Analytical results for random walks in the presence of disorder and traps

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(Received 17 February 1999)

In this paper, we study the dynamics of a random walker diffusing on a disordered one-dimensional lattice with random trappings. The distribution of escape probabilities is computed exactly for any strength of the disorder. These probabilities do not display any multifractal properties, contrary to previous numerical claims. The explanation for this apparent multifractal behavior is given, and our conclusions are supported by numerical calculations. These exact results are exploited to compute the large time asymptotics of the survival probability (or the density) which is found to decay as $\exp[-Ct^{1/3}\ln^{2/3}(t)]$. An exact lower bound for the density is found to decay in a similar way. [S1063-651X(99)07608-4]

PACS number(s): 05.40.−a, 05.60.−k, 02.50.Ey, 61.43.Hv

I. INTRODUCTION

The dynamics of the survival probability of particles diffusing in the presence of traps is a rich problem which has been widely discussed in the physical and mathematical literature within the past two decades [1–5]. The simplest system is that of diffusing particles in the presence of perfect static traps [1–4]. This problem (which we will call the Donsker-Varadhan problem) has been solved using very different technics. The main result is that the density does not decay exponentially (as a simple mean-field argument would predict), but as

$$n(t)\sim\exp[-C_d(-\ln(1-c))^{2/(d+2)}t^{d/(d+2)}], \tag{1}$$

where c is the trapping site density. The physical interpretation in d dimensions is that the process is dominated by particles standing in very large trap-free regions of linear size L [these regions have a probability of order $\exp(-cL^d)$ for small c]. In such a region, the density decays as $\exp(-t/L^2)$. A saddle-point argument then leads to the result of Eq. (1), with the relevant regions being of typical size $L\sim(t/c)^{1/(d+2)}$, at time t.

In another class of models [5], the traps are allowed to move. When these traps undergo free diffusion, the density of particles decays as

$$n(t)\sim\exp[-C_d c t^{d/2}] \tag{2}$$

for $d<2$, and decays exponentially for $d>2$. This result holds in the case of static or diffusing particles [5].

It would be interesting to introduce the effects of hopping disorder on the trapping process. Even without trapping, quenched disorder in the particle hopping probabilities is known to have very important effects on the diffusion and first return properties [6–8]. In the case of symmetric hopping probabilities ($w_{i,i+1}=w_{i+1,i}$) [6,7], anomalous diffusion is observed, with an exponent depending on the properties of the disorder. In the generic nonsymmetric case (see Ref. [7] for a more precise criterion), as in the Sinai model [8] where a particle diffuses in a random (Brownian) potential, the diffusion is dramatically suppressed, the particle being effectively trapped in deeper and deeper valleys of the potential as time goes on. In the present paper, we study the dynamics of particles diffusing in a symmetric or nonsymmetric disorder, in the presence of a random finite trapping probability at each site.

II. MODEL AND KNOWN RESULTS

Consider a particle moving on a one-dimensional lattice with random barriers (or hopping probabilities) and random trapping probabilities. More precisely, a particle at site i has a probability $w_{i,i+1}<\frac{1}{2}$ (respectively $w_{i,i-1}<\frac{1}{2}$) to hop on site $i+1$ (respectively $i-1$), and a probability $(1-\gamma)(1-w_{i,i-1}-w_{i,i+1})$ to disappear ($\gamma<1$). With residual probability $\gamma(1-w_{i,i-1}-w_{i,i+1})$, it just stays on site i. The hopping probabilities can be taken to be symmetric ($w_{i,i+1}=w_{i+1,i}$) or nonsymmetric, and will be chosen according to the typical probability distribution

$$\rho(w)=2^{1-\beta}(1-\beta)w^{-\beta}\theta(w)\theta(1/2-w), \tag{3}$$

where $\beta<1$ measures the quenched disorder strength.

The case $\gamma=1$ (no trapping) has been extensively studied [6–8]. In the symmetric case [6,7] one observes anomalous diffusion, $\langle x^2(t)\rangle\sim t^{2\nu}$, with ν depending continuously on β. The return probability $P_s(t)=\langle p_{i,i}(t)\rangle$, which is the probability of being at site i at time t having started at site i, decays as $P_s(t)\sim t^{-d_s/2}$, where d_s is the spectral dimension [6,7]. The Sinai model [8] describes the generic nonsymmetric case, and displays logarithmically slow diffusion and other peculiar properties.

In the presence of trapping ($0\leqslant\gamma<1$), the problem has been studied essentially by numerical means [9,10]. In addition to $P_s(t)$, one can define the normalized return probability $P(t)$ as

$$P(t)=\left\langle\frac{p_{i,i}(t)}{\sum_j p_{i,j}(t)}\right\rangle. \tag{4}$$

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Note that in order to keep the notations simple, it is understood that $\langle\cdots\rangle$ involves an average over the disorder *and* the considered site i. The proper way of defining $\langle x^2(t)\rangle$ is now

$$\langle x^2(t)\rangle=\left\langle \frac{\sum_j p_{i,j}(t)(j-i)^2}{\sum_j p_{i,j}(t)} \right\rangle, \tag{5}$$

only taking into account surviving particles. With these new definitions, $\langle x^2(t)\rangle \sim t^{2\nu}$, with $2\nu\approx1.25$, seemingly independent of γ and the disorder strength β [10]. $P(t)$ decays as a power law, $P(t)\sim t^{-\alpha}$, with $\alpha\approx0.59$, also independent of γ and β. Due to trapping, the survival return probability $P_s(t)=\langle p_{i,i}(t)\rangle$ decays much faster, and the authors of Ref. [10] gave a heuristic argument leading to

$$\ln P_s(t)\sim-\sqrt{t}, \tag{6}$$

in qualitative agreement with numerical simulations. Moreover the probability distributions of quantities such as $p_{i,j}(t)$ have been shown numerically to be very broad, leading to non-self-averaging effects.

Some of the peculiar properties of this model have been related to the possible existence of multifractal distributions for quantities such as the escape probability (see below) [9], in analogy [11] to what has been observed for the Sinai model [8]. For instance, let us consider the probability $G_{i,i+1}(t)$ that a particle makes a first passage from a site i to a site $i+1$ in t steps. This obeys the master equation [9]

$$G_{i,i+1}(t)=w_{i,i+1}\delta_{t,1}+w_{i,i-1}G_{i-1,i+1}(t) + \gamma(1-w_{i,i+1}-w_{i,i-1})G_{i,i+1}(t-1), \tag{7}$$

with the boundary conditions

$$G_{0,1}(t)=w_{0,1}\delta_{t,1}+\gamma(1-w_{0,1})G_{0,1}(t-1), \tag{8}$$

and $G_{i,i+1}(t)=0$, for $i<0$. Now the escape probability is defined as

$$x_i=\sum_{t=0}^{+\infty} G_{i,i+1}(t), \tag{9}$$

and the authors of Ref. [9] claimed to have found a multifractal distribution for this quantity, after averaging over i and the quenched disorder.

In Sec. III we show analytically that this escape probability has in fact a well defined distribution, and that the apparent multifractality can be ascribed to peculiar features of this distribution. In addition, we use some of these results in Sec. IV to derive the exact large time behavior of $P_s(t)$, by computing the properties of the survival probability distribution. We also give an exact bound on the density of particles, which fully confirms our result:

$$\ln P_s(t)\sim-t^{1/3}\ln^{2/3}(t). \tag{10}$$

Sections III C and IV B, where a rigorous method to analyze Lifshitz-like tails is introduced, are rather technical. The reader more interested in the physical consequences of these results could skip these technicalities and take the qualitative arguments given in the beginning of these sections for granted.

