Multicomponent photorefractive cnoidal waves: Stability, localization, and soliton asymptotics

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(Received 29 July 1998; revised manuscript received 29 March 1999)

An algorithm of building up a different class of stable self-consistent multicomponent periodical solutions of the nonlinear Schrödinger equation—multicomponent cnoidal waves—has been formulated by the example of a nonlinear wave propagating through a photorefractive crystal with a drift nonlinear response. Exact analytical expressions, describing distribution of light field in the components, have been obtained for solutions, which include up to three mutually incoherent components. It has been shown that such cnoidal waves are stable and their spatial structure is robust to collisions with the same cnoidal waves and to stochastic perturbations of the components' intensity distributions in a sufficiently wide range of changing spatial period.
[S1063-651X(99)03607-7]

PACS number(s): 42.65.Tg, 42.65.Hw, 42.65.Jx, 42.65.Wi

I. INTRODUCTION

One of the most exciting problems of modern laser physics is the investigation of self-organization in systems consisting of a nonlinear medium and a light field. After predictions of the possibility of the existence [1] and first demonstration [2] of stable self-consistent distributions of light field (optical solitons) in systems of such kind, the simplest case of a nonlinear medium with a spatially localized (local) cubic nonlinear response (so-called ''Kerr-type'' nonlinearity [3]) has been studied in detail. The concept of one-component [4] and two-component (''vector'') [5] solitons as stable self-consistent solutions of nonlinear problems has solidly clamped many fields of modern physics. Fiber optics and the optics of ultrashort laser pulses [3,6], nonlinear optics and laser spectroscopy [7,8], physics of one-dimensional (1D) chains and two-dimensional (2D) atomic planes in molecular systems [9], ferromagnetics [10], high-temperature superconductors (HTSC's) [11], conjugated polymers [12], etc., could be listed among these fields. From the model point of view, results obtained in recent investigations of solitons, ''multisolitons'' (high-order solitons), and stable soliton pairs in photorefractive crystals (PRC's), are of great importance. The main specific feature of PRC's is a very strong optical nonlinearity. Perceptible nonlinear effects can be observed here in laser beams with intensities of only a few mW/cm^2 [13]. Starting from the pioneering papers of [14] related to PRC's with a drift (local) nonlinear response [15], so-called photorefractive ''bright'' [16,17], ''dark'' [17,18], ''gray'' [19], vector [20] and ''vortex'' [21] solitons, spatial shock waves [22], and multisolitons [23], as well as some questions of such solitons propagation, interaction [24], spatial dimensionality [25], and stability [26,27] were intensively studied. It is known that in a Kerr-type nonlinear medium one can ''write'' a stable solitonlike waveguide by an intensive light beam and capture relatively weak (in intensity) light beams. In PRC's, intensive light beams can be captured in waveguides written by beams with lower intensity [23,28,29] when the wavelengths of the intensive beams do not fall within the photorefractive sensitivity range. Stable pairs of two incoherent spatial photorefractive solitons of any listed above types were predicted and experimentally observed [30].

Recently, we presented a different class of multicomponent screening solitons, which can stably propagate through PRC's with a drift nonlinear response [31,32]. A particular (two-component) case of such solitons was considered before for Kerr-type nonlinear media [33]. Multicomponent solitons of this class consist of some mutually incoherent light beams, coupled by cross-modulation interaction. In relation to the character of the light field distribution, their components look like some zero- and higher-order modes, confined within the nonlinear waveguide written in PRC's by themselves. We formulated an analytical algorithm of constructing of such solitons and showed that they are robust to collisions and appreciable stochastic perturbations.

The approach, used in [31,32], is based on three points that have been discussed before.

(1) A long transient time of the PRC nonlinear response enables one to neglect by interference of incoherent or shifted in carrier frequency light field components [31,32,34,35]. The situation is fully analogous to the case of a two-component soliton with cross-polarized components [33], while the maximal number of frequency-shifted components is obviously limited by the photorefractive sensitivity bandwidth.

(2) If both components of the two-component soliton (the particular case of a multicomponent soliton) do not interfere, light field distributions of its components are described by a set of coupled nonlinear Schrödinger equations [36].

(3) To find steady-state solutions of a self-consistent problem, one may reduce it to a special auxiliary linear problem [37–40].

