5]. This result naturally includes the lattice theorem [1], since Euclidean lattices in one and two dimensions turn out to be ROA. However, up to now a sufficient condition has been lacking.

In this paper we study the case $\overline{F}<1$, i.e., transient on the average (TOA) graphs and we give a rigorous proof of the existence of spontaneous magnetization at $T>0$ for classical spin models with $O(n)$ symmetry. This result is the exact inversion of theorem [5] for the classical case and a generalization to graphs of Refs. [3,4], since lattices with $d>2$ are TOA. Now, each graph can be classified either as ROA or TOA and therefore this theorem completes the picture for classical spin models on graphs. Moreover, as in the lattice case [3,4], the proof also holds for $n=1$, i.e., for the Ising model.

In the following G is a graph consisting of N_g sites, $i=1,2,\ldots,\ N_g$, and of links (ij) joining them; we say that two sites connected by a bond are nearest neighbors. A graph is connected if, given any two points in G, there exists a path joining them. Here we will consider connected graphs. The chemical distance between sites i and j is the length (number of links) of the shortest path joining them. The graph topology is algebraically described by its adjacency matrix A_{ij}, given by $A_{ij}=1$ if i and j are nearest neighbors, $A_{ij}=0$ otherwise. $O(n)$ models on $\mathcal{G}$ with $n\geqslant 1$ are defined by the Hamiltonian

$$\mathcal{H}=-\frac{1}{2}\sum_{ij} J_{ij}\vec{\sigma}_i\cdot\vec{\sigma}_j-\vec{h}\cdot\sum_i \vec{\sigma}_i, \qquad (1)$$

where J_{ij} are bounded ferromagnetic interactions on G:

$$J_{ij}=J_{ji}=\begin{cases} J_{ij} & \text{with } 0<\epsilon\leqslant J_{ij}\leqslant J<\infty \text{ if } A_{ij}=1,\\ 0 & \text{if } A_{ij}=0,\end{cases} \qquad (2)$$

and $z_i=\Sigma_j J_{ij}\leqslant z<\infty$. $\vec{\sigma}_j$ are n-dimensional real unit vectors $\vec{\sigma}_i\equiv(\sigma^1,\ldots,\sigma^n)$ defined on each vertex satisfying the constraints: $|\vec{\sigma}_i|^2=1\forall i$. For $n=1$ $\mathcal{H}$ describes the Ising model which is invariant under the discrete symmetry group $\mathbf{Z}_2$,

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while for $n \geqslant 2$ $\mathcal{H}$ represents a model with an $O(n)$ continuous symmetry. Finally $\vec{h} \equiv (h,0,\ldots,0)$, $h>0$, is an external magnetic field coupled to $\vec{\sigma}_i$.

The average magnetization is the order parameter of the model and it is defined by

$$M(h) \equiv \frac{1}{N_g} \sum_i \langle \sigma_i^1 \rangle, \tag{3}$$

where the average $\langle \cdots \rangle$ is taken with respect to the usual Boltzmann weight $\exp(-\beta\mathcal{H})$ with $\beta = 1/KT$.

The Mermin-Wagner theorem will be inverted by proving a lower positive bound for the magnetization at sufficiently low T in the thermodynamic limit $N_g \to \infty$, on an infinite graph $\mathcal{G}$. Namely, we will show that if $\overline{F}<1$, there exists a small enough temperature T for which $\lim_{h\to 0} M(h) > c(T)$ with $c(T)>0$. In order to define the thermodynamic limit let us introduce the Van Hove spheres $S_{r,o}$ as the subsets of sites in $\mathcal{G}$ whose chemical distance from o is $\leqslant r$, N is the number of sites in $S_{r,o}$. In our proof to explore the behavior of the model on the infinite graph, first we will obtain inequalities for thermal averages on the finite subgraphs $\mathcal{S}_{r,o}$ given by the sites of the sphere $S_{r,o}$ and the bounds (i,j) of $\mathcal{G}$ with i,j, $\in S_{r,o}$. Then we will take the thermodynamic limit $N_g \to \infty$, letting $r\to\infty$; finally we will take the limit $h \to 0$.

Let us first consider graphs for which the average of F_i is smaller than 1 in every positive measure subset S of the sites of $\mathcal{G}$, where the measure of the subset S is given by $|S| = \lim_{N_g\to\infty}[\Sigma_i \chi_S(i)]/N_g$ and $\chi_S(i)$ is the characteristic function of S: $\chi_S(i)=1$ if $i\in S$ and $\chi_S(i)=0$ if $i \notin S$. We will call these graphs pure TOA. For these graphs in the thermodynamic limit $N_g\to\infty$ [7]:

$$\lim_{\mu\to 0} \lim_{N_g\to\infty} \frac{1}{N_g} \mathrm{Tr}(L+\mu)^{-1} = v < \infty, \tag{4}$$

where L_{ij} is the Laplacian operator given by $L_{ij} = z_i\delta_{ij} - J_{ij}$ and $\mu_{ij} = \mu\delta_{ij}$, $\mu>0$.

