Microscopic analysis of currency and stock exchange markets

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Recently it was shown that distributions of short-term price fluctuations in foreign-currency exchange exhibit striking similarities to those of velocity differences in turbulent flows. Similar profiles represent the spectral-diffusion behavior of impurity molecules in disordered solids at low temperatures. It is demonstrated that a microscopic statistical theory of the spectroscopic line shapes can be applied to the other two phenomena. The theory interprets the financial data in terms of information which becomes available to the traders and their reactions as a function of time. The analysis shows that there is no characteristic time scale in financial markets, but that instead stretched-exponential or algebraic memory functions yield good agreement with the price data. For an algebraic function, the theory yields truncated Lévy distributions which are often observed in stock exchange markets. [S1063-651X(99)07008-7]

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I. INTRODUCTION

Distributions of price fluctuations in currency and stock exchange markets have been the subject of interdisciplinary studies for several years. In particular the fact that they are usually of non-Gaussian shape with distinctly more pronounced wings has attracted much interest. The long, slowly decaying tails are the origin of frequent turbulences or even crashes of the markets, because extreme price fluctuations occur with much higher probability as compared to Gaussian statistics. The leptokurtic shapes can often be empirically desribed by Lévy distributions [1,2] or truncated Lévy distributions [3,4]. An analysis of a very large data base which comprised all stock transactions in three major U.S. stock markets during a two-year period yielded an inverse-power-law distribution with an exponent close to -3 [5]. These distributions are mainly valid for short-term price fluctuations with time intervals of some minutes. Similar histograms of price changes were also obtained from the numerical study of a prototype model of a self-organized stock market, although in this case they exhibited a sharp tip at the center [6]. The influence of interactions between the traders on non-Gaussian shapes of price distributions was investigated by several authors [6–8]. A different approach was based on the Langevin equation [9].

Recently it was found that the short-term price fluctuations (from 10 min up to two days) in currency exchange between U.S.$ and German marks show a remarkably similar behavior as velocity differences in turbulent flows [10]. In both fields, the distributions change their shapes in a characteristic manner between very short time differences (currency exchange) or spatial distances (turbulence) and longer periods: The wings become less and less pronounced, and finally Gaussian profiles are approached. The histograms could be well reproduced with superpositions of Gaussians whose variances obey a log-normal distribution. The convergence toward a Gaussian shape corresponds to a decrease of the log-normal variance for increasing time intervals or spatial distances. The authors explained the similarity in the behavior as being due to turbulent cascades in both phenomena: Turbulent flows are characterized by *energy* cascades from larger to smaller vortices, and for the financial data a transfer of *information* from long- to short-term traders was invoked [10] (see also Ref. [11]).

Although the stochastic nature of turbulent flows and financial data was shown to be quantitatively different [12,13], the analogy in the behavior of the respective distribution functions is striking. Similar statistical distributions are also known in a completely different field of physics, namely, inhomogeneous spectroscopic line shapes of impurity molecules in solids. It will be shown in the following that a microscopic statistical theory of the line shapes can be used to describe the financial and turbulent-flow data as well, and that the similarities between the three phenomena have a very fundamental statistical origin. For the case of the financial data, the distribution of price fluctuations is obtained as the functional of a memory function, according to which the influence of new information on the decisions of the traders decreases with time. All observed distributions of real market data can be reproduced with different forms of the memory function. Algebraic functions, for example, yield the analytical result of truncated Lévy profiles. The paper is organized as follows. Section II contains a brief review of the spectroscopic line shape theory. In Sec. III the theory is transformed to describe financial data, and in Sec. IV the resulting distributions for some specific memory functions are discussed. The scaling behavior of these distributions is examined in Sec. V. Finally, the conclusions are given in Sec. VI.

II. MARKOFF-STONEHAM THEORY OF INHOMOGENEOUS SPECTRAL LINE SHAPES

We consider a transparent solid matrix containing an ensemble of dye molecules at low concentration such that interactions between the dye molecules can be neglected and only the coupling to the surrounding matrix units plays a

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role. The dopant molecules are assumed to have an isolated optical absorption (or emission) line whose homogeneous linewidth is negligibly narrow at low temperatures. In any real solid there are imperfections such as point defects and dislocations, so that the dye molecules are located in environments which differ from molecule to molecule to some extent. We are interested in the resulting inhomogeneous distribution of the molecular transition frequencies. A comprehensive review article on inhomogeneous-broadening effects was published by Stoneham [14]; the basic ideas date back to Markoff [15].

According to the Markoff-Stoneham theory, the inhomogeneous line shape (i.e., the distribution of molecular absorption frequencies) can be expressed by the ansatz

$$I(\nu)=\frac{1}{V^N}\int_{(V)} d\mathbf{R}_1\cdots\int_{(V)} d\mathbf{R}_N P(\mathbf{R}_1,\ldots,\mathbf{R}_N)\times\delta\left(\nu-\sum_{n=1}^{\infty}\tilde{\nu}(\mathbf{R}_n)\right). \tag{1}$$

ν is the shift of the absorption frequency with respect to the value when the molecule is in vacuum, and the function $\tilde{\nu}(\mathbf{R}_n)$ describes the interaction with a matrix molecule at position $\mathbf{R}_n$ when the dopant is located at the coordinate origin. The contributions of the matrix units to the line shift are assumed to be additive. $P(\mathbf{R}_1,\ldots,\mathbf{R}_N)$ is the combined probability of finding matrix molecule n at position $\mathbf{R}_n$ for $n=1,\ldots,N$. It is normalized as

$$\int_{(V)} d\mathbf{R}_1\cdots\int_{(V)} d\mathbf{R}_N P(\mathbf{R}_1,\ldots,\mathbf{R}_N)=V^N, \tag{2}$$

with N being the total number of constituents of the solid sample of volume V. $I(\nu)$ is then normalized to unity,

$$\int_{-\infty}^{+\infty} I(\nu)\,d\nu=1. \tag{3}$$

Since $P(\mathbf{R}_1,\ldots,\mathbf{R}_N)$ is a very complicated function which is usually not known, it is often factorized into N equal two-particle solute-solvent distribution functions:

$$P(\mathbf{R}_1,\ldots,\mathbf{R}_N)=\prod_{n=1}^{N} g(\mathbf{R}_n). \tag{4}$$

This simplification is based on the assumption that the positions of the matrix molecules are statistically independent, which means that the nonzero volume of the molecules is neglected. The index n can then be omitted. It is possible to introduce correction terms to avoid this problem [16–18], but since we want to summarize only the basic ideas of the line shape theory we will not use them here. With approximation (4), the expression for the inhomogeneous line profile (1) can be cast in the form

$$I(\nu)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, e^{i\nu x}e^{-J(x)}, \tag{5}$$

with

$$J(x)=\varrho\int_{(V)} d\mathbf{R}\, g(\mathbf{R})(1-e^{-i\tilde{\nu}(\mathbf{R})x}), \tag{6}$$

where $\varrho=N/V$ is the number density of the matrix molecules. Details of the calculation can be found in Refs. [14,19]. The distribution of absorption frequencies is given by the negative exponential of a characteristic function $J(x)$ which is a functional of the dye-matrix interaction potential $\tilde{\nu}(\mathbf{R})$.

