Wigner rotations and Iwasawa decompositions in polarization optics

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Wigner rotations and Iwasawa decompositions are manifestations of the internal space-time symmetries of massive and massless particles, respectively. It is shown to be possible to produce combinations of optical filters which exhibit transformations corresponding to Wigner rotations and Iwasawa decompositions. This is possible because the combined effects of rotation, phase-shift, and attenuation filters lead to transformation matrices of the six-parameter Lorentz group applicable to Jones vectors and Stokes parameters for polarized light waves. The symmetry transformations in special relativity lead to a set of experiments which can be performed in optics laboratories. [S1063-651X(99)08907-2]

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I. INTRODUCTION

In our earlier papers [1,2], we formulated Jones vectors and Stokes parameters in terms of the 2×2 and 4×4 matrix representations of the six-parameter Lorentz group [3]. It was seen there that, for every 2×2 transformation matrix for the Jones vector, there is a corresponding 4×4 matrix for the Stokes parameters. It was also found that Stokes parameters are like the components of Minkowskian four-vectors, and two-component Jones vectors are like two-component spinors in the relativistic world. This enhances our capacity to approach polarization optics in terms of the kinematics of special relativity.

Indeed, we can now design specific experiments which will test some of the consequences derivable from the principles of special relativity. The most widely known example is the Wigner rotation. This has been extensively discussed in the literature in connection with the Thomas effect [4], Berry's phase [5,6], and squeezed states of light [7].

In our earlier papers, we discussed an optical filter which exhibits the matrix form of

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \tag{1.1}$$

applicable to two transverse components of the light wave, where u is a controllable parameter. When applied to a two-component system, this matrix performs a superposition in the upper channel while leaving the low channel invariant. The question is whether it is possible to produce optical filters with this property.

In Ref. [1], we approached this problem in terms of the generators of the Lorentz group. It is very difficult, if not impossible, to manufacture optical devices performing the function of group generators. In the case of optical filters, this means an infinite number of layers of zero thickness. In the present paper, we deal with the same problem from an experimental point of view. We will present a specific design for optical filters performing this function. We will of course present our case in terms of a combination of three filters of finite thickness.

In order to achieve this goal, we use the fact that polarization optics and special relativity share the same mathematics. This aspect was already noted in the literature for the case of the Wigner rotation [6]. The concept of the Wigner rotation comes from the kinematics of special relativity, in which two successive noncollinear Lorentz boosts do not end up with a boost. The result is a boost followed or preceded by a rotation. Thus we can achieve a rotation from three noncollinear boosts starting from a particle at rest. Since each boost corresponds to an attenuation filter, it requires three attenuation filters to achieve a Wigner rotation in polarization optics.

While the Wigner rotation is based on Lorentz transformations of massive particles, there are similar transformations for massless particles. Here two noncollinear Lorentz boosts do not result in one boost. They become one boost preceded or followed by a transformation which corresponds to a gauge transformation. In 2×2 formalism, the transformation takes the form of Eq. (1.1). We shall show in this paper that the filter possessing the property of Eq. (1.1) can be constructed from one rotation filter and one attenuation filter. In mathematics, this type of decomposition is called the Iwasawa decomposition [8,9].

While the primary purpose of this paper is to discuss filters and their combinations in polarization optics, we also provide concrete illustrative examples of Wigner's ''little group'' [10]. The little group is the maximal subgroup of the

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Lorentz group whose transformations leave the four-momentum of a given particle invariant, and has a long history [11]. The Wigner rotation and the Iwasawa decomposition are transformations of the little groups for massive and massless particles, respectively. It is interesting to note that these transformations can also be achieved in optics laboratories.

In Sec. II, we review the formalism for optical filters based on the Lorentz group, and explain why filters are like Lorentz transformations. It is shown in Sec. III that a rotation can be achieved by three noncollinear Lorentz boosts. In Sec. IV, we spell out in detail how the Iwasawa decomposition can be achieved from the combination of two optical filters.

II. FORMULATION OF THE PROBLEM

In studying polarized light propagating along the z direction, the traditional approach is to consider the x and y components of the electric fields. Their amplitude ratio and the phase difference determine the degree of polarization. Thus we can change the polarization either by adjusting the amplitudes, by changing the relative phases, or both. For convenience, we call the optical device which changes amplitudes an ''attenuator'' and the device which changes the relative phase a ''phase shifter.''