Our proof will follow the following main steps. (a) We introduce for the constraints $|\vec{\sigma}_i|^2=1$ an integral representation with Lagrange multipliers α_i and perform the Gaussian integration on $\vec{\sigma}_i$, (b) we determine the asymptotic behavior of the integrals over α_i for $\beta\to\infty$ by saddle point technique, (c) we establish the lower bound on $M(h)$ exploiting (b) and the identity

$$1 = \frac{1}{N_g} \sum_i \langle |\vec{\sigma}_i|^2 \rangle. \tag{5}$$

Let us start with step (a). In the expressions (3) and (5) we introduce the integral representation for the constraints $|\vec{\sigma}|_i^2 = 1$:

$$\delta(|\vec{\sigma}_i|^2 - 1) = \frac{e^{\epsilon/2}}{2\pi} \int d\alpha_i e^{[-i\alpha_i(|\vec{\sigma}_i|^2-1)/2 - \epsilon|\vec{\sigma}_i|^2/2]}, \tag{6}$$

where ϵ is a real arbitrary constant. We will choose $\epsilon = h\beta$. We now perform the Gaussian integration over the variables $\vec{\sigma}_i$, obtaining for Eqs. (3) and (5):

$$M(h) = \frac{1}{Z} \int \prod_{i\in S_{r,o}} d\alpha_i e^{iS_{\beta h}(\alpha)} \frac{h}{N_g} \sum_{kj} (L+H+i\alpha)^{-1}_{kj}, \tag{7}$$

$$1 = \frac{1}{Z}\int \prod_{i\in S_{r,o}} d\alpha_i e^{iS_{\beta h}(\alpha)} \left[\frac{n}{\beta N_g} \mathrm{Tr}(L+H+i\alpha)^{-1} + \frac{h^2}{N_g} \sum_{ij} (L+H+i\alpha)^{-2}_{ij} \right], \tag{8}$$

where

$$iS_{\beta h}(\alpha) \equiv -\frac{n}{2}\mathrm{Tr}[\ln(L+H+i\alpha)] + \frac{\beta}{2}\left[i\sum_i \alpha_i + h^2 \sum_{ij} (L+H+i\alpha)^{-1}_{ij} \right],$$

$$Z \equiv \int \prod_{i\in S_{r,o}} d\alpha_i e^{iS_{\beta h}(\alpha)}, \alpha_{ij} = \alpha_i\delta_{ij}, \text{ and } H_{ij} = h\delta_{ij}.$$

Notice that the order of the symmetry group n becomes a parameter of the integration.

Let us now study the behavior of Eqs. (7) and (8) for large β, which is point (b) of our plan. By saddle point theorem, the leading asymptotic behavior of Eqs. (7) and (8) is given by the $\overline{\alpha_i}$, which satisfy the stationary conditions

$$\frac{\partial}{\partial \overline{\alpha_i}} \left[i\sum_k \overline{\alpha}_k + h^2 \sum_{kj} (L+H+i\overline{\alpha})^{-1}_{kj} \right] = 0 \forall i, \tag{9}$$

where $\overline{\alpha}_{ij} = \overline{\alpha}_i \delta_{ij}$. Equation (9) is satisfied for all $h \geqslant 0$ only if $\overline{\alpha}_i = 0 \forall i$, so that [7]

$$M(h) = \frac{1}{Z}\int_\Gamma \prod_{i\in S_{r,o}} d\alpha_i \mathrm{Re}[e^{iS_{\beta h}(\alpha)}] \frac{h}{N_g} \times \mathrm{Re}\left[\sum_{kj} (L+H+i\alpha)^{-1}_{kj}\right] + o(1/\beta), \tag{10}$$

$$1 = \frac{1}{Z}\int_\Gamma \prod_{i\in S_{r,o}} d\alpha_i \mathrm{Re}[e^{iS_{\beta h}(\alpha)}] \mathrm{Re}\left[\frac{n}{\beta N_g} \mathrm{Tr}(L+H+i\alpha)^{-1} + \frac{h^2}{N_g}\sum_{ij} (L+H+i\alpha)^{-2}_{ij} \right] + o(1/\beta), \tag{11}$$

where Γ is the region around the saddle point $\overline{\alpha_i}$ in which $\mathrm{Re}\{\exp[iS_{\beta h}(\alpha)]\} > 0$. Here we exploited the property that $\exp[iS_{\beta h}(\overline{\alpha_i})]$ is real and positive and that we are evaluating real quantities.