In order to evaluate Eqs. (5) and (6), it is necessary to specify the functions $g(\mathbf{R})$ and $\tilde{\nu}(\mathbf{R})$. For the two-molecule distribution function we use the simple step form

$$g(\mathbf{R})=\begin{cases}1 & \text{for } R\geqslant R_c\\ 0 & \text{for } R<R_c,\end{cases} \tag{7}$$

which states that matrix units can be found anywhere outside a spherical cavity of radius R_c around a dye molecule but not within this cavity. For the interaction potential, we insert the dipole-dipole form

$$\tilde{\nu}(\mathbf{R})=-A\cos\theta\left(\frac{R_c}{R}\right)^3, \tag{8}$$

with θ being the polar angle of the coordinate frame, and the constant A depending on the dipole moments of the solute and the solvent molecule [19,20]. In weakly polar organic systems, electrostatic interactions are usually not the main origin of the solvent shift of dopant molecules; instead, the dispersive forces yield the main contribution [21]. Nevertheless, the dipole-dipole interaction plays an important role in a special kind of inhomogeneous broadening, namely, spectral diffusion [22,23] of hole-burning [24] spectra in glasses or polymers. Spectral diffusion consists in an increasing spreading of an ensemble of originally degenerate molecular absorption lines with time; it is due to the coupling of the dye molecules to the tunneling systems (TLS's) of the glass [25,26]. Hence in this case the parameter ϱ in Eq. (6) is not the density of the matrix molecules but the density of the TLS's which is much lower [20]. This has important consequences for the line shape $I(\nu)$, as will be discussed below.

After inserting Eqs. (7) and (8) into Eq. (6), the angular part of the $\mathbf{R}$ integration can be carried out analytically. The expression for the characteristic function then reads

$$J(x)=\frac{4\pi}{3}FA|x|\int_0^{A|x|}\left(1-\frac{\sin u}{u}\right)\frac{du}{u^2}, \tag{9}$$

with $F=\varrho R_c^3$. The remaining u integral must be evaluated numerically. Equation (9) together with Eq. (5) describes the shape of the inhomogeneous line profile $I(\nu)$ for the case of dipole-dipole interaction. The important parameter which determines the shape is F, whereas the interaction strength A influences the scaling of the frequency axis. For $F\to\infty$, only a narrow region around $x=0$ contributes significantly to the Fourier integral in Eq. (5). Since the Taylor expansion of Eq. (9) yields, for the lowest-order term,

$$\lim_{x\to 0} J(x)=\frac{2\pi}{9}F(Ax)^2, \tag{10}$$

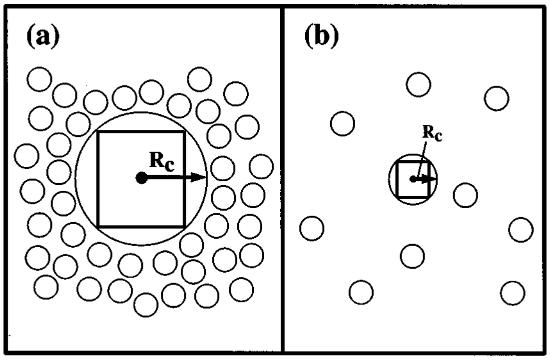

FIG. 1. Schematic representation of the two limiting cases that a dye molecule (central square) in a solid is surrounded by closely packed [part (a)] or dilute [part (b)] perturbers which determine the solvent shift of its absorption line. No perturbers can be located within a sphere of radius R_c around the dye molecule.

the resulting line shape $I(\nu)$ is Gaussian in this case. In the opposite limit $F\to 0$, on the other hand, the Fourier integral is largely determined by the asymptotic behavior of $J(x)$ for $|x|\to\infty$ which reads

$$\lim_{|x|\to\infty} J(x)=\frac{\pi^2}{3}FA|x|. \tag{11}$$

The corresponding line profile $I(\nu)$ is then of Lorentzian (or Cauchy) shape. For intermediate values of F, $I(\nu)$ can be characterized as a Lorentzian profile with cutoff wings, as was discussed in Ref. [20].

Figure 1 contains a schematic representation of the physical situations corresponding to the two limiting cases of $F \gg 1$ [part (a)] and $F \ll 1$ [part (b)]. In the first case, a large, bulky dye molecule is surrounded by much smaller matrix units so that there are many interaction partners already in the first few solvent shells. Hence the central limit theorem applies and the inhomogeneous line shape is Gaussian, irrespective of the form of the interaction potential $\tilde{\nu}(\mathbf{R})$. This situation is usually (approximately) given for regular inhomogeneous absorption profiles. It also holds true for the Brownian motion of small dust particles in a liquid due to collisions with the (even smaller) molecules of the liquid. If the perturbers are very dilute, on the other hand, there are only a few nearest neighbors which yield the predominant contributions to the line shift so that the wings of the inhomogeneous profile decay much more slowly as compared to a normal distribution. The truncation of the wings which is obtained for any nonzero value of $F=\varrho R_c^3$ has the consequence that all moments of the distribution are finite, as should be the case for physical systems.

The Lorentz distribution is a special case of the more general Lévy distribution which corresponds to

$$\lim_{|x|\to\infty} J(x)\propto|x|^{\alpha}, \tag{12}$$

with $0<\alpha<2$ [cf. Eq. (11)] and is obtained when the interaction potential $\tilde{\nu}(\mathbf{R})$ varies as $R^{-3/\alpha}$. Also a general Lévy profile would be truncated in its wings due to the quadratic variation of $J(x)$ around $x=0$. A similar model was published by Zumofen and Klafter for the spectral-diffusion propagator of single molecules in a solid [27].