Let us write these electric fields as

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} A \exp\{i(kz-\omega t+\phi_1)\} \\ B \exp\{i(kz-\omega t+\phi_2)\} \end{pmatrix}, \tag{2.1}$$

where A and B are amplitudes which are real and positive numbers, and ϕ_1 and ϕ_2 are the phases of the x and y components, respectively. This column matrix is called the Jones vector. In dealing with light waves, we have to realize that the intensity is the quantity we measure. Then there arises the question of coherence and time average. We are thus led to consider the following parameters:

$$\begin{aligned} S_{11} &= \langle E_x^* E_x \rangle, \quad S_{22} = \langle E_y^* E_y \rangle, \\ S_{12} &= \langle E_x^* E_y \rangle, \quad S_{21} = \langle E_y^* E_x \rangle. \end{aligned} \tag{2.2}$$

Then, we are naturally invited to write down the 2×2 matrix:

$$C = \begin{pmatrix} \langle E_x^* E_x \rangle & \langle E_y^* E_x \rangle \\ \langle E_x^* E_y \rangle & \langle E_y^* E_y \rangle \end{pmatrix}, \tag{2.3}$$

where $\langle E_i^* E_j \rangle$ is the time average of $E_i^* E_j$. The above form is called the coherency matrix [12].

It is sometimes more convenient to use the following combinations of parameters:

$$\begin{aligned} S_0 &= S_{11}+S_{22}, \\ S_1 &= S_{11}-S_{22}, \\ S_2 &= S_{12}+S_{21}, \\ S_3 &= -i(S_{12}-S_{21}). \end{aligned} \tag{2.4}$$

These four parameters are called the Stokes parameters in the literature [12].

We showed in our earlier papers that the Jones vectors and the Stokes parameters can be formulated in terms of the 2×2 spinor and 4×4 vector representations of the Lorentz group. This group theoretical formalism allows us to discuss three different sets of physical quantities using one mathematical device. In our earlier publications, we used the concept of Lie groups extensively, and used their generators based on infinitesimal generators.

In this paper, we avoid the Lie groups and work only with explicit transformation matrices. For this purpose, we start with the following two matrices:

$$\begin{aligned} B &= \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ R &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.5}$$

If the above matrices are applied to the Minkowskian space of (ct,z,x,y), the matrix B performs a Lorentz boost:

$$\begin{aligned} t' &= (\cosh\chi)t + (\sinh\chi)z, \\ z' &= (\sinh\chi)t + (\cosh\chi)z, \end{aligned} \tag{26}$$

while R leads to a rotation:

$$\begin{aligned} z' &= (\cos\phi)z - (\sin\phi)x, \\ x' &= (\sin\phi)z + (\cos\phi)x. \end{aligned} \tag{2.7}$$

In our previous paper, we discussed in detail what these matrices do when they are applied to the Stokes four-vectors.

In the two-component spinor space, the above transformation matrices take the forms

$$\begin{pmatrix} e^{\chi/2} & 0 \\ 0 & e^{-\chi/2} \end{pmatrix}, \quad \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix}. \tag{2.8}$$

We discussed the effect of these matrices on the Jones spinors in our earlier publications.

In this paper, we discuss some of nontrivial consequences derivable from the algebra generated by these two sets of matrices. We shall study Wigner rotations and Iwasawa decompositions. The Wigner rotation has been discussed in optical science in connection with Berry's phase, but the Iwasawa decomposition is a relatively new word in optics. We would like to emphasize here that both the Wigner rotation and Iwasawa decomposition come from the concept of subgroup of the Lorentz groups, whose transformations leave the momentum of a given particle invariant.

III. WIGNER ROTATIONS

There are many different versions of the Wigner rotation in the literature. Basically, this rotation is a product of two noncollinear Lorentz boosts. The result of these two boosts is

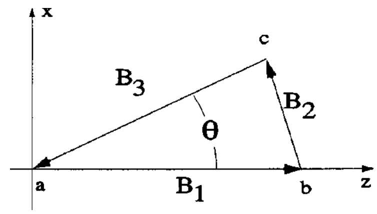

FIG. 1. Closed Lorentz boosts. Initially, a massive particle is at rest with its four momentum P_a. The first boost B_1 brings P_a to P_b. The second boost B_2 transforms P_b to P_c. The third boost B_3 brings P_c back to P_a. The particle is again at rest. The net effect is a rotation around the axis perpendicular to the plane containing these three transformations. We may assume for convenience that P_b is along the z axis, and P_c in the zx plane. The rotation is then made around the y axis.

not a boost, but a boost preceded or followed by a rotation. This rotation is called the Wigner rotation.