III. ESCAPE PROBABILITY DISTRIBUTION: NONSYMMETRIC CASE

A. Preliminaries

Let us show that x_{i+1} can be simply evaluated from the knowledge of x_i, a calculation already appearing in Refs. [12,9]. We define the generating functions

$$\hat{G}_{i,j}(z)=\sum_{t=0}^{+\infty} z^t G_{i,j}(t), \tag{11}$$

with $x_i=\hat{G}_{i,i+1}(z=1)$. The convolution theorem ensures that

$$\hat{G}_{i-1,i+1}(z)=\hat{G}_{i-1,i}(z)\hat{G}_{i,i+1}(z). \tag{12}$$

Using the master equation (7) this straightforwardly leads to the iterated map

$$x_i=\frac{p_i}{1-\gamma(1-p_i-q_i)-q_i x_{i-1}}, \quad i\geqslant1 \tag{13}$$

and

$$x_0=\frac{p_0}{1-\gamma(1-p_0)}, \quad i\geqslant1, \tag{14}$$

where $p_i=w_{i,i+1}$ and $q_i=w_{i,i-1}$ are random variables of identical distribution ρ given in Eq.(3). Note that these variables are independent only in the nonsymmetric model, as the extra constraint $q_i=p_{i-1}$ holds in the symmetric case.

At the cost of lengthier calculations, we have checked that the asymptotic behaviors of the observables of interest are not affected by the value of γ, provided trapping is not suppressed ($\gamma<1$). This is in agreement with the numerical simulations performed in Refs. [9,10]. From now on, we therefore restrict our analytic study to $\gamma=0$. In the case $\gamma=0$, the maximum possible value for x should satisfy

$$x_{\max}=\frac{p_{\max}}{1-q_{\max}x_{\max}}, \tag{15}$$

with $p_{\max}=q_{\max}=\frac{1}{2}$, leading to $x_{\max}=1$.

B. Equation for the stationary distribution: Nonsymmetric case

As noted above, the p_i's and q_i's are independent variables in the nonsymmetric case. Let us now assume that for large i the probability distribution of x_i exists and becomes independent of i, in contradiction with the numerical claim of Ref. [9]. The stationarity of the distribution is exploited by expressing that the distribution of x_i should be the same as that of x_{i-1} (at least for large i), leading to

$$f(x)=\int_0^1 dy\int_0^{1/2}dp\int_0^{1/2}dq f(y)\rho(p)\rho(q)\,\delta\!\left(x-\frac{p}{1-qy}\right). \tag{16}$$

This equation, though apparently very complicated, can be in fact exploited quite precisely. First consider $x\leqslant\frac{1}{2}$, $p=x(1-qy)$ is then always within the integration domain $[0,\frac{1}{2}]$, for any $q\in[0,1/2]$ and any $y\in[0,1]$. We then find

$$f(x)=\int_0^1 dy\int_0^{1/2}dq\, f(y)\rho(x(1-qy))\rho(q)(1-qy), \tag{17}$$

$$=2^{1-\beta}(1-\beta)x^{-\beta}\times\int_0^1 dy\int_0^{1/2}dq f(y)(1-qy)^{1-\beta}\rho(q), \tag{18}$$

that is a pure power-law behavior.

For $x>\frac{1}{2}$, imposing $p=x(1-qy)\in[0,\frac{1}{2}]$ leads to new integration bounds:

$$f(x)=2^{2(1-\beta)}(1-\beta)^2x^{-\beta}\int_{2-x^{-1}}^1 dy\times\int_{(1-(2x)^{-1})y^{-1}}^{1/2}dq\, f(y)(1-qy)^{1-\beta}q^{-\beta}, \tag{19}$$

$$=2^{2(1-\beta)}(1-\beta)^2x^{-\beta}\int_{2-x^{-1}}^1 dy\times\int_{1-(2x)^{-1}}^{y/2}dq\, f(y)y^{\beta-1}(1-q)^{1-\beta}q^{-\beta}. \tag{20}$$

Introducing $g(x)=x^{\beta}f(x)$, one can differentiate this equation twice with respect to the variable x, leading to the following nonlocal differential equation for g, valid for $x\in[\frac{1}{2},1]$:

$$g''(x)+\left[\frac{3-\beta}{x}+\frac{\beta}{x^2(2-x^{-1})}\right]g'(x)=(1-\beta)^2x^{\beta-5}[2-x^{-1}]^{-(1+\beta)}g(2-x^{-1}). \tag{21}$$

The structure of this equation is quite unfamiliar as the left-hand-side (LHS) linear differential operator is determined by the value of g at $x'=2-x^{-1}$ on the right-hand side (RHS).

$g(x)=x^{\beta}f(x)$ being zero for $x'<0$ (that is, $x\in[0,\frac{1}{2}[$), this equation actually reproduces that $g'(x)$ should be zero for $x\in[0,\frac{1}{2}[$, in agreement with Eq. (18). Then, the knowledge of $g(x')$ for $x'\in[0,\frac{1}{2}[$ determinates the LHS on the interval $x\in[\frac{1}{2},\frac{2}{3}]$. g is then determined by imposing that it is continuous at $x=\frac{1}{2}$, and a consistency equation at $x=\frac{2}{3}$ (see below). This procedure can be iterated. Consider $u_0=0$ and $u_{n+1}=1/(2-u_n)$ (that is $u_n=2-u_{n+1}^{-1}$); the knowledge of g on the interval $[u_{n-1},u_n[$, and the continuity condition at $x=u_n$ for g and its first derivative, fully determine the function g on the next interval $[u_n,u_{n+1}[$. u_n can be exactly computed by induction, leading to

$$\varepsilon_n=1-u_n=\frac{1}{n+1}. \tag{22}$$

As can be expected, $u_n\to x_{\max}=1$ when $n\to+\infty$.

This recursion process and the form of Eq. (21) ensures that g is infinitely differentiable on the interval $]u_n,u_{n+1}[$. Moreover, if g is d_{n+1} times differentiable at $x=u_{n+1}$, it is at least $d_{n+1}+1$ times differentiable at $x=u_{n+2}$. This shows that the continuity condition for g and its first derivative suffices to determine g on the interval $[u_n,u_{n+1}[$ for $n>1$. However, the knowledge of g on the interval $[0,\frac{1}{2}[$ is not sufficient to determine g on the next interval $[\frac{1}{2},\frac{2}{3}[$, as the derivative of g is not continuous at $x=\frac{1}{2}$. For instance, for $\beta>0$, one has $g'(\frac{1}{2}^-)=0$, whereas it can be shown that $g'(1/2+\varepsilon)\sim\varepsilon^{-\beta}\ln(\varepsilon)$. As the differential equation in Eq. (21) is of the second order type, one of the integration constants remains unknown, the other being determined by the continuity condition at $x=1/2$. We are thus left with a classical shooting problem, where $g'(\frac{2}{3})$ will be fixed by asking that the distribution vanishes at $x=1$.