As for (c), we first introduce the following inequalities, which can be proven exploiting the boundedness and the non-negativity of the Laplacian operator [7]:

$$1 \geqslant \mathrm{Re}\left[\frac{h}{N_g}\sum_{ij}(L+H+i\alpha)^{-1}_{ij}\right] \geqslant \mathrm{Re}\left[\frac{h^2}{N_g}\sum_{ij}(L+H+i\alpha)^{-2}_{ij}\right], \tag{12}$$

$$0 \leqslant \frac{1}{N_g}\mathrm{Re}[\mathrm{Tr}(L+H+i\alpha)^{-1}] \leqslant \frac{1}{N_g}\mathrm{Tr}(L+H)^{-1}. \tag{13}$$

Using Eq. (12) we compare the expressions (10) and (11), obtaining for the magnetization

$$M(h) \geqslant 1 - o(1/\beta) - \frac{1}{Z}\int_\Gamma \prod_{i\in S_{r,o}} d\alpha_i \mathrm{Re}[e^{iS_{\beta h}(\alpha)}]\frac{n}{\beta N_g} \times \mathrm{Re}[\mathrm{Tr}(L+H+i\alpha)^{-1}]. \tag{14}$$

Now with Eq. (13) we obtain for $M(h)$ the following inequality:

$$M(h) \geqslant 1 - o(1/\beta) - \frac{1}{N_g}\mathrm{Tr}(L+H)^{-1}. \tag{15}$$

Using property (4) of pure TOA graphs, we finally get in the thermodynamic limit

$$\lim_{h\to 0}\lim_{N_g\to\infty} M(h) \geqslant 1 - \frac{v}{\beta} - o(1/\beta) \tag{16}$$

and this complete the proof for pure TOA graphs.

Let us consider now the most general case of a graph $\mathcal{G}$ which is not pure TOA. In this case $\mathcal{G}$ must have a positive measure subset where the average of F_i is 1, i.e., a ROA subgraph. We call such a graph mixed TOA. A mixed TOA graph can always be decomposed in a pure subgraph $\mathcal{S}$ and a ROA subgraph, connected by a zero measure set of links. This implies that the total free energy per site $f_{\mathcal{G}}$ is given by $f_{\mathcal{G}} = |\mathcal{S}| f_{\mathcal{S}} + |\mathcal{G}-\mathcal{S}| f_{\mathcal{G}-\mathcal{S}}$ and as a consequence

$$\lim_{N_g\to\infty} M(h) \geqslant \lim_{N_g\to\infty} \frac{1}{N_g}\sum_{i\in\mathcal{S}} \langle \sigma_i^1 \rangle. \tag{17}$$

On $\mathcal{S}$,

$$\lim_{h\to 0} \frac{1}{N_g}\sum_{i\in\mathcal{S}} (L+H)^{-1}_{ii} = v' < \infty \tag{18}$$

and using Eqs. (16), (17), and (18) we get

$$\lim_{h\to 0}\lim_{N_g\to\infty} M(h) \geqslant |\mathcal{S}| - \frac{v'}{\beta} - o(1/\beta). \tag{19}$$

Inequality (19) proves the existence of a lower positive bound at low enough temperature for the magnetization of an $O(n)$ model defined on a TOA graph. In this way we obtain the inversion of Ref. [5] and we generalize the Frölich-Simon-Spencer result to generic inhomogeneous discrete structures.

A few comments follow from our result. The condition $\bar{F} < 1$ turns out to be a condition on the spectral density at low eigenvalues of the Laplacian operator L on $\mathcal{G}$ and provides the link between the physical properties of the $O(n)$ model and the topology of the discrete space. In particular it includes the lattice *geometrical superuniversality class* $d>2$, i.e., the result of Refs. [3,4]. More generally for ROA and pure TOA graphs, if one can define the spectral dimension $\bar{d}$ [8], the condition becomes $\bar{d} > 2$. However, we point out that the present result is far more general, holding also for graphs without spectral dimension. This is the case of the Bethe lattice, which is a pure TOA graph with finite temperature phase transitions.

Our result completes the description of the behavior of continuous classical spin models on generic networks. On the other hand, it also provides a rigorous and very general sufficient condition for spontaneous magnetization of the Ising model ($n=1$) on graphs. Obviously this condition is not necessary. A simple counterexample is the two-dimensional Ising model, which has spontaneous magnetization. The study of the Ising model on ROA graphs is therefore a key step to obtain a complete picture of the behavior of spin models on general discrete structures.

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