In this paper, we approach the problem by using three boosts described in Fig. 1. Let us start with a particle at rest, with its four-momentum

$$P_a=(m,0,0,0), \tag{3.1}$$

where we use the metric convention (ct,z,x,y). Let us next boost this four-momentum along the z direction using the matrix

$$B_1=\begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

resulting in the four-momentum

$$P_b=m(\cosh\eta,\sinh\eta,0,0). \tag{3.3}$$

Let us rotate this vector around the y axis by an angle θ. Then the resulting four-momentum is

$$P_c=m(\cosh\eta,(\sinh\eta)\cos\theta,(\sinh\eta)\sin\theta,0). \tag{3.4}$$

Instead of this rotation, we propose to obtain this four-vector by boosting the four-momentum of Eq.(3.3). The boost matrix in this case is

$$B_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh\lambda & \sinh\lambda & 0 & 0 \\ \sinh\lambda & \cosh\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & \sin\psi & 0 \\ 0 & -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.5}$$

with

$$\lambda=2\tanh^{-1}\{[\sin(\theta/2)]\tanh\eta\},\qquad \psi=\frac{\theta}{2}+\frac{\pi}{2}. \tag{3.6}$$

If we carry out the matrix multiplication,

$$B_2=\begin{pmatrix} \cosh\lambda & -\sin(\theta/2)\sinh\lambda & \cos(\theta/2)\sinh\lambda & 0 \\ -\sin(\theta/2)\sinh\lambda & 1+\sin^2(\theta/2)(\cosh\lambda-1) & -\sin\theta\sinh^2(\lambda/2) & 0 \\ \cos(\theta/2)\sinh\lambda & -\sin\theta\sinh^2(\lambda/2) & 1+\cos^2(\theta/2)(\cosh\lambda-1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.7}$$

Next we boost the four-momentum of Eq. (3.4) to that of Eq. (3.1). The particle is again at rest. The boost matrix is

$$B_3=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.8}$$

After the matrix multiplication,

$$B_3=\begin{pmatrix} \cosh\eta & -\cos\theta\sinh\eta & -\sin\theta\sinh\eta & 0 \\ -\cos\theta\sinh\eta & 1+\cos^2\theta(\cosh\eta-1) & \sin\theta\cos\theta(\cosh\eta-1) & 0 \\ -\sin\theta\sinh\eta & \sin\theta\cos\theta(\cosh\eta-1) & 1+\sin^2\theta(\cosh\eta-1) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.9}$$

The net result of these transformations is $B_3B_2B_1$. This leaves the initial four-momentum of Eq. (3.1) invariant. Is it going to be an identity matrix? The answer is ''No.'' The result of the matrix multiplications is

$$W=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Omega & -\sin\Omega & 0 \\ 0 & \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

with

$$\Omega=2\sin^{-1}\left\{\frac{(\sin\theta)\sinh^2(\eta/2)}{\sqrt{\cosh^2\eta-\sinh^2\eta\sin^2(\theta/2)}}\right\}. \tag{3.11}$$

This matrix performs a rotation around the y axis, and leaves the four-momentum of Eq. (3.1) invariant. This rotation is an element of Wigner's little group, whose transformations leave the four-momentum invariant. This is precisely the Wigner rotation.

This relativistic effect manifests itself in atomic spectra as the Thomas precession. Otherwise, the experiments on Wigner rotation in special relativity is largely academic. On the other hand, as noted in the literature, this effect could be tested in optics laboratories. As for the Stokes parameters, the above 4×4 matrices are directly applicable. Indeed, each 4×4 matrix corresponds to one optical filter applicable to polarized light.

In order to see this effect more clearly, let us use the Jones matrix formalism. The 2×2 squeeze matrix corresponding to the boost matrix B_1 of Eq. (3.2) is

$$S_1=\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}. \tag{3.12}$$

The 2×2 squeeze matrix corresponding to the boost matrix of Eq. (3.5) is now

$$S_2=\begin{pmatrix} \cos(\psi/2) & -\sin(\psi/2) \\ \sin(\psi/2) & \cos(\psi/2) \end{pmatrix}\begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix}\begin{pmatrix} \cos(\psi/2) & \sin(\psi/2) \\ -\sin(\psi/2) & \cos(\psi/2) \end{pmatrix}, \tag{3.13}$$

where the parameters ψ and λ are given in Eq. (3.6). After the matrix multiplication, S_2 becomes

$$S_2=\begin{pmatrix} \cosh(\lambda/2)-\sin(\theta/2)\sinh(\lambda/2) & \cos(\theta/2)\sinh(\lambda/2) \\ \cos(\theta/2)\sinh(\lambda/2) & \cosh(\lambda/2)+\sin(\theta/2)\sinh(\lambda/2) \end{pmatrix}. \tag{3.14}$$