Let us make this point clearer in the case $\beta=0$, for which the first iteration can be explicitly performed starting from $g(x)=a=g(0)$ for $x\in[0,\frac{1}{2}[$, where the constant a is given in Eq. (18). This constant can be eventually calculated once the full distribution is known up to this overall constant, as it will ensure the proper normalization of the distribution $f(x)=x^{-\beta}g(x)$. For $x\in[\frac{1}{2},\frac{2}{3}]$, we obtain

$$\frac{g(x)}{g(0)}=\frac{1}{4x^2}+\frac{1}{x}-2-2\left(1-\frac{1}{4x^2}\right)\left(\ln\left[\frac{x}{2(2x-1)}\right]+c\right), \tag{23}$$

where $c=-\frac{8}{27}\times g'(2/3)/g(0)$. Note that in this case, we indeed find that $g'(1/2+\varepsilon)\sim\ln(\varepsilon)$, leading to an infinite derivative at $x=(\frac{1}{2})^+$. This form for g, valid on the interval $[\frac{1}{2},\frac{2}{3}]$, the result of Eq. (21) and the continuity condition for g and g' at u_n (for $n>1$) leads to the full determination of g. Then a proper choice of the constant c ensures that $f(1)=0$.

In Fig. 1, $g(x)$ [equal to $f(x)$ for $\beta=0$] is shown for different values of c. For the optimal choice for c, it coincides perfectly with the distribution obtained by directly iterating Eq. (13). In Fig. 2, $g(x)=x^{\beta}f(x)$ is shown for $\beta=-\frac{1}{2}$ (weak disorder) and $\beta=\frac{1}{2}$ (strong disorder). The small x behavior for f is confirmed, and the distributions are again in perfect agreement with the numerical integration of Eq. (21).

C. Lifshitz tail at the edge of the spectrum

Note that all these distributions seem to vanish well before $x_{\max}=1$. We will show below that this is not the case, and that this apparent behavior can be accounted by the fact that

$$f(1-\varepsilon)\sim\varepsilon^{-4/\varepsilon}\quad\text{when}\quad\varepsilon\to0. \tag{24}$$

We shall see in Sec. III E that the apparent multifractal properties of the x_i's observed in Ref. [9] are partly due to this phenomenon.

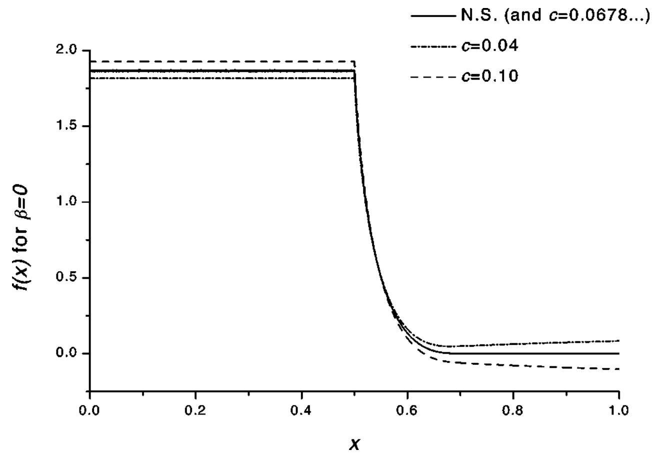

FIG. 1. In the case $\beta=0$ $[f(x)=g(x)]$, we plot the distribution obtained by iterating Eq. (13) 2×10^9 times (N.S.), and the solution of Eq. (21) obtained by imposing $f(1)=0$, which leads to $c=0.0678\ldots$ (see text). The two curves are indiscernable. We also plot the normalized solution of Eq. (21) for $c=0.04$ and 0.10.

Let us give a qualitative justification of Eq. (24). Taking $x=1-\varepsilon$, and expecting a very fast decay of g at $x=1$, the two most singular terms in Eq. (21) should be g'' and the RHS. It can then be checked that the *ansatz* $g(1-\varepsilon)\sim\varepsilon^{-\alpha/\varepsilon}$ is the solution of Eq. (21) (up to subleading multiplicative logarithmic terms) if one takes $\alpha=4$.

Still, actually solving Eq. (21), even in the limit $x\to1$, remains a formidable task and the previous argument should be taken with care. However, we can justify rigorously the fast decaying tail of f at $x=1$, finding a result fully compatible with Eq. (24).

Consider $P(\varepsilon)=\int_{1-\varepsilon}^{1}f(x)\,dx$, the probability to have $x_i>1-\varepsilon$. In a way similar to that leading to Eq. (20), we find

$$P(\varepsilon)=\int_{1-h(\varepsilon)}^{1}dy\int_{(1-h(\varepsilon))/2y}^{1/2}dq \times\int_{(1-\varepsilon)(1-qy)}^{1/2}dp\, f(y)\rho(p)\rho(q), \qquad (25)$$

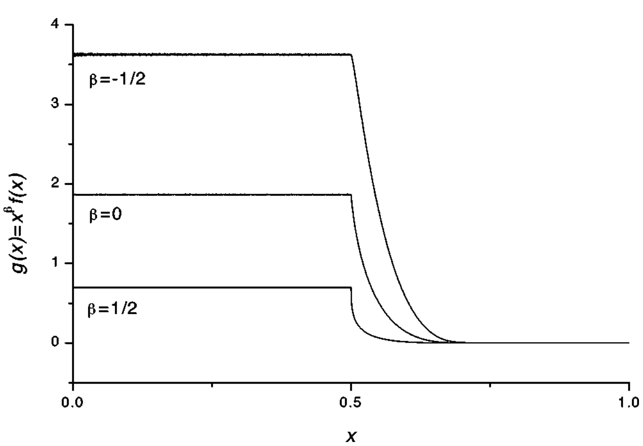

FIG. 2. We plot $g(x)=x^{\beta}f(x)$ for $\beta=-\frac{1}{2}$, 0, and $\frac{1}{2}$. In each case, the distribution is obtained by iterating the map of Eq. (13) 2×10^9. The agreement with the numerical solution of Eq. (21) is perfect.

where $h(\varepsilon)=\varepsilon/(1-\varepsilon)$. For small ε, we expect $P(\varepsilon)$ to be very small as p and q are to be taken close to $\frac{1}{2}$ and, simultaneously, y must be close to 1 [see Eq. (13)]. In the vicinity of $(p,q)=\frac{1}{2}$, the distribution ρ is smooth and roughly constant [see Eq. (3)]. We thus obtain

$$P(\varepsilon)\sim4(1-\beta)^2\int_{1-h(\varepsilon)}^{1}dy\int_{(1-h(\varepsilon))/2y}^{1/2}dq \times\int_{(1-\varepsilon)(1-qy)}^{1/2}dp f(y), \qquad (26)$$

$$\sim\frac{(1-\beta)^2}{2}\int_{1-h(\varepsilon)}^{1}dy\, f(y)[1-y-h(\varepsilon)]^2. \qquad (27)$$

From now on, we use the symbol $\sim$ in its true mathematical sense, such that $a(\varepsilon)\sim b(\varepsilon)$ means that $a(\varepsilon)/b(\varepsilon)\to1$, when $\varepsilon\to0$. For sufficiently small ε, this leads to the upper bound

$$P(\varepsilon)\leqslant a_+\int_{1-h(\varepsilon)}^{1}dy\, f(y)[1-y-h(\varepsilon)]^2, \qquad (28)$$

valid for any constant $a_+>(1-\beta)^2/2$. We then obtain

$$P(\varepsilon)\leqslant a_+\int_{1-h(\varepsilon)}^{1-\varepsilon}dy\, f(y)[\varepsilon-h(\varepsilon)]^2 +a_+\int_{1-\varepsilon}^{1}dy\, f(y)h(\varepsilon)^2, \qquad (29)$$

where we have used the fact that

$$[1-y-h(\varepsilon)]^2\leqslant[\varepsilon-h(\varepsilon)]^2\sim\varepsilon^4 \quad \text{for } y\in[1-h(\varepsilon),1-\varepsilon], \qquad (30)$$

$$[1-y-h(\varepsilon)]^2\leqslant h(\varepsilon)^2\sim\varepsilon^2 \quad \text{for } y\in[1-\varepsilon,1]. \qquad (31)$$