Experimental hole-burning data show that the spectral-diffusion kernel is nearly always of Lorentzian shape [20,23]. Clear deviations from this shape were observed only in very few cases [28,29]. Hence, it can be concluded that the TLS density in a glass at low temperatures is much smaller than the reciprocal volume of typical organic dye molecules [20] and that the above approximation of neglecting correlations between the TLS's [Eq. (4)] is justified.

III. APPLICATION TO FINANCIAL MARKETS

The basic ideas of the Markoff-Stoneham theory of inhomogeneous spectral line shapes can be applied to distributions of price changes in financial markets. Here we must find a model to describe the impact on the traders to change the bid and ask prices. This influence can certainly not be a function of spatial coordinates, but will instead depend on the time which has elapsed since new information on the market has become available. Hence we are interested in a memory function or temporal impact function $af(t-t')$ describing the influence of a piece of information which has become available at time t' on the decision of a trader to propose or accept a price change at a later time $t>t'$. The prefactor a denotes the magnitude of the influence for fixed time delay; it will depend on the type of information and will, therefore, itself be subject to a distribution. Also the time function $f(t-t')$ will probably depend on the information and the personality of the trader; however, we make the simplifying assumption that one general effective function $f(t-t')$ can be used to describe the market. Consequences of this simplification will be discussed in Sec. VI. The function $af(t-t')$ replaces the interaction potential $\tilde{\nu}(\mathbf{R})$ in the spectroscopic problem. The distribution of price changes can then be written as

$$P(s)=\frac{1}{T^N}\int_{-\infty}^{+\infty}da_1\int_{t-T}^{t}dt_1'\cdots\int_{-\infty}^{+\infty}da_N\int_{t-T}^{t} \times dt_N'\,\eta(a_1,t_1';\ldots;a_N,t_N')\,\delta\left(s-\sum_{n=1}^{N}a_nf(t-t_n')\right) \tag{13}$$

[cf. Eq. (1)], where T is a very long period of time. In the following we will consider the limit $T\to\infty$. The ''multi-event'' distribution $\eta(a_1,t_1';\ldots;a_N,t_N')$ is again approximated by the product of N equal ''one-event'' distributions,

$$\eta(a_1,t_1';\ldots;a_N,t_N')=\prod_{n=1}^{N}g^*(a_n,t_n'), \tag{14}$$

which denote the probability that a piece of information having an impact of magnitude a_n becomes available at time t_n'. With these assumptions we obtain the following expression for the distribution of price changes during an elementary time step Δt_0:

$$p_0(s)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}dx\,e^{isx}e^{-j_0(x)}, \tag{15}$$

with

$$j_0(x) = \varrho \int_{-\infty}^{t} dt' \int_{-\infty}^{+\infty} da\, g^*(a,t') \times \{1 - \exp[-iaf(t-t')x]\}. \quad (16)$$

$\varrho = \lim_{T\to\infty} N/T$ is now the average temporal density of new information. We do not discriminate between different types of information. News coming from the market itself is included as well as news of other origins such as, for example, the collapse of a large company or political changes in a country. Also, possible temporal variations of the information density (for instance with periods of one day or one week) are neglected.

The price change of a special item during a longer time interval ΔT is the sum of the individual elementary changes in the intervals Δt_0. Under the assumption that the changes from one step Δt_0 to the next are statistically independent, the distribution $P(s)$ during ΔT is given by repeated convolutions of $p_0(s)$ with itself [30]. Applying the convolution theorem leads to the result

$$P(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx\, e^{isx} e^{-J(x)}, \quad (17)$$

with

$$J(x) = \frac{\Delta T}{\Delta t_0} \varrho \int_{-\infty}^{t} dt' \int_{-\infty}^{+\infty} da\, g^*(a,t') \times \{1 - \exp[-iaf(t-t')x]\}. \quad (18)$$

Equations (17) and (18) describe the distribution of price changes for an ensemble of independently acting traders, when also successive changes are statistically independent of each other. Both assumptions are approximations. In particular the temporal independence is at odds with the presence of a memory function $f(t-t')$ in the system. The consequences of these approximations are discussed in Sec. VI.

In order to evaluate the integrals in Eq. (18), we must specify the functions $g^*(a,t')$ and $f(t-t')$. For $g^*(a,t')$ we choose a Gaussian distribution of a and a step function in t',

$$g^*(a,t') = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(a-b)^2}{2\sigma^2} \right] g(t-t'), \quad (19)$$

with

$$g(t-t') = \begin{cases} 1 & \text{for } t' \le t - \Delta t_c \\ 0 & \text{for } t' > t - \Delta t_c. \end{cases} \quad (20)$$

The shape of $P(s)$ does not depend very sensitively on the form of $g^*(a,t')$, however. The basic ideas of the model are sketched in Fig. 2. The decision of a trader for a price change at time t (thick bar) is influenced by information obtained at earlier times $t' < t$ with the thin curves indicating its relative impact at time t. $t - \Delta t_c$ is the latest moment at which a new piece of information can contribute to the price change.

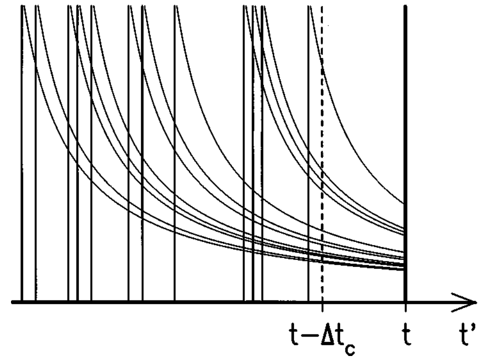

FIG. 2. Schematic representation of the proposed model of price finding in financial markets. A trader makes a price decision at time t (thick bar), based upon information which was received earlier at statistical times t' (thin solid lines). The decaying curves indicate the memory function $f(t-t')$ which describes the relative impact of a piece of information that depends on the time difference $t - t'$. $t - \Delta t_c$ is the latest moment at which information must be obtained in order to contribute to the price decision. The distribution of the impact strength a [Eq. (19)] has been omitted for clarity.