Notice, that it is the set of points that enables us to construct a different class of multicomponent solitons by means of eigenfunctions (modes) of an auxiliary linear problem. This problem describes a beam propagating through the gradient optical waveguide with a fixed (auxiliary) spatial profile, induced by all the components. A similar approach to a problem of such kind was independently described by the authors of [34]. Though they used a more realistic model of PRC nonlinearity, taking into account saturation, a quite different iteration algorithm was here suggested. Self-consistent solutions, obtained by this way, should be an infinite series of light field components. Cutting off the series reduces the accuracy of the description of the self-consistent structure and, therefore, leads to a limitation on the distance of the stable propagation of such a soliton.

The main goal of this paper is to construct multicomponent photorefractive cnoidal waves, i.e., multicomponent periodical distributions of light field that can stably propagate through PRC's with a drift nonlinear response. We describe here the physical model used (Sec. II), the algorithm of solving the self-consistent problem (Secs. III–V), the analytical solutions obtained, consisting of up to three mutually incoherent components (Sec. VI). An analysis of stability of the solutions obtained (Sec. VII) is illustrated by examples of stochastic perturbations and collisions. Notice that, because to build up the auxiliary problem we use a number of one- and two-component cnoidal waves [9,11,37,38,41,42], known in fiber-optics and condensed matter physics, these solutions are a particular case of presented class of multicomponent cnoidal waves.

II. MODEL

The basis of our model is a well-known set of material equations [15] for the internal electric field E_{sc}, formed in PRC's due to a spatially nonuniform redistribution of charge carriers under illumination and a standard shortened wave equation [3] in a paraxial approximation with no regard for absorption. We consider the case of partial steady-state screening [43] of the external static electric field E_0, applied to PRC's in the transverse direction, by the internal field $E_{sc} \ll E_0$ in experiments with so-called "slit" beams [44], which are widely used due to a large anisotropy of PRC nonlinearity. For example, $E_0 \sim 7$ kV/cm in BTO crystals [24], but the E_0 value can be significantly reduced if one uses a more efficient photorefractive crystal such as SBN [13] with larger electro-optical constants. That means all types of photorefractive multicomponent solitons and cnoidal waves considered here, can be referred to the class of so-called 2D screening solitons [17,19]. Without taking into account the photovoltaic effect in PRC's illuminated by p mutually incoherent light field components, their spatial distributions are described by a normalized set of coupled nonlinear Schrödinger equations [31,32]

$$d^2\rho_i(\xi)/d\xi^2 \pm 2\left[\sum_{i=1}^{p} \rho_i^2(\xi) - \beta_i\right]\rho_i(\xi) = 0, \quad i = 1, \ldots, p. \tag{1}$$

Here $\rho_i(\xi)$ is the real dimensionless amplitude of the ith component, the positive constants β_i define the components' nonlinear phase shifts along the ζ axis as $\beta_i\zeta$, ξ and ζ are the dimensionless transverse and longitudinal coordinates and the signs "+" and "−" correspond to the cases of focusing and defocusing nonlinearities. Further, we will consider the former case, but both of them can be realized under a corresponding choice of PRC and E_0 orientations [15].

III. AUXILIARY LINEAR PROBLEM

The order of our further consideration will be analogous to the following [31,32].

(a) In the first stage, we remove the set (1) from the class of self-consistent problems. We build up an auxiliary linear problem, which describes a beam propagating in a periodical gradient waveguide with fixed (auxiliary) profile of the refractive index. We describe this problem in terms of its eigenvalues and eigenfunctions: the modes.

(b) In the second stage, we reset the problem to the class of self-consistent problems. That means we construct self-consistent solutions of (1) from the modes of the auxiliary problem, found on the first stage. We determine the constants of such solution decomposition on the modes, allowable spatial periods, etc.

It seems a similar procedure, though not so clearly formulated, was used by the authors of [37,38] to construct self-consistent periodical solutions of the set of two coupled nonlinear Schrödinger equations. It enabled one to describe two cross-polarized cnoidal waves propagating through a Kerr-type nonlinear medium. With respect to [37,38], we go forward and formulate an algorithm of constructing of cnoidal waves composed of an arbitrary number of components.