This is a matrix which squeezes along the direction which makes an angle $(\pi+\theta)/2$ with the z axis. The 2×2 squeeze matrix corresponding to B_3 of Eq. (3.8) is

$$S_3=\begin{pmatrix} \cosh(\eta/2)-\cos\theta\sinh(\eta/2) & -\sin\theta\sinh(\eta/2) \\ -\sin\theta\sinh(\eta/2) & \cosh(\eta/2)+\cos\theta\sinh(\eta/2) \end{pmatrix}. \tag{3.15}$$

Now the matrix multiplication $S_3S_2S_1$ corresponds to the closure of the kinematical triangle given in Fig. 1. The result is

$$S_3S_2S_1=\begin{pmatrix} \cos(\Omega/2) & -\sin(\Omega/2) \\ \sin(\Omega/2) & \cos(\Omega/2) \end{pmatrix}, \tag{3.16}$$

where Ω is given in Eq. (3.11).

IV. IWASAWA DECOMPOSITIONS

In Sec. III, the Lorentz kinematics was based on a massive particle at rest. If the particle is massless, there are no Lorentz frames in which the particle is at rest. Thus we start with a massless particle whose momentum is in the z direction,

$$K_a=(k,k,0,0), \tag{4.1}$$

where k is the magnitude of the momentum. We can rotate this four-vector to

$$K_b=(k,-k\sin\alpha,k\cos\alpha,0) \tag{4.2}$$

by applying to K_a the rotation matrix

$$R_+=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_+ & -\sin\alpha_+ & 0 \\ 0 & \sin\alpha_+ & \cos\alpha_+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.3}$$

with $\alpha_+=\alpha+\pi/2$.

If we rotate K_b around the y axis by -2α, the resulting four-momentum will be

$$K_c=(k,k\sin\alpha,k\cos\alpha,0). \tag{4.4}$$

It is possible to transform K_b to K_c by applying to K_b the boost matrix

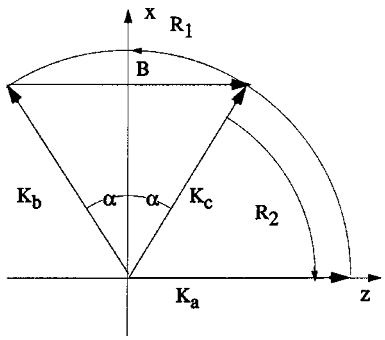

FIG. 2. Two rotations and one Lorentz boost which preserve the four-momentum of a massless particle invariant. The four-momentum K_a is rotated to K_b by R_+. It is then boosted to K_c by the boost matrix B. The rotation matrix R_- brings back the four-momentum to K_a. The initial momentum is along the z direction, and the boost B is also made along the same direction. The rotations are performed around the y axis.

$$B=\begin{pmatrix} \cosh\gamma & \sinh\gamma & 0 & 0 \\ \cosh\gamma & \sinh\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.5}$$

with

$$\sinh\gamma=\frac{2\sin\alpha}{\cos^2\alpha}, \quad \cosh\gamma=\frac{1+\sin^2\alpha}{\cos^2\alpha}. \tag{4.6}$$

We can transform K_c to K_a by rotating it around the y axis by $(\alpha-\pi/2)$ (see Fig. 2). The rotation matrix takes the form

$$R_-=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_- & -\sin\alpha_- & 0 \\ 0 & \sin\alpha_- & \cos\alpha_- & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.7}$$

with $\alpha_-=\alpha-\pi/2$. Thus the multiplication of the three matrices, R_-BR_+, gives

$$T=\begin{pmatrix} 1+u^2/2 & -u^2/2 & u & 0 \\ u^2/2 & 1-u^2/2 & u & 0 \\ u & -u & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.8}$$

with

$$u=-2\tan\alpha.$$

This T matrix plays an important role in studying space-time symmetries of massless particles. If this matrix is applied to the four-momentum K_a given in Eq. (4.1), the four-momentum remains invariant. If this matrix is applied to the electromagnetic four-potential for the plane wave propagating along the z direction with the frequency k, the result is a gauge transformation.