After inserting Eqs. (19) and (20) into Eq. (18), the a integral can be solved analytically. The characteristic function $J(x)$ then reads

$$\mathrm{Re}[J(x)] = \frac{\Delta T}{\Delta t_0} \varrho \int_{-\infty}^{t-\Delta t_c} \left\{ 1 - \cos[bxf(t-t')] \times \exp\left[-\frac{1}{2}\sigma^2 x^2 f^2(t-t') \right] \right\} dt', \quad (21)$$

$$\mathrm{Im}[J(x)] = \frac{\Delta T}{\Delta t_0} \varrho \int_{-\infty}^{t-\Delta t_c} \sin[bxf(t-t')] \times \exp\left[-\frac{1}{2}\sigma^2 x^2 f^2(t-t') \right] dt'. \quad (22)$$

Equations (17), (21), and (22) are the general result of the theory. According to these equations, the distribution $P(s)$ of price changes is a functional of the memory function $f(t-t')$. In Sec. IV, $P(s)$ will be evaluated for several forms of the memory function. Since the drift in high-frequency financial data is much smaller than the statistical fluctuations [30], we will always set $b=0$ so that $\mathrm{Im}[J(x)]=0$ and $P(s)$ is symmetrical. The calculation for $b \neq 0$ would be straightforward, however.

IV. SPECIFIC FORMS OF THE MEMORY FUNCTION

A. Exponential function

As the first example we consider the case of an exponential impact function,

$$f(t-t') = \exp\left(-\frac{t-t'}{\tau} \right), \quad (23)$$

which means that the whole market has a characteristic memory time τ. With Eq. (23) and $b=0$, Eq. (21) can be transformed to

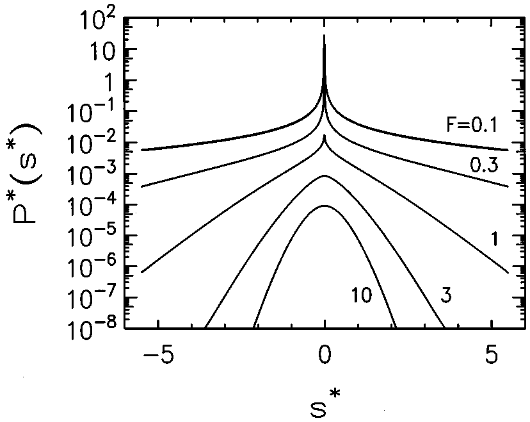

FIG. 3. Distribution of price changes for an exponential memory function and different values of the parameter F in a semilogarithmic representation.

$$J(x)=F\int_0^{G|x|}(1-e^{-u^2})\frac{du}{u}, \qquad (24)$$

with

$$F=\frac{\Delta T}{\Delta t_0}\varrho\tau, \quad G=\frac{\sigma}{\sqrt{2}}\exp\left(-\frac{\Delta t_c}{\tau}\right).$$

Figure 3 shows the distribution $P(s)$ for different values of F in a semilogarithmic representation. For each curve the coordinate axes were rescaled with the respective half-width [full width at half maximum (FWHM)]; hence, $P^*(s^*)$ are normalized dimensionless distributions with a FWHM of 1. As was discussed in Sec. II for the spectroscopic problem, the shape of $P(s)$ depends solely on F, whereas G determines the scaling of the s axis. Therefore, the normalized form $P^*(s^*)$ ist not influenced by G. For $F\rightarrow 0$ the shape is distinctly non-Gaussian, but it tends toward Gaussian as F increases, corresponding to the quadratic behavior of $J(x)$ around $x=0$. In the non-Gaussian limit, the distribution is characterized by a sharp δ-like spike at $s=0$. When F becomes sufficiently large so that the spike disappears, $P(s)$ quickly approaches a normal distribution. A central spike was observed in a numerical prototype model of stock exchange markets [6], but it does not occur in real market data [3,5,10]. Hence we can conclude that real stock and foreign-currency exchange markets are not characterized by one certain memory time—a result which appears reasonable, given the diversity of the involved people and of the information determining their behavior.

Mathematically, the spike is due to the fact that for an exponential memory function only information which becomes available at times $t'>t-\tau$ has a noticeable influence on the decisions of the traders. If the information density ϱ is so low that F is much smaller than 1, there is a sizable fraction of the traders who receive no impact for a price change at all during the memory time τ. Consequently, a large number of price changes of amount zero will occur.

B. Stretched-exponential function

Now we use a stretched-exponential (or Kohlrausch-Williams-Watts) memory function of the form

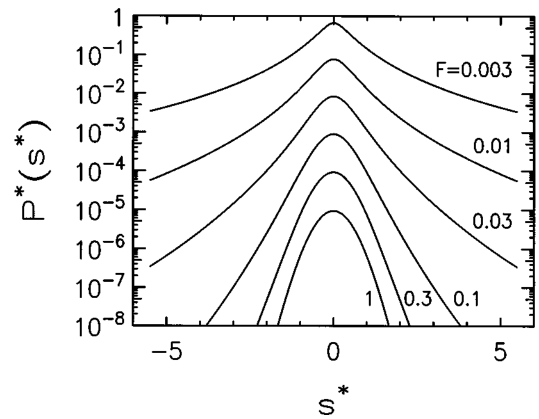

FIG. 4. Same as Fig. 3, for a stretched-exponential memory function with exponent $\beta=0.2$.

$$f(t-t')=\exp\left[-\left(\frac{t-t'}{\tau}\right)^{\beta}\right], \qquad (25)$$

with $0<\beta<1$. This yields for the characteristic function

$$J(x)=F\int_0^{G|x|}(1-e^{-u^2})\left[\left(\frac{\Delta t_c}{\tau}\right)^{\beta}+\ln\frac{G|x|}{u}\right]^{(1/\beta)-1}\frac{du}{u}. \qquad (26)$$

Here the parameters are

$$F=\frac{\Delta T}{\Delta t_0}\frac{\varrho\tau}{\beta}, \quad G=\frac{\sigma}{\sqrt{2}}\exp\left[-\left(\frac{\Delta t_c}{\tau}\right)^{\beta}\right].$$

We will only consider the limit $\Delta t_c \ll \tau$ in which the first term in the square brackets vanishes. The corresponding profiles of $P(s)$ are shown in Fig. 4 for $\beta=0.2$. Since there is now a broad distribution of time constants in the system, the central spike is absent even for very small values of F. The calculated curves for F values between approximately 0.01 and 0.3 are very similar to the distributions of foreign-currency exchange data as published by Ghashghaie *et al.* (Ref. [10], Fig. 1). Hence a stretched-exponential memory function with an exponent around 0.2 seems to yield an appropriate description for this type of financial market. A simple mathematical description of the asymptotic behavior of $P(s)$ does not seem to be possible in this case (in contrast to the case of an algebraic memory function to be discussed below).