There is a simple and universal way to build up an auxiliary linear problem. Actually, a definite profile of the refractive index distribution corresponds to known one-component self-consistent solutions of the set (1). Let us suppose that this profile shape is the same for a certain class of multicomponent self-consistent solutions of (1). In this case, one can build up a proper linear equation (the auxiliary problem) by replacing $\Sigma_{i=1}^{p}\rho_i^2(\xi)$ with a kernel function in (1). It should be noticed that the auxiliary problem must give a possibility to scale the refractive index profile, i.e., to change its depth. After this, the initial problem (1), looking like a set of equations of motion for some coupled nonlinear oscillators, comes to the required auxiliary linear problem, i.e., to the equation of motion for the only oscillator in a field of known restoring force. One needs to find steady-state solutions (eigenfunctions and eigenvalues) of the built auxiliary problem. These eigenfunctions (the modes) can be used in the second stage to construct self-consistent multicomponent solutions of (1). Notice that one can build up some classes of multicomponent self-consistent solutions of the same problem (1) which differ in the choice of kernel function of the auxiliary problem [38,42].

To build up multicomponent solitons in [31,32], we used the auxiliary problem based on a well-known equation for Legendre associated polynomials [45]. To build up the auxiliary problem for periodical self-consistent solutions of (1), we will use the so-called Lame equation [46]. The point is that well-known [9] one-component periodical self-consistent solutions of (1) are proportional to the Jacobi elliptic functions $\mathrm{cn}(\xi)$, $\mathrm{dn}(\xi)$, and $\mathrm{sn}(\xi)$. Their periods are determined by an additional parameter: the modulus k ($1 \geq k \geq 0$) [46]. These functions form a fundamental set of solutions of the first-order Lame equation and can be expressed in terms of each other [46]:

$$\mathrm{dn}^2(\xi) = 1 - k^2\,\mathrm{sn}^2(\xi) = 1 - k^2 + k^2\,\mathrm{cn}^2(\xi). \tag{2}$$

According to the procedure described above, the kernel function of our auxiliary problem must be proportional to $\mathrm{cn}^2(\xi)$, $\mathrm{dn}^2(\xi)$, or $\mathrm{sn}^2(\xi)$. Notice that the first- and second-order Lame equations (in so-called Weierstrass form) have been

TABLE I. A full set of the eigenfunctions $\Lambda_i^{(n)}(\xi)$ and the eigenvalues $B_i^{(n)}(k)$ of the nth-order Lame equation for $n=1,2,3$.

i	$\Lambda_i^{(n)}(\xi)$	$B_i^{(n)}(k)$	$B_i^{(n)}(k\to 1)$	$B_i^{(n)}(k\to 0)$
		$n=1$		
3	$\mathrm{dn}(\xi)$	$2-k^2$	1	2
2	$\mathrm{cn}(\xi)$	1	1	1
1	$\mathrm{sn}(\xi)$	$1-k^2$	0	1
		$n=2$		
5	$\mathrm{dn}^2(\xi)+\gamma_5^{(2)}$	$2(k^2-1)/\gamma_5^{(2)}$	4	6
4	$\mathrm{cn}(\xi)\mathrm{dn}(\xi)$	$5-k^2$	4	5
3	$\mathrm{sn}(\xi)\mathrm{dn}(\xi)$	$5-4k^2$	1	5
2	$\mathrm{sn}(\xi)\mathrm{cn}(\xi)$	$2-k^2$	1	2
1	$\mathrm{dn}^2(\xi)+\gamma_1^{(2)}$	$2(k^2-1)/\gamma_1^{(2)}$	0	2
		$n=3$		
7	$\mathrm{dn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_7^{(3)}]$	$5(2-k^2)+2\sqrt{1-k^2+4k^4}$	9	12
6	$\mathrm{cn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_6^{(3)}]$	$7-2k^2+2\sqrt{4-k^2+k^4}$	9	11
5	$\mathrm{sn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_5^{(3)}]$	$7-5k^2+2\sqrt{4-7k^2+4k^4}$	4	11
4	$\mathrm{sn}(\xi)\mathrm{cn}(\xi)\mathrm{dn}(\xi)$	$4(2-k^2)$	4	8
3	$\mathrm{dn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_3^{(3)}]$	$5(2-k^2)-2\sqrt{1-k^2+4k^4}$	1	8
2	$\mathrm{cn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_2^{(3)}]$	$7-2k^2-2\sqrt{4-k^2+k^4}$	1	3
1	$\mathrm{sn}(\xi)[\mathrm{dn}^2(\xi)+\gamma_1^{(3)}]$	$7-5k^2-2\sqrt{4-7k^2+4k^4}$	0	3

used before to find two-component cnoidal waves in Kerr-type nonlinear medium [37,38]. Moreover, eigenfunctions of the Lame equation of the same orders appear almost in all one- and two-component periodical self-consistent solutions of the nonlinear Schrödinger equation known today [9,11,37,38,41,42].