Thus there exist two constants a_1 and a_2, such that

$$P(\varepsilon)\leqslant a_1\varepsilon^4P(h(\varepsilon))+a_2\varepsilon^2P(\varepsilon). \qquad (32)$$

Finally, this last inequality shows that for ε sufficiently small, there exists a constant $A_+>0$ such that

$$P(\varepsilon)\leqslant A_+\varepsilon^4P(h(\varepsilon)). \qquad (33)$$

On the other hand, using again Eq. (27) and choosing a sufficiently small δ to be determined below, we have

$$P(\varepsilon)\geqslant\frac{(1-\beta)^2}{2}\int_{1-h(\varepsilon)(1+\delta)}^{1}dy\, f(y)[1-y-h(\varepsilon)]^2, \qquad (34)$$

$$\geqslant A_-P(h(\varepsilon)(1+\delta))\varepsilon^2\delta^2, \qquad (35)$$

again valid for any constant $A_-<(1-\beta)^2/2$, for small enough ε. In the following, we take $\delta=\varepsilon^{\alpha}$, and will fix the constraint on α later.

Let us now start from a small enough ε such that the inequalities in Eqs. (33) and (35) hold. Taking $\varepsilon_0^+=\varepsilon_0^-=\varepsilon$, we then define ε_n^+ and ε_n^- by the recursion relations

$$\varepsilon_n^+=h(\varepsilon_{n+1}^+)=\frac{\varepsilon_{n+1}^+}{1-\varepsilon_{n+1}^+},\quad \varepsilon_n^-=\varepsilon_{n+1}^-\frac{1+(\varepsilon_{n+1}^-)^\alpha}{1-\varepsilon_{n+1}^-}. \tag{36}$$

Both sequences go to $\varepsilon=0$. These recursion relations can also be rewritten

$$\frac{1}{\varepsilon_{n+1}^+}-\frac{1}{\varepsilon_n^+}=1, \tag{37}$$

$$\frac{1}{\varepsilon_{n+1}^-}-\frac{1}{\varepsilon_n^-}=1+O(\varepsilon_{n+1}^{-\,(\alpha-1)}). \tag{38}$$

Thus, for any $\alpha>1$, both sequences are equivalent to n^{-1} (by applying Cesaro's mean theorem):

$$\varepsilon_n^+\sim\varepsilon_n^-\sim\frac{1}{n}. \tag{39}$$

Then, by iterating the recursion relations of Eqs. (33) and (35), we obtain

$$\begin{aligned}\ln(P(\varepsilon_n^+))&\leqslant 4\sum_{k=0}^{n-1}\ln(\varepsilon_n^+)+n\ln(A_+)+\ln(P(\varepsilon))\\ &\sim -4n\ln(n),\end{aligned} \tag{40}$$

$$\begin{aligned}\ln(P(\varepsilon_n^-))&\geqslant 2(1+\alpha)\sum_{k=0}^{n-1}\ln(\varepsilon_n^-)+n\ln(A_-)+\ln(P(\varepsilon))\\ &\sim -2(1+\alpha)n\ln(n).\end{aligned} \tag{41}$$

Finally, using Eqs. (37) and (38) and the fact that $P(\varepsilon)$ is a continuous and increasing function of ε, and since α can be arbitrary close to 1, we obtain that for any arbitrary small $\eta>0$, there exists $\hat{\varepsilon}>0$, such that for any $0<\varepsilon<\hat{\varepsilon}$,

$$-\frac{4(1-\eta)}{\varepsilon}\ln(\varepsilon)\leqslant-\ln(P(\varepsilon))\leqslant-\frac{4(1+\eta)}{\varepsilon}\ln(\varepsilon), \tag{42}$$

which leads to our final result,

$$-\ln(P(\varepsilon))\sim-\ln(f(1-\varepsilon))\sim-\frac{4}{\varepsilon}\ln(\varepsilon)\quad\text{when }\ \varepsilon\to 0, \tag{43}$$

which is a more precise and rigorous statement than that of Eq. (24). It can also be shown that the subleading terms in Eq. (43) are *a priori* of order $\ln(\ln(1/\varepsilon))/\varepsilon$. In practice, these strong subleading corrections and the very fast decay of the distribution near $x=1$ make the quantitative numerical confirmation of Eq. (43) quite difficult.

D. Escape probability distribution: symmetric case

In this section we are interested in the symmetric version of our model for which $q_i=p_{i-1}$. The approach is slightly different from that of the previous case, but is definitively in the same spirit. As a consequence, less attention will be paid to rigorous arguments, although they can be adapted without any difficulty to this problem.

The map now becomes

$$x_i=\frac{p_i}{1-p_{i-1}x_{i-1}}, \tag{44}$$

which shows that the novel variable $y_i=p_ix_i$ satisfies the recursion

$$y_i=\frac{p_i^2}{1-y_{i-1}}. \tag{45}$$

$u=p^2$ has a distribution $\sigma(u)$ satisfying

$$\begin{aligned}\sigma(u)&=\int_0^{1/2}dw\,\rho(w)\,\delta(u-w^2)\\ &=2^{-\beta}(1-\beta)u^{-(1+\beta)/2}\theta(u)\,\theta(1/4-u).\end{aligned} \tag{46}$$

Using Eq. (45), we find that y is always in $[0,\frac{1}{2}]$, and that for $y\in[0,\frac{1}{4}]$, the probability distribution $F(y)$ of y satisfies

$$F(y)=2^{-\beta}(1-\beta)y^{-(1+\beta)/2}\int_0^{1/2}F(y')(1-y')^{(1-\beta)/2}\,dy', \tag{47}$$

which is a pure power-law behavior. For $y>\frac{1}{4}$, and proceeding along the same line as in the nonsymmetric case, we find that $F(y)$ satisfies the following nonlocal differential equation:

$$F'(y)+\frac{1+\beta}{2y}F(y)=\frac{1-\beta}{8y^3}F\left(1-\frac{1}{4y}\right). \tag{48}$$

Again, the knowledge of $F(y)$ on the interval $[0,\frac{1}{4}]$ permits the determination of the distribution on the next interval $[\frac{1}{4},\frac{1}{3}]$ and by recursion on each of the intervals of the form $[u_n,u_{n+1}]$, with $u_n=n/2(n+1)$.