C. Algebraic function

The last example that we consider explicitly for $f(t-t')$ is an inverse-power-law function

$$f(t-t')=\left(\frac{\tau}{t-t'}\right)^{\gamma}, \qquad (27)$$

with exponent $\gamma>0$. The characteristic function then reads

$$J(x)=F(G|x|)^{1/\gamma}\int_0^{G|x|}(1-e^{-u^2})\frac{du}{u^{1+1/\gamma}}, \qquad (28)$$

with the parameters

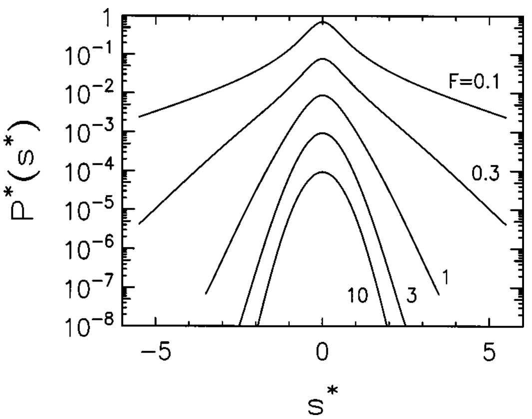

FIG. 5. Same as Fig. 3, for an algebraic memory function with exponent $\gamma = 1.0$.

$$F = \frac{\Delta T}{\Delta t_0} \frac{\varrho \Delta t_c}{\gamma}, \quad G = \frac{\sigma}{\sqrt{2}} \left(\frac{\tau}{\Delta t_c} \right)^{\gamma}.$$

Whereas in the first two cases the cutoff time Δt_c is not very important and can be set equal to zero, it plays a decisive role in the algebraic model due to the divergence for $t' \rightarrow t$. Both F and G depend critically on Δt_c. As discussed in Sec. II, the upper integration bound in Eq. (28) can be shifted to infinity for large $|x|$, so that

$$\lim_{|x| \rightarrow \infty} J(x) \propto |x|^{1/\gamma}. \tag{29}$$

In the opposite limit $x \rightarrow 0$, the function depends quadratically on x. Hence for $\gamma > 0.5$ the distribution $P(s)$ is again a Lévy profile with cutoff wings. Its index is now given by $1/\gamma$. Truncated Lévy distributions are not stable with respect to convolution, but tend toward Gaussians [31,32]. This is in contrast to true Lévy distributions without cutoff (''Lévy stable distributions''). Typical curves of $P(s)$ are plotted in Fig. 5 for the case $\gamma = 1.0$. The agreement with the foreign-currency exchange data of Ref. [10] is fair but is not as good as for the stretched-exponential memory function.

The short-term price changes of the New York Stock Exchange [3], on the other hand, are very well represented by a truncated Lévy distribution. Figure 6 shows the distribution for $\gamma = 0.7$ and $F = 0.02$; cf. Fig. 2 of Ref. [3]. $\gamma = 0.7$ corresponds to the Lévy index $1/\gamma = 1.40 \pm 0.05$ within the error margins which was obtained from a fit to the central part of the distribution of the stock exchange data [3]. The narrow central part, the shoulders in the intermediate region, and also the fast decay for large $|s|$ are well reproduced by the model, although the shoulders are somewhat less pronounced than in the market data.

For a very large data base of stock price fluctuations which was compiled from three major U.S. stock markets, an inverse-power-law distribution with an exponent close to -3 was recently found [5]. The statistical model in conjunction with an algebraic memory function is also able to reproduce such a behavior, as Fig. 7 shows. The curves correspond to $\gamma = 0.61$ and $F = 0.0003$; they represent the same distribution in a semilogarithmic plot [part (a)] and a double-logarithmic plot [part (b)]. The dashed straight line in Fig. 7(b) has a slope of -2.77; it yields a good fit to the calculated distribution over at least 1.5 orders of magnitude in the price changes.

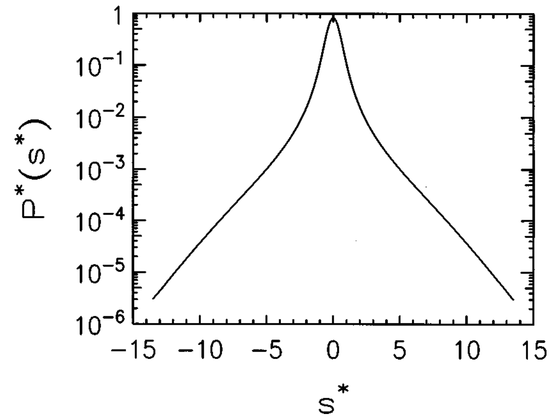

FIG. 6. Distribution of price changes for an algebraic memory function with exponent $\gamma = 0.7$ and $F = 0.02$ in a semilogarithmic representation.

At present there is no ''microscopic'' argument for a specific choice of the memory function (stretched exponential or algebraic). These two functions were empirically found to yield the best description for different types of financial markets. Possible interpretations (e.g., in the framework of human psychology) require further studies.