We will use a form of the Lame equation with a kernel function, expressed in terms of $\mathrm{dn}^2(\xi)$,

$$d^2\Lambda_i^{(n)}(\xi)/d\xi^2+[n(n+1)\mathrm{dn}^2(\xi)-B_i^{(n)}]\Lambda_i^{(n)}(\xi)=0,$$

$$n=1,2,\ldots,\quad i=1,2,\ldots,(2n+1). \tag{3}$$

Here $\Lambda_i^{(n)}(\xi)$ is the ith eigenfunction of the nth-order Lame equation, corresponding to the eigenvalue $B_i^{(n)}$. Changing n scales the profile of the nonlinear waveguide, written down in PRC's.

The modulus k substantially affects solutions of the problem (3). When $k\to 0$ or 1, Jacobi elliptic functions asymptotically go into trigonometric or hyperbolic ones [46]:

$$\mathrm{sn}(\xi)\to\sin(\xi),\quad \mathrm{cn}(\xi)\to\cos(\xi),\quad \mathrm{dn}(\xi)\to 1\quad \text{when}\quad k\to 0,$$

$$\mathrm{sn}(\xi)\to\tanh(\xi),\quad \mathrm{cn}(\xi)\to 1/\cosh(\xi),$$

$$\mathrm{dn}(\xi)\to 1/\cosh(\xi)\quad \text{when}\quad k\to 1. \tag{4}$$

When $k\to 0$, the Lame equation goes into the equation of harmonic oscillators. It means that there are no nonlinear terms in the initial set (1). With increasing k, the nonlinear terms ''switch on'' and gradually increase. When $k\to 1$, the Lame equation goes into the equation for Legendre-associated polynomials [45], periods of elliptic functions tend to infinity, and all periodical solutions asymptotically go into spatially localized ones. That is why multicomponent solitons, described in [31,32], are the asymptotic ($k\to 1$) of the multicomponent cnoidal waves presented here.

The nth-order Lame equation has $(2n+1)$ eigenfunctions [46] and, in the second stage (Secs. IV–VI), we will use them to construct nth-order p-component cnoidal waves. For $n=1-3$, a full set of eigenfunctions $\Lambda_i^{(n)}(\xi)$ and eigenvalues $B_i^{(n)}(k)$ of Eq. (3) is given in Table I.
The following designations are used here:

$$\gamma_{1,5}^{(2)}(k)=(k^2-2\mp\sqrt{1-k^2+k^4})/3,$$

$$\gamma_{2,6}^{(3)}=(k^2-3\mp\sqrt{4-k^2+k^4})/5,$$

$$\gamma_{1,5}^{(3)}=(2k^2-3\mp\sqrt{4-7k^2+4k^4})/5,$$

$$\gamma_{3,7}^{(3)}=(2k^2-4\mp\sqrt{1-k^2+4k^4})/5. \tag{5}$$

When $k=1$, we obtain three new (with respect to [31,32]) spatially localized solutions

$$\Lambda_1^{(1)}(\xi)=\tanh(\xi),\quad \Lambda_1^{(2)}(\xi)=1/\cosh^2(\xi)-2/3,$$

$$\Lambda_1^{(3)}(\xi)=\tanh(\xi)[1/\cosh^2(\xi)-2/5]. \tag{6}$$

All corresponding eigenvalues are equal to zero. In this case, Eq. (3) goes into the equation for Legendre polynomials and modes of this type relate to the defocusing case. In contrast to [31,32], the modes (6) do not decrease at infinity. The first one ($n=1$) is a well-known dark soliton [17,18]. In the limit of $k=1$, the remaining $2n$ eigenfunctions of Eq. (3) go into the full set of solutions of Legendre-associated polynomial equations that consists of n doubly degenerated eigenfunctions.