Again, the above 4×4 matrices are directly applicable to the Stokes parameters. On the other hand, if we are interested in designing optical filters, we need 2×2 representations corresponding to the 4×4 matrices given so far. The 2×2 squeeze matrix corresponding to the boost matrix B of Eq. (4.5) is

$$S=\begin{pmatrix} e^{\gamma/2} & 0 \\ 0 & e^{-\gamma/2} \end{pmatrix}, \tag{4.9}$$

while the 2×2 matrices corresponding to R_+ of Eq. (4.3) and R_- of Eq. (4.7) are

$$R_\pm=\begin{pmatrix} \cos(\alpha_\pm/2) & -\sin(\alpha_\pm/2) \\ \sin(\alpha_\pm/2) & \cos(\alpha_\pm/2) \end{pmatrix}, \tag{4.10}$$

where α_+ and α_- are given in Eqs. (4.3) and (4.7), respectively. They satisfy the equations

$$\alpha_++\alpha_-=2\alpha, \quad \alpha_+-\alpha_-=\pi.$$

The relation between γ and α given in Eq. (4.6) can also be written as $\cosh(\gamma/2)=1/\cos\alpha$, which is more useful for carrying out the 2×2 matrix algebra.

The matrix multiplication R_-SR_+ leads to

$$T=R_-SR_+=\begin{pmatrix} 1 & -2\tan\alpha \\ 0 & 1 \end{pmatrix}. \tag{4.11}$$

Conversely, we can write

$$\begin{pmatrix} 1 & -2\tan\alpha \\ 0 & 1 \end{pmatrix}=R_-SR_+. \tag{4.12}$$

The T matrix can be decomposed into rotation and squeeze matrices. This possibility is called the Iwasawa decomposition. In the present case, T of Eq. (4.11) can also be written as

$$T=R_-S\{(R_-)^{-1}R_-\}R_+=\{R_-S(R_-)^{-1}\}(R_-R_+). \tag{4.13}$$

The matrix chain $R_-S(R_-)^{-1}$ is one squeeze matrix whose squeeze axis is rotated by $\alpha_-/2$, and the matrix product $R_{-1}R_+$ becomes one rotation matrix. The result is

$$T=S(\alpha_-)R(2\alpha), \tag{4.14}$$

with

$$S(\alpha_-)=\begin{pmatrix} \cosh(\gamma/2)+\cos\alpha_-\sinh(\gamma/2) & \sin\alpha_-\sinh(\gamma/2) \\ \sin\alpha_-\sinh(\gamma/2) & \cosh(\gamma/2)-\cos\alpha_-\sinh(\gamma/2) \end{pmatrix},$$
$$R(2\alpha)=\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{4.15}$$

It is indeed gratifying to note that the T matrix can be decomposed into one rotation and one squeeze matrix. The squeeze is made along the direction which makes an angle of $\alpha_{-}/2$ or $-(\pi/2-\alpha)/2$ with the z axis. The angle α is smaller than $\pi/2$.

In our earlier papers [1,2] we discussed optical filters with the property given in Eq. (4.11). We said there that filters with this property can be produced from an infinite number of infinitely thin filters. This argument was based on the theory of Lie groups where transformations are generated by infinitesimal generators. This may be possible these days, but the method presented in this paper is far more practical. We need only two filters [13].

We are able to achieve this improvement because here we used the analogy between polarization optics and Lorentz transformations which share the same mathematical framework.

V. CONCLUDING REMARKS

In this paper, we noted first that both the Wigner rotation and the Iwasawa decomposition come from Wigner's little group whose transformations leave the four-momentum of a given particle invariant. Since the Lorentz group is also applicable to the Jones vector and the Stokes parameters, it is possible to construct corresponding transformations in polarization optics. We have shown that both the Wigner rotation and the Iwasawa decomposition can be realized in optics laboratories.

The matrix of Eq. (1.1) performs a shear transformation when applied to a two-dimensional object, and has a long history in physics and engineering. It also has a history in mathematics. The fact that a shear can be decomposed into a squeeze and rotations is known as the Iwasawa decomposition [8].

Among the many interesting applications of shear transformations, there is a special class of squeezed states of photons or phonons having the symmetry of shear [14, 15]. The wave-packet spread can be formulated in terms of shear transformations [16].

As we can see from this paper, a set of shear transformations can be formulated as a subset of Lorentz transformations. This set plays an important role in understanding internal space-time symmetry of massless particles, such as gauge transformation and neutrino polarizations [11,17,18].

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