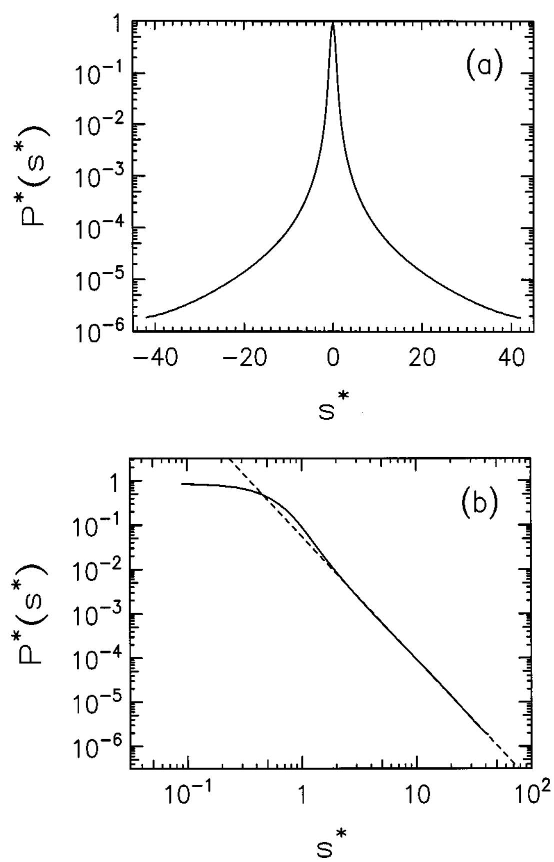

FIG. 7. Distribution of price changes for an algebraic memory function with exponent $\gamma = 0.61$ and $F = 0.0003$ in semilogarithmic [part (a)] and double-logarithmic [part (b)] representations. The dashed straight line in (b) indicates a power-law behavior with exponent -2.77.

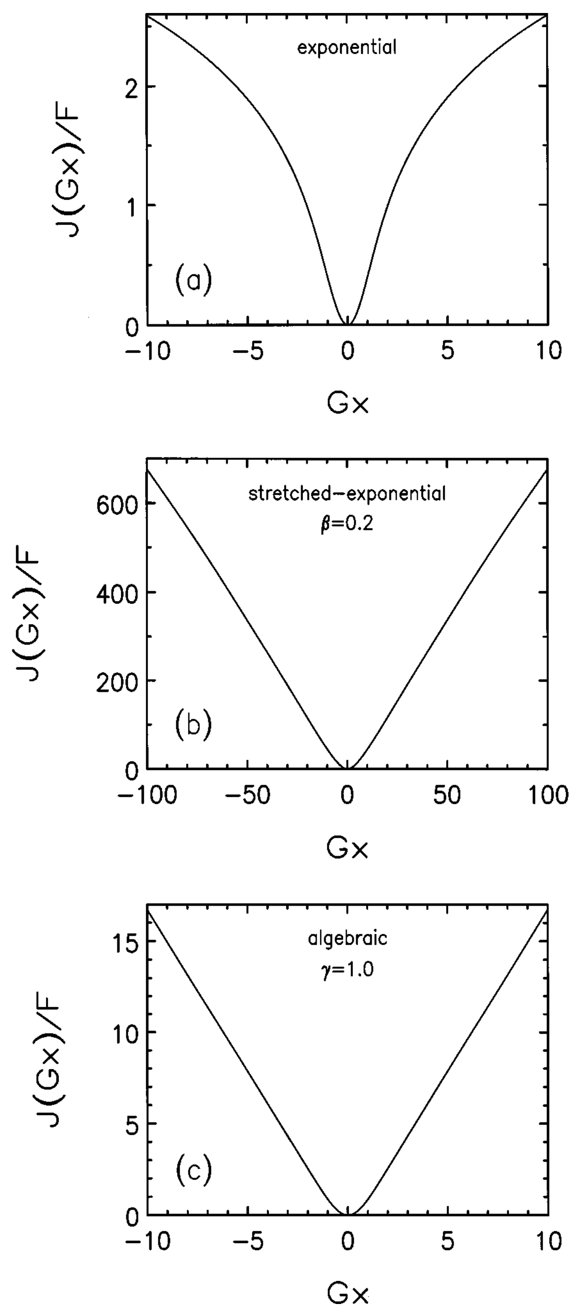

FIG. 8. Characteristic function $J(Gx)$ (divided by F) for the three cases of exponential [part (a)], stretched-exponential [part (b); $\beta=0.2$], and algebraic [part (c); $\gamma=1.0$] memory functions.

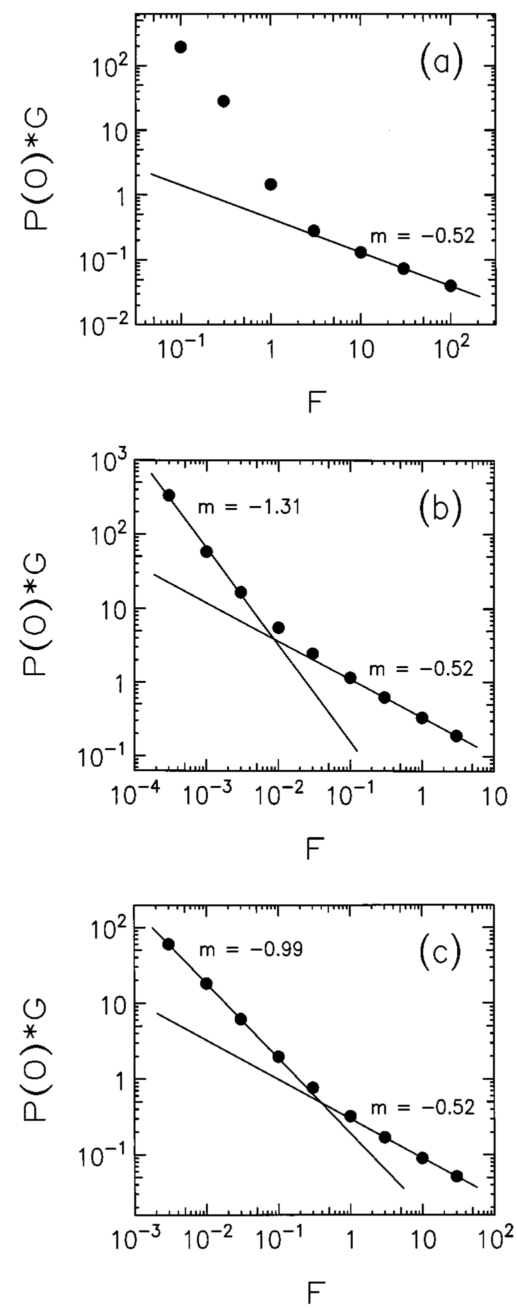

FIG. 9. Probability of return to the origin $P(s=0)$ (multiplied by G) as a function of the dimensionless parameter F for the three cases of exponential [part (a)], stretched-exponential [part (b); $\beta=0.2$], and algebraic [part (c); $\gamma=1.0$] memory functions. The data points were obtained from Eqs. (17) and (21) for discrete values of F. The straight lines represent linear fits to the last three points and—in (b) and (c)—the first three points in each part; their slopes are indicated.