IV. CONSTRUCTION OF NONDEGENERATE SELF-CONSISTENT SOLUTIONS

Let the components of any multicomponent self-consistent solution of Eq. (1) be proportional to eigenfunctions of the nth-order auxiliary problem (3) with unknown amplitudes $A_i^{(n)}$:

$$\rho_i^{(n)}(\xi)=A_i^{(n)}\Lambda_i^{(n)}(\alpha\xi), \quad i=1,\ldots,(2n+1). \tag{7}$$

Here α is the scale factor, and $n=1,2,\ldots$. Let us define a p-component solution of the problem (1) as the solution (7), in which only $p\leqslant(2n+1)$ amplitudes $A_i^{(n)}$ are not equal to zero for a fixed set of i numbers. To check that the sought self-consistent solutions (7) form the kernel function of the Lame equation (3), we expand $[\Lambda_i^{(n)}(\alpha\xi)]^2$ in a power series of $\mathrm{dn}^2(\alpha\xi)$ by means of Eq. (2):

$$[\Lambda_i^{(n)}(\alpha\xi)]^2=\sum_{j=0}^{n} a_{ij}^{(n)}(k)\mathrm{dn}^{2j}(\alpha\xi),$$

$$n=1,2,\ldots, \quad i=1,\ldots,(2n+1). \tag{8}$$

Here $a_{ij}^{(n)}(k)$ can be easily determined for fixed i and n. Further, substituting Eqs. (7) and (8) into Eq. (1), comparing the equations obtained with Eq. (3), and equating the coefficients at the same degrees of $\mathrm{dn}^2(\alpha\xi)$, we obtain the set of $(3n+1)$ independent linear equations in $[A_s^{(n)}]^2$ and $\beta_s^{(n)}$:

$$\beta_s^{(n)}=(\alpha^2/2)B_s^{(n)}+\Delta\beta^{(n)},$$

$$\Delta\beta^{(n)}=\sum_{i=1}^{2n+1}[A_i^{(n)}]^2 a_{i0}^{(n)}(k), \quad j=0,$$

$$s=1,\ldots,(2n+1), \tag{9}$$

$$\sum_{i=1}^{2n+1}[A_i^{(n)}]^2 a_{i1}^{(n)}(k)=(\alpha^2/2)n(n+1), \quad j=1,$$

$$\sum_{i=1}^{2n+1}[A_i^{(n)}]^2 a_{ij}^{(n)}(k)=0, \quad j=2,3,\ldots,n. \tag{10}$$

Here α and k are considered as the given parameters. It is easy to see that the first $(2n+1)$ equations (9) of the obtained set determine $\beta_s^{(n)}$ and the next n equations (10) determine $[A_i^{(n)}]^2$.

For each n value, one can construct not more than

$$C_{2n+1}^{p}=\frac{(2n+1)\times(2n)\times\cdots(2n-p+2)}{1\times2\cdots p}$$

linearly independent p-component self-consistent solutions of the problem (1). Since the set (10) is linear in $[A_i^{(n)}]^2$ and contains n independent equations, nonzero squared amplitudes $[A_i^{(n)}]^2$ are determined ambiguously for $p>n$. When $p=n$, nonzero squared amplitudes $[A_i^{(n)}]^2$ are determined uniquely. It means that there are C_{2n+1}^{n} self-consistent n-component nth-order solutions, which differ from each other by a concrete choice of the set of the modes with $[A_i^{(n)}]^2\neq0$. Moreover, this number $(p=n)$ of nonzero independent self-consistent nth-order components is minimal, because when $p<n$ the set (10) is overdetermined and has no solutions.

Not all self-consistent solutions, obtained by the described algorithm, have physical meaning because $[A_i^{(n)}]^2$ and $\beta_i^{(n)}$ must be positive. The first requirement decreases the number of allowable mode combinations. For example, if $n>1$, multicomponent self-consistent solutions cannot be composed of the only even $[a_{in}^{(n)}(k)>0]$ or odd $[a_{in}^{(n)}(k)<0]$ nth-order modes because the last equation of Eqs. (10) cannot be satisfied. The second requirement follows from consideration of focusing nonlinearity and results in a narrowing of the allowable range of k.

When $k\rightarrow1$, the number of independent eigenfunctions of Eq. (3) falls to n. In this case, the only solution of the sets (9) and (10) can be written as

$$[A_i^{(n)}]^2=2\alpha^2 i^2(n-i)!/(n+i)!, \quad \beta_i^{(n)}=(\alpha^2/2)B_i^{(n)},$$

$$i=1,\ldots n. \tag{11}$$

V. "DEGENERATE" MULTICOMPONENT SELF-CONSISTENT SOLUTIONS

We described the algorithm of constructing of multicomponent cnoidal waves with linearly independent components. However, one can construct self-consistent solutions in which some