Now, let us analyze the behavior of $P(\varepsilon)=\int_{1/2-\varepsilon}^{1/2}F(y)\,dy$, for small ε. Defining $h(\varepsilon)=\varepsilon/(1-2\varepsilon)$, we easily find that there exist two constants c and C (which can be actually determined) such that

$$P(\varepsilon)\sim c\int_0^{h(\varepsilon)}(h(\varepsilon)-z)F(1/2-z)\,dz\sim C\varepsilon^2P(h(\varepsilon)). \tag{49}$$

The last estimate is obtained using the same types of inequalities as in the nonsymmetric case. Again defining, $\varepsilon_{n+1}=\varepsilon_n/(1+2\varepsilon_n)$, and using the fact that $\varepsilon_n\sim(2n)^{-1}$, we find that

$$\ln(P(\varepsilon_n))\sim 2\sum_{k=1}^{n}\ln(\varepsilon_k)\sim\frac{\ln(\varepsilon_n)}{\varepsilon_n}, \tag{50}$$

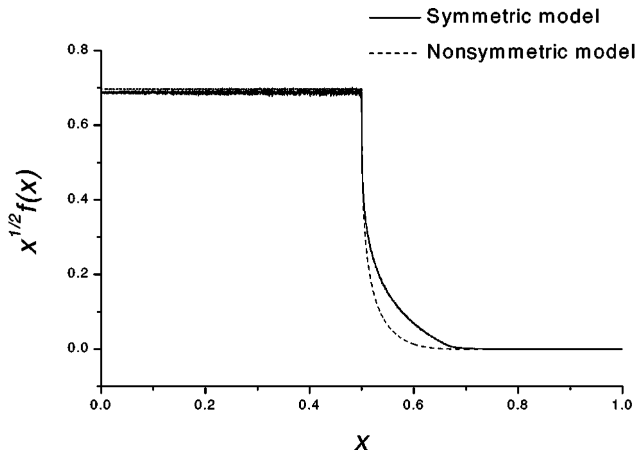

FIG. 3. We plot $g(x)=x^{\beta}f(x)$, for $\beta=\frac{1}{2}$, and after 2×10^9 iterations of the symmetric and nonsymmetric maps. As explained in the text, the distribution decays faster in the nonsymmetric case.

which finally shows that

$$\ln(P(\varepsilon))\sim\ln(F(1/2-\varepsilon))\sim\frac{\ln(\varepsilon)}{\varepsilon}. \tag{51}$$

From the knowledge of the properties of $F(y)$, we have access to that of $f(x)$, the stationary distribution of the x_i's, using the relation

$$f(x)=\int_0^{1/2}dp\int_0^{1/2}dy\,\rho(p)F(y)\,\delta\left(x-\frac{p}{1-y}\right). \tag{52}$$

Let us exhibit the main properties of f, which will be completely similar to those obtained in the nonsymmetric version of the model. For $x\in[0,1/2]$, Eq. (52) leads to

$$f(x)=2^{1-\beta}(1-\beta)x^{-\beta}\int_0^{1/2}F(y)(1-y)^{1-\beta}dy, \tag{53}$$

which shows that $f(x)$ is simply proportional to $x^{-\beta}$ as in the nonsymmetric case. For $x\in[\frac{1}{2},1]$, we obtain

$$g'(x)=(x^{\beta}f(x))'=\frac{1-\beta}{2x^{3-\beta}}F\left(1-\frac{1}{2x}\right). \tag{54}$$

Using Eq. (51), we finally conclude that

$$\ln(f(1-\varepsilon))\sim\ln\left(F\left(1-\frac{1}{2(1-\varepsilon)}\right)\right)\sim\frac{2}{\varepsilon}\ln(\varepsilon), \tag{55}$$

which is a similar behavior to that found in the nonsymmetric case [see Eq. (43)] up to the factor 4 which is replaced by a factor 2. The physical interpretation of this smaller coefficient is quite clear: in the nonsymmetric case, for x_i to be close to 1, one must have x_{i-1}, p_i, and q_i close to their maximal value. In the symmetric case, for x_i to be close to 1, only x_{i-1} and p_i must be close to their maximal value, as $q_i=p_{i-1}$ is automatically close to 1/2 as x_{i-1} is close to 1. This extra constraint explains why $f(x)$ decays faster in the nonsymmetric case than in the symmetric case, which is confirmed numerically in Fig. 3. Note that the nonsymmetric case could have been treated by the same techniques as in this subsection by replacing the distribution σ of p_i^2 with the distribution $\hat{\sigma}$ of $p_i q_{i+1}$.

Finally, we can conclude that up to a few irrelevant details, the symmetric and nonsymmetric models seem to share exactly the same properties. This is apparently surprising, since the corresponding models without trapping are drastically different [7]. This intriguing property is confirmed and explained physically in Sec. IV D.

E. Explanation of the apparent multifractality

In the preceding subsections, we have obtained a puzzling numerical result (see Figs. 1–3): although we have shown that the maximum value $x_{\max}=1$ must be attained, the numerical maximum value effectively obtained after 2×10^9 iterations of the map Eq. (13) is typically of order $x_{\max,\text{eff}}\approx 0.73\sim 78$, for the three values of β actually tested in the nonsymmetric case. This apparent paradox can be explained by the sharp decay of the distribution $f(x)$. $x_{\max,\text{eff}}$ can be estimated by considering that after N iterations of the map,

$$\int_{x_{\max,\text{eff}}}^1 f(x)\,dx\sim N^{-1}, \tag{56}$$

which is the smallest nonzero value that this integral can take (the integrated distribution increasing by elementary steps of height N^{-1}). Using Eq. (43), we find that $\varepsilon=1-x_{\max,\text{eff}}$ must satisfy

$$-\frac{\ln(\varepsilon)}{\varepsilon}\sim\frac{1}{4}\ln(N). \tag{57}$$

If we now take $N=2\times 10^9$, the above estimate gives $x_{\max,\text{eff}}\approx 0.75$, in fair agreement with the observed range of numerical effective values for $x_{\max}$.

In the limit of very large N, Eq. (43) also leads to the leading order estimate

$$1-x_{\max,\text{eff}}\approx\frac{4\ln(\ln(N))}{\ln(N)}, \tag{58}$$

which goes to zero very slowly.

In Ref. [9], the fact that $x_{\max}=1$ was not recognized. The authors actually computed the multifractal distribution of the x_i's in the interval $[0,x_{\max,\text{eff}}]$. This interval was cut into n equal length intervals, and ρ_j was defined as the fraction of the total number of the x_i's that belongs to the jth interval. One then defines [13]

$$Z(q,n)=\sum_{i=1}^{n}\rho_i^q\sim n^{-\tau(q)}, \tag{59}$$

the last equivalent defining the scaling exponent $\tau(q)$ associated with the qth moment. Let us first derive the exact expression of $\tau(q)$ for a given choice of $\beta>0$ (strong disorder), and taking the actual value of $x_{\max}=1$. For $0\leqslant q<1/\beta$, the function $f(x)^q$ is integrable, so that

$$Z(q,n)=n^{-(q-1)}\times\frac{1}{n}\sum_{i=1}^{n}(n\rho_i)^q\sim n^{-(q-1)}\int_0^1 f(x)^q\,dx. \tag{60}$$

This shows that, in this regime, $\tau(q)=q-1$. For $q>1/\beta$, the function $f(x)^q$ is no longer integrable due to the power-law divergence at $x=0$:

$$Z(q,n)=n^{-(q-1)}\times\frac{1}{n}\sum_{i=1}^{n}(n\rho_i)^q\sim n^{-(q-1)}\int_{1/n}^1 f(x)^q\,dx \sim n^{-(1-\beta)q}. \tag{61}$$

In this regime, we thus find $\tau(q)=(1-\beta)q$. Moreover, for $q=1/\beta$, we find

$$Z(q=1/\beta,n)\sim n^{-(1/\beta-1)}\int_{1/n}^1 f(x)^{1/\beta}\,dx\sim n^{-(1/\beta-1)}\ln(n), \tag{62}$$

which shows that in the region $q\approx 1/\beta$, the numerical determination of $\tau(q)$ will be strongly affected by a logarithmic slow crossover. Finally, strictly speaking, $\tau(q)$ is not defined for negative q, due to the essential singularity at $x_{\max}=1$. But if one computes the multifractal scaling exponents by restraining the study on the interval $[0,x_{\max,\mathrm{eff}}]$ (as in Ref. [9]), we then recover $\tau(q)=q-1$, for negative values of q as well.