At the end of this section it is illustrative to compare the characteristic function $J(x)$ for the three cases of an exponential function, a stretched-exponential function, and an algebraic memory function $f(t-t')$. This is done in Fig. 8. For the latter two cases the same parameters as above have been used, i.e., $\beta=0.2$ (stretched exponential) and $\gamma=1.0$ (algebraic). In each part J/F has been plotted versus Gx. J, F, and Gx are dimensionless quantities. The general quadratic behavior around $x=0$ is very visible, which leads to the truncation (i.e., Gaussian decay) of $P(s)$ at large $|s|$ values.

V. SCALING BEHAVIOR

Mantegna and Stanley pointed out that the maximum $P(0)$ of the probability distribution of short-term price fluctuations often shows clear non-Gaussian scaling as a function of the time interval ΔT, i.e., that $P(0)\propto(\Delta T)^{-\kappa}$ with $\kappa>0.5$ [3,12,13]. The scaling behavior of the above theoretical distributions is depicted in Fig. 9, where $P(0)$ has been plotted versus F in a double-logarithmic fashion. Parts (a), (b), and (c) again represent the three cases of exponential, stretched-exponential (with $\beta=0.2$), and algebraic (with $\gamma=1.0$) memory functions. The dimensionless parameter F is proportional to the time interval ΔT (see above). $P(0)$ has been multiplied by G, so that the data are independent of the absolute volatility of the market.

Each part of Fig. 9 contains linear-regression lines for the data points with the three largest and—except for part (a)—three smallest F values. In the limit of large F, the maximum of the distribution shows Gaussian scaling for all memory functions $f(t-t')$, i.e., the slope of the regression lines tends to -0.5. In the opposite limit $F\rightarrow 0$, on the other hand, the behavior depends on the memory function. For the exponential function, the slope becomes very large due to the δ-functionlike spike around $s=0$ (a). In this case it does not make sense to investigate the scaling behavior and, correspondingly, a linear fit has not been performed. With the stretched-exponential function, $P(0)$ shows approximate scaling for F values between 3×10^{-4} and 3×10^{-3}, although the data points are not perfectly located on the regression line (b). In the algebraic case, however, the scaling is perfect for $F\rightarrow 0$ (c), as can also be shown analytically [3]. For a Lévy distribution with index $1/\gamma$ the regression line has a slope of $-\gamma$.

It should be emphasized that the calculations presented in this paper are solely based on a statistical ensemble model. The temporal evolution of specific prices is not considered. Hence it is not possible to calculate the autocorrelation of the price changes or of their absolute values. The price changes in real markets were found to have very short correlation times on the order of a few minutes, but their absolute values (or squares) exhibit long correlations with slow algebraic decay [30,33]. This is in accordance with the slowly decaying memory functions which were discussed in the context of the present theory.

VI. SUMMARY AND CONCLUSIONS

It was shown that a microscopic statistical theory, which has long been used in optical spectroscopy to model inhomogeneous spectral line shapes, can be applied to describe distributions of price changes in financial markets. In this model, the behavior of a set of independently acting traders is analyzed in terms of information which has become available in the past, and whose impact on current decisions is described by a memory function of the elapsed time. The distribution of price changes is obtained as a functional of the memory function. For a variety of memory functions the distributions are distinctly non-Gaussian with long tails (i.e., leptokurtic). This results from the fact that a comparatively small number of most recent pieces of information have the strongest impact on the price decisions so that the central limit theorem is not applicable. Far out in the wings, however, the distributions do exhibit a Gaussian-like decay and, hence, all their moments and in particular the variance are finite. For the special case of algebraic memory functions $f(t-t')\propto(t-t')^{-\gamma}$ with exponent $\gamma>0.5$, truncated Lévy distributions are obtained. Recently published data of real stock [3,5,13] and foreign-currency exchange markets [10] can be reproduced very well with algebraic or stretched-exponential memory functions.

The finding that there is no characteristic time scale in the judgement of information by the traders corresponds to the often observed scaling behavior of financial data [3,12,13]. It is not clear whether an algebraic or stretched-exponential law is immanent to human psychology or whether different types of information (and different types of people) are connected with very different time constants so that the superposition of exponential functions leads to the slow decay. The latter case would be analogous, for instance, to the algebraic decay of transient photocurrents in disordered photoconductors after pulsed optical excitation [34]. This question can perhaps be addressed by comparing large sets of financial data [5] with smaller subsets which belong to shorter time intervals or certain branches of a stock market. Such a ''site-selective'' analysis of financial data might be similarly successful as site-selective spectroscopic methods [24]. The latter yield information about dye-matrix interactions which is otherwise obscured by the large inhomogeneous ensemble of dopant molecules in a solid.

The above statistical model is related to continuous-time random walk theories which are used to describe transport phenomena in disordered and in nonlinear systems, and which also yield probability distributions with long tails [35–37]. Similar ideas can be applied to model velocity distributions in turbulent flows [38,39]. In this case the perturbers that yield additive contributions to the velocity component of a volume element along a given direction are the vortices in the liquid or gas which are assumed to be statistically independent. If the velocity in an individual vortex decreases algebraically from the core toward the periphery, Lévy laws are obtained. Truncated Lévy laws result if the vortex cores are smooth rather than having a singularity [38,39]. The similarity of the velocity distributions with those of price changes in foreign-currency exchange [10] suggests that the radial velocity dependence in turbulent vortices may be better described by stretched-exponential laws—perhaps due to the presence of vortices of different sizes.

Very recently, distributions of flow velocities and flow velocity gradients of ocean currents were calculated from satellite data [40]. In this example of large-scale, two-dimensional turbulence the distributions were either Gaussian or they had similar leptokurtic shapes as the distributions that were obtained in the laboratory experiment with water flowing through a nozzle [10]. Leptokurtic distributions were found in particular in those parts of the oceans in which the eddy activity is high [40].