The two linear regimes for $\tau(q)$ are clearly visible in Fig. 4 of Ref. [9], with the predicted slopes and transition point $q=1/\beta$. Of course, numerically, the change in slope [from $\tau'(q)=1$ to $\tau'(q)=1-\beta$] is found to be smooth, partly due to the logarithmic correction around $q=1/\beta$. Then, the spectrum of singularities defined as the Legendre transform of $\tau(q)$ [13],

$$s(\alpha)=\alpha q-\tau(q),\qquad \alpha=\tau'(q) \tag{63}$$

apparently yields a nontrivial spectrum, whereas the actual one is concentrated on two points: $\alpha=1$, with a support of fractal dimension $s(1)=1$, reflecting that for almost all values of x the distribution $f(x)$ is actually continuous, and $\alpha=1-\beta$, with a support of fractal dimension $s(1-\beta)=0$, which simply results from the $x=0$ singularity of the distribution $f(x)$.

The moral that we can draw from this is that one must be very careful when dealing with multifractal analysis, especially if there is no special reason to expect a multifractal spectrum. Similar problems were encountered by the author of Ref. [14], who obtained an apparent multifractal spectrum in a model which can actually be solved exactly [15], and for which it can be shown that the true multifractal spectrum is of the same type as above. Similar doubts can be raised on the findings of multifractal spectra in certain biological systems or in the field of finance [16], where simple power-law distributions can lead to such apparent behaviors.

IV. LARGE TIME BEHAVIOR OF THE SURVIVAL RETURN PROBABILITY

A. General results

Following the tracks of Ref. [10], let us evaluate the survival return, or more exactly, its discrete time Laplace transform. We consider the symmetric model but the nonsymmetric case can be treated in the same spirit, leading to exactly the same results.

Thus consider

$$\hat{p}_{0,i}(\omega)=\sum_{t=0}^{+\infty}\frac{p_{0,i}(t)}{(1+\omega)^{t+1}}. \tag{64}$$

It can be shown [10] that $\hat{p}_{0,i}(\omega)$ satisfies the equation [see Eq. (7)]

$$\hat{p}_{0,i}(\omega)=w_{i-1,i}\hat{p}_{0,i-1}(\omega)+w_{i,i+1}\hat{p}_{0,i+1}(\omega)+(1+\omega)^{-1}\delta_{i,0}. \tag{65}$$

Then, the variables $\phi_i^+(\omega)$ and $\phi_i^-(\omega)$, defined, respectively, for $i>0$ and $i<0$ by

$$\phi_i^+(\omega)=\frac{w_{i-1,i}}{1+\omega}\cdot\frac{\hat{p}_{0,i}(\omega)}{\hat{p}_{0,i-1}(\omega)},$$

$$\phi_i^-(\omega)=\frac{w_{i,i+1}}{1+\omega}\cdot\frac{\hat{p}_{0,i}(\omega)}{\hat{p}_{0,i-1}(\omega)}, \tag{66}$$

satisfy the same recursion, reminiscent of that of Eq. (13). For instance,

$$\phi_i^+(\omega)=\frac{\mu_{i-1}^2}{1-\phi_{i+1}^+(\omega)}\quad\text{with}\quad \mu_{i-1}=\frac{w_{i-1,i}}{1+\omega}, \tag{67}$$

with a similar equation for $i<0$. It can then be shown that

$$\langle p_{0,0}(\omega)\rangle=\int_0^{\phi_{\max}(\omega)}d\phi^+ \times\int_0^{\phi_{\max}(\omega)}d\phi^-\,\Pi_\omega(\phi^+)\Pi_\omega(\phi^-) \times\frac{\theta(1-\phi^+-\phi^-)}{1-\phi^+-\phi^-}, \tag{68}$$

where $\Pi_\omega(\phi)$, is the expected stationary distribution of ϕ^+ and ϕ^-.

From now on, we follow the method of Sec. III to evaluate the behavior of the distribution $\Pi_\omega(\phi)$ close to $\phi=\phi_{\max}$, and for small ω [as we are mainly interested in the large time behavior of $P_s(t)$]. From Eq. (67), $\phi_{\max}(\omega)$ can be easily calculated:

$$\phi_{\max}(\omega)=\frac{1}{2}(1-\sqrt{1-(1+\omega)^{-2}})=\frac{1}{2}-\sqrt{\frac{\omega}{2}}+O(\omega^{3/2}). \tag{69}$$

Moreover, using again Eq. (67), we find that $\Pi_\omega(\phi)$ satisfies the following self-consistent equation:

$$\Pi_\omega(\phi)=\int_0^{1/4}dx\int_0^{\phi_{\max}(\omega)}d\phi'\ \sigma(x)\Pi_\omega(\phi')\delta\times\left(\phi-\frac{x}{(1+\omega)^2(1-\phi')}\right), \tag{70}$$

where the distribution $\sigma(x)$ of $x=\mu^2(1+\omega)^2$ is given by Eq. (46).

Equation (67) can be first solved for $\phi<1/4(1+\omega)^2$, leading to a pure power-law behavior (as encountered in Sec. III):

$$\Pi_\omega(\phi)=2^{-\beta}(1-\beta)(1+\omega)^{-(1+\beta)}\phi^{-(1+\beta)/2}\times\int_0^{\phi_{\max}(\omega)}d\phi'\frac{\Pi_\omega(\phi)}{(1-\phi')^{(1+\beta)/2}}. \tag{71}$$

For $\phi>1/4(1+\omega)^2$, one can differentiate Eq. (70) (noticing that $\phi'\in[1-[1/4(1+\omega)^2\phi]]$), which leads to the following differential equation for $\Pi_\omega(\phi)$:

$$[\phi^{(1+\beta)/2}\Pi_\omega(\phi)]'=-\frac{1-\beta}{2}(1+\omega)^{-(5+\beta)/2}\phi^{-(3-\beta)/2}\Pi_\omega\times\left(1-\frac{1}{4(1+\omega)^2\phi}\right). \tag{72}$$

The leading asymptotics of $\Pi_\omega(\phi)$ close to $\phi=\phi_{\max}(\omega)$ can be calculated rigorously, adapting the method of Sec. III C. Here we first derive this result by a less rigorous method already mentioned in Sec. III A, consisting of keeping only the most singular terms in the differential equation (72), leading to

$$\Pi'_\omega(\phi)\sim A\Pi_\omega\left(1-\frac{1}{4(1+\omega)^2\phi}\right)\sim A\Pi_\omega\left(\phi_{\max}(\omega)-\frac{1-\phi_{\max}(\omega)}{\phi_{\max}(\omega)}\varepsilon\right), \tag{73}$$

where A is a computable positive constant, and the explicit equation for $\phi_{\max}(\omega)$ was used. One can try an ansatz form for $\Pi_\omega(\phi)$, which satisfies this equation up to multiplicative logarithmic terms. We find

$$\ln[\Pi_\omega(\phi_{\max}(\omega)-\varepsilon)]\sim-\frac{\ln^2(\varepsilon)}{2\ln(r)}\quad\text{with}\quad r=\frac{1-\phi_{\max}(\omega)}{\phi_{\max}(\omega)}. \tag{74}$$

B. Tail of the distribution

In fact, this complicated ansatz was originally found by applying a method similar to that of Sec. III C, which we present now. Again, let us define $P_\omega(\varepsilon)=\int_{\phi_{\max}(\omega)-\varepsilon}^{\phi_{\max}(\omega)}\Pi_\omega(\phi)\,d\phi$, which satisfies

$$P_\omega(\varepsilon)=\int_0^{1/4}dx\int_0^{\phi_{\max}(\omega)}d\,\phi\,\sigma(x)\Pi_\omega(\phi)\theta\times\left(\frac{x}{(1+\omega)^2(1-\phi)}+\varepsilon-\phi_{\max}(\omega)\right). \tag{75}$$

Once we express the actual domain of integration by imposing that the argument of the θ function be positive, we find that the variable x remains very close to $\frac{1}{4}$, where the distribution $\sigma(x)$ is essentially constant. Exploiting this fact, we find after a few elementary manipulations that

$$P_\omega(\varepsilon)\sim C\int_0^{r\varepsilon}P_\omega(u)\,du, \tag{76}$$

with $r=[1-\phi_{\max}(\omega)/\phi_{\max}(\omega)]>1$, and $C=\sigma(1/4)/4r\phi_{\max}(\omega)$. This equation is a rigorous integrated version of Eq. (73). Note that, for small ω, we have

$$r=1+2\sqrt{2\omega}+O(\omega^{3/2}). \tag{77}$$

We can now proceed in the same spirit as we did in Sec. III C, and find exact inequalities for $P_\omega(\varepsilon)$. For any $c_+>C$, and sufficiently small ε

$$P_\omega(\varepsilon)\leqslant c_+\int_\varepsilon^{r\varepsilon}P_\omega(u)\,du+c_+\int_0^{\varepsilon}P_\omega(u)\,du, \tag{78}$$

$$\leqslant c_+(r-1)\varepsilon P_\omega(r\varepsilon)+c_+\varepsilon P_\omega(\varepsilon), \tag{79}$$

which finally leads, for small enough ε, to the existence of a constant C_+ of order $O(1)$, such that

$$P_\omega(\varepsilon)\leqslant C_+(r-1)\varepsilon P_\omega(r\varepsilon). \tag{80}$$

On the other hand, for any $r'<1$ close to 1, to be specified later, we can write that

$$P_\omega(\varepsilon)\geqslant C_-\int_{r'r\varepsilon}^{r\varepsilon}P_\omega(u)\,du, \tag{81}$$

$$\geqslant C_-(1-r')\varepsilon P_\omega(r'r\varepsilon). \tag{82}$$

Let us now start from a small enough $\varepsilon_0=\varepsilon$ such that the inequalities of Eqs. (80) and (82) hold, and define

$$\varepsilon_n^+=r^{-1}\varepsilon_{n-1}^+=r^{-n}\varepsilon, \tag{83}$$

$$\varepsilon_n^-=(r'r)^{-1}\varepsilon_{n-1}^-=(r'r)^{-n}\varepsilon. \tag{84}$$

In the following, we will choose r' such that $r'r>1$, so that $\varepsilon_n^-\to 0$, when $n\to+\infty$.

By iterating the recursion inequalities we obtain

$$-\ln(P_\omega(\varepsilon_n^+))\geqslant-n\ln(C_+(r-1))-\sum_{k=0}^{n-1}\ln(\varepsilon_k^+)-\ln(P_\omega(\varepsilon)), \tag{85}$$

$$\sim-n\ln(C_+(r-1))+\frac{n^2}{2}\ln(r), \tag{86}$$

where, after using $n\sim -\ln(\varepsilon_n^+)/\ln(r)$, the last line can also be written

$$\frac{\ln^2(\varepsilon_n^+)}{2\ln(r)}+\frac{\ln(C_+(r-1))}{\ln(r)}\ln(\varepsilon_n^+). \tag{87}$$

Similarly,

$$-\ln(P_\omega(\varepsilon_n^-))\leqslant -n\ln(C_-(1-r'))-\sum_{k=0}^{n-1}\ln(\varepsilon_k^-) -\ln(P_\omega(\varepsilon)), \tag{88}$$

$$\sim -n\ln(C_-(1-r'))+\frac{n^2}{2}\ln(r'r), \tag{89}$$

where the last line can also be written under the form

$$\frac{\ln^2(\varepsilon_n^-)}{2\ln(r'r)}+\frac{\ln(C_-(1-r'))}{\ln(r'r)}\ln(\varepsilon_n^-). \tag{90}$$

Since r' can be arbitrarily close to 1, we thus find that the leading order of Eq. (74) is exactly recovered.

Now, we restrict ourselves to the case of small ω, and analyze the effect of the subleading term. To be specific, let us take $1-r'=\omega^{1/2+\delta}$, with δ arbitrarily small, such that the condition $r'r>1$ remains satisfied [see Eq. (77)]. Our final results are that $P_\omega^-(\varepsilon)\leqslant P_\omega(\varepsilon)\leqslant P_\omega^+(\varepsilon)$, with

$$-\ln(P_\omega^+(\varepsilon))\sim\frac{\ln^2(\varepsilon)}{4\sqrt{2\omega}}+\frac{\ln(\omega)}{4\sqrt{2\omega}}\ln(\varepsilon), \tag{91}$$

$$-\ln(P_\omega^-(\varepsilon))\sim\frac{\ln^2(\varepsilon)}{4\sqrt{2\omega}}+(1+2\delta)\frac{\ln(\omega)}{4\sqrt{2\omega}}\ln(\varepsilon), \tag{92}$$

which strongly suggests that

$$-\ln[\Pi_\omega(\phi_{\max}(\omega)-\varepsilon)]\sim -\ln(P_\omega(\varepsilon))\sim\frac{\ln(\varepsilon)\ln(\omega\varepsilon)}{4\sqrt{2\omega}}. \tag{93}$$

Note that the case $\omega=0$ falls exactly in the class of problem studied in Sec. III C, leading to the exact asymptotics [with $\phi_{\max}(0)=\frac{1}{2}$]

$$\ln(\Pi_{\omega=0}(1/2-\varepsilon))\sim\ln(P_{\omega=0}(\varepsilon))\sim\frac{\ln(\varepsilon)}{\varepsilon}. \tag{94}$$

C. Survival return probability

The results of this section will not rely on mathematical grounds as firm as that of the preceding sections, but will appear to be quite reasonable. Comparison of Eqs. (93) and (94) suggests that the asymptotics of Eq. (93) should be correct at least up to ε of order $\sqrt{\omega}\sim 1/2-\phi_{\max}(\omega)$ [or more exactly $\sqrt{\omega}/|\ln(\omega)|$]. Using Eq. (68), we find that there should be a singular contribution in $\langle p_{0,0}(\omega)\rangle$ of order [see Eq. (93)]

$$[\langle p_{0,0}(0)\rangle-\langle p_{0,0}(\omega)\rangle]_{\text{sing}}\sim\int_{\phi_{\max}(\omega)}^{1/2}\Pi_{\omega=0}(x)\,dx \sim\exp\left(\frac{c_1\ln(\omega)}{\sqrt{\omega}}\right), \tag{95}$$

where the last estimate comes from Eq. (94). In principle, the leading singular correction to Eq. (95) should come from the contribution of Π_ω to the integral of Eq. (68) on the interval $[0,\phi_{\max}]$ and should be of order $\exp[-c_2\ln^2(\omega)/\sqrt{\omega}]$.

A contribution of the form $\exp(-c|\ln(\omega)|^{\alpha'}/\omega^{\alpha})$ in a Laplace transform generally originates from a large time decay of the form (as found by a steepest descent type of argument)

$$\exp(-Ct^{\alpha/(1+\alpha)}\ln^{\alpha'/(1+\alpha)}(t)). \tag{96}$$

If we take the logarithmic corrections in Eq. (95) seriously, we thus find

$$\langle p_{0,0}(t)\rangle\sim\exp(-Ct^{1/3}\ln^{2/3}(t)), \tag{97}$$

in disagreement with the heuristic argument given in Ref. [10] [see Eq. (6)].

D. Exact bound for the survival return probability

In this subsection, we give an exact lower bound for the number of surviving particles using an argument which can be easily generalized to obtain a similar bound for $\langle p_{0,0}(t)\rangle$. The resulting bound is fully consistent with Eq. (97), and contradicts the heuristic estimate of Ref. [10] [see Eq. (6)].

Consider the symmetric model $(w_{i,i+1}=w_{i+1,i})$. The probability of having a region of L sites on which all $w_{1,2},\ldots,w_{L-2,L-1}>1/2-\varepsilon$ is

$$\mathcal{P}_L(\varepsilon)=[1-(1-2\varepsilon)^{1-\beta}]^{L-2}\sim[2(1-\beta)\varepsilon]^L. \tag{98}$$

If we were interested in the nonsymmetric model, the corresponding probability of having $w_{1,2},w_{2,1}\ldots,w_{L-2,L-1},w_{L-1,L-2}>1/2-\varepsilon$ would be simply

$$\mathcal{P}_L(\varepsilon)=[1-(1-2\varepsilon)^{1-\beta}]^{2(L-2)}\sim[2(1-\beta)\varepsilon]^{2L}, \tag{99}$$

and the rest of the argument would be essentially identical. On such a region, the density cannot decay faster than that of the following problem where we consider $w_{1,2},\ldots,w_{L-1,L}=1/2-\varepsilon$, with fully absorbing boundary conditions $(w_{0,1}=w_{1,0}=w_{L-1,L}=w_{L,L-1}=0)$. This simple property can be shown by induction using the master equation for $p_i(t)$, the density at site i:

$$p_i(t+1)=w_{i+1,i}p_{i+1}(t)+w_{i-1,i}\,p_{i-1}(t),\quad p_i(t=0)=1. \tag{100}$$

This simpler problem can be solved exactly (on the lattice or in the continuum), leading to the following bound for the average decay of the total density:

$$n_L(t)>\frac{8}{\pi^2}\exp\left(-\frac{\pi^2 t}{2L^2}-2\varepsilon t\right). \tag{101}$$

Finally, we find that the total density $n(t)$ is bounded for any L and ε by the exact lower bound

$$n(t) > C_0 \exp\left(-\frac{\pi^2 t}{2L^2} - 2\varepsilon t + L \ln(2(1-\beta)\varepsilon) \right), \qquad (102)$$

where C_0 is a positive constant. We can now take the maximum of this lower bound over L and ε. Expressing this condition, we obtain the following optimal values for L and ε defined implicitly by

$$L = 2\varepsilon t = \left[-\frac{\pi^2 t}{\ln(2(1-\beta)\varepsilon)} \right]^{1/3} \sim \left[\frac{3\pi^2 t}{2\ln(t)} \right]^{1/3}, \qquad (103)$$

where the last estimate is valid for large time.

Finally, we have shown that

$$n(t) > \exp - S(t) \quad \text{with } S(t) \sim \left[\frac{3\pi^2 t}{2} \right]^{1/3} \ln^{2/3}(t), \qquad (104)$$

in perfect agreement with the above analytical argument. The physical interpretation of this result is that surviving particles are living in large regions where the $w_{i,i\pm1}$ are very close to $\frac{1}{2}$, and annihilates with a large probability outside these regions. This explains why the symmetry of the hopping probabilities is irrelevant, and why the result is essentially similar to that of perfectly diffusing particles with randomly distributed perfect traps. The $\ln^{2/3}(t)$ corrections are due to the fact that for large time, $w_{i,i\pm1}$ in regions where the surviving particles stand must be closer and closer to $\frac{1}{2}$, with an allowed fluctuation decreasing as $\varepsilon \sim t^{-2/3}$. Moreover, within these large regions there is an extra probability of leaking per site of order ε. Note, finally, that if $0<\gamma<1$, the argument can be repeated with εt replaced by $(1-\gamma)\varepsilon t$ in Eq. (101), leading to the same decaying behavior.

Let us now exhibit the flaw(s) in the argument given in Ref. [10], leading to Eq. (6). On a large region of size L with $w_{i,i\pm1} > 1/2 - \varepsilon$, the authors of Ref. [10] estimated the probability decay as

$$n_L(t) \sim \exp\left(-\frac{t}{L^{1/\nu}} - \varepsilon L^{1/\nu} \right), \qquad (105)$$

where ν is the effective diffusion exponent defined in Eq. (5) and below. This estimate is to be compared to our Eq.(101). The first term is supposed to represent the probability decay due to the absorption of particles at the boundaries of the considered region. It is not correct to consider that particles in this region display anomalous diffusion, as $w_{i,i\pm1}$ are in fact very close to $\frac{1}{2}$. This fact is confirmed by the exact bound, Eq. (101). The second term in the exponential is supposed to reflect the fact that there is a small trapping probability (of order ε) on each site of the considered region. The authors of Ref. [10] assumed that time can be replaced by $L^{1/\nu}$. This is obviously wrong, as the decay due to this small trapping probability explicitly depends on the time spent in the region but not on its size. Finally, the authors did not realize that in the final expression that they obtained, ε (and not only L) should also be treated as a variational parameter.

Note that for intermediate times, we expect that the density should decay as

$$n(t) \sim \exp(-cN(t)) \sim \exp(-Ct^{1/2}), \qquad (106)$$

where $N(t)$ is the number of different sites visited by a random walker. This phenomenon is also well known for the Donsker-Varadhan problem, for which this behavior can actually dominate the numerically accessible time regime [4,3].

Finally, the generalization to higher dimensions of this model is straightforward. On a periodic lattice of coordination number z, the hopping probabilities are bounded by z^{-1}, and particles disappear if they do not move. The above argument suggests that

$$n(t) \sim \exp(-Ct^{d/(d+2)} \ln^{2/(d+2)}(t)). \qquad (107)$$

V. CONCLUSION

In this paper, we have considered a model where the trapping probabilities are strongly correlated with the hopping probabilities of the walker. We have shown that the escape probabilities have a well defined distribution which has been analyzed in great detail in Sec. III. We have also explained why this quantity displays an apparent multifractal distribution. In Sec. IV, we generalized our approach to the study of the survival return probability distribution. We deduced from this exact analysis that this survival probability (and the density) decays as $\exp(-Ct^{1/3}\ln^{2/3}(t))$. To support this result, we have obtained an exact bound for the density which even reproduces the correct power of the logarithmic correction. Moreover, we have explained the independence of the model properties with respect to the disorder strength β, the trapping rate $\gamma>0$ and, more surprisingly, the symmetry of the hopping probabilities.

A challenging problem is the understanding of the diffusion properties in this model. The fact that the effective spreading of the survivors is apparently faster than diffusive [see Eq. (5)] remains to be explained.

Note added: After this paper was released on cond-mat, the authors of Refs. [9,10] mentioned to me that they had later realized that multifractality of the first return probabilities was dubious and that their heuristic argument for $n(t)$ was wrong, as they claimed to have also found the correct one presented here (unpublished).

ACKNOWLEDGMENTS

I am grateful to F. van Wijland, C. Monthus, and S. Redner for helpful discussions on this problem, and D. Dean for useful comments on the manuscript.

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