There is an important quantitative difference between the price changes in financial markets and the theoretical calculations, which was not discussed so far. In the real market data of Ref. [10], the transition from the strongest non-Gaussian to almost Gaussian shape stretches over more than 2.5 orders of magnitude in the time difference ΔT (from $\Delta T=640$ s to beyond $\Delta T=163\,840$ s). According to the theoretical profiles of Fig. 4, however, this change should not cover more than about 1.5 orders of magnitude; the distributions corresponding to the data of Ref. [10] are approximately those between $F=0.01$ and 0.3. The slow change can be ascribed to ''coherence'' or ''herding effects'' [7,8] which are not taken into account in the model: In reality, the traders in financial markets do not act completely independently of each other, and their actions during successive time steps will also be correlated to a certain degree (corresponding to the presence of a memory function). Some events or pieces of information give rise to similar reactions of a large number of traders, thereby stretching the time span during which the shape of the distribution changes. This seems to be true in particular for very short time intervals. For fluctuations of stock prices ranging from 1 to 1000 min, no significant change of the shape of the distributions was observed, although they can be well described by truncated Lévy distributions [3]. It should be emphasized that in the present model the leptokurtic distributions are not intrinsically due to coherence effects but coherence effects are only assumed to be responsible for the slow transition to Gaussians. In the case of the turbulent-flow data, the transition from non-Gaussian to Gaussian shape occurs between 3.3η and 138η (in the units of Ref. [10]); i.e., it really covers about 1.5 orders of magnitude of the distance parameter. This problem must be treated in three-dimensional space, however, so that the situation is not completely equivalent.

The theoretical investigation of distributions of price changes in financial markets began in 1900 when Bachelier calculated that the price changes should follow a Gaussian distribution whose width increases as the square root of time [41]. In this way he found a description of Brownian motion five years before Einstein published his famous paper on particle diffusion [42]. Starting in the 1960s, it was empirically discovered that actual market data do not obey Gaussian statistics, but usually follow leptokurtic distributions with much longer tails [1,2]. The present work suggests that financial markets probably have a closer analogy with spectral diffusion of dye molecules in disordered solids than with particle diffusion.

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[1] B. B. Mandelbrot, J. Business **36**, 394 (1963).
[2] E. F. Fama, J. Business **38**, 34 (1965).
[3] R. N. Mantegna and H. E. Stanley, Nature (London) **376**, 46 (1995).
[4] J.-Ph. Bouchaud and M. Potters, *Théorie des Risques Financiers* (Aléa-Saclay, Paris, 1997).
[5] P. Gopikrishnan, M. Meyer, L. A. N. Amaral, and H. E. Stanley, Eur. Phys. J. B **3**, 139 (1998).
[6] G. Caldarelli, M. Marsili, and Y.-C. Zhang, Europhys. Lett. **40**, 479 (1997).
[7] R. Cont and J.-Ph. Bouchaud, e-print cond-mat/9712318.
[8] Th. Lux and M. Marchesi, Nature (London) **397**, 498 (1999).
[9] J.-Ph. Bouchaud and R. Cont, Eur. Phys. J. B **6**, 543 (1998).
[10] S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, and Y. Dodge, Nature (London) **381**, 767 (1996).
[11] A. Arnéodo, J.-F. Muzy, and D. Sornette, Eur. Phys. J. B **2**, 277 (1998).
[12] R. N. Mantegna and H. E. Stanley, Nature (London) **383**, 588 (1996).
[13] R. N. Mantegna and H. E. Stanley, Physica A **239**, 255 (1997).
[14] A. M. Stoneham, Rev. Mod. Phys. **41**, 82 (1969).
[15] A. A. Markoff, *Wahrscheinlichkeitsrechnung* (Teubner, Leipzig, 1912).
[16] I. Messing, B. Raz, and J. Jortner, J. Chem. Phys. **66**, 2239 (1977).
[17] S. H. Simon, V. Dobrosavljević, and R. M. Stratt, J. Chem. Phys. **93**, 2640 (1990).
[18] H. M. Sevian and J. L. Skinner, Theor. Chim. Acta **82**, 29 (1992).
[19] L. Kador, J. Chem. Phys. **95**, 5574 (1991).
[20] L. Kador, J. Lumin. **56**, 165 (1993).
[21] L. Kador, S. Jahn, D. Haarer, and R. Silbey, Phys. Rev. B **41**, 12 215 (1990).
[22] T. L. Reinecke, Solid State Commun. **32**, 1103 (1979).
[23] W. Breinl, J. Friedrich, and D. Haarer, J. Chem. Phys. **81**, 3915 (1984).
[24] *Persistent Spectral Hole Burning: Science and Applications*, edited by W. E. Moerner (Springer, Berlin, 1988).
[25] P. W. Anderson, B. I. Halperin, and C. M. Varma, Philos. Mag. **25**, 1 (1972).
[26] W. A. Phillips, J. Low Temp. Phys. **7**, 351 (1972).
[27] G. Zumofen and J. Klafter, Chem. Phys. Lett. **219**, 303 (1994).
[28] A. Kurita, Y. Shibata, and T. Kushida, Phys. Rev. Lett. **74**, 4349 (1995).
[29] Y. Shibata, A. Kurita, and T. Kushida, J. Chem. Phys. **104**, 4396 (1996).
[30] R. Cont, e-print cond-mat/9705075.
[31] R. N. Mantegna and H. E. Stanley, Phys. Rev. Lett. **73**, 2946 (1994).
[32] I. Koponen, Phys. Rev. E **52**, 1197 (1995).
[33] Y. Liu, P. Cizeau, M. Meyer, C.-K. Peng, and H. E. Stanley, Physica A **245**, 437 (1997); e-print cond-mat/9706021.
[34] H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975).
[35] G. Zumofen and J. Klafter, Phys. Rev. E **47**, 851 (1993).
[36] J. Klafter and G. Zumofen, J. Phys. Chem. **98**, 7366 (1994).
[37] *Lévy Flights and Related Topics in Physics*, edited by M. F. Shlesinger, G. M. Zaslavsky, and U. Frisch (Springer, Berlin, 1995).
[38] I. A. Min, I. Mezić, and A. Leonard, Phys. Fluids **8**, 1169 (1996).
[39] B. Dubrulle and J.-Ph. Laval, Eur. Phys. J. B **4**, 143 (1998).
[40] S. G. Llewellyn Smith and S. T. Gille, Phys. Rev. Lett. **81**, 5249 (1998).
[41] L. Bachelier, Ann. Sci. École Norm. Super. **3**, 21 (1900).
[42] A. Einstein, Ann. Phys. (N.Y.) **17**, 549 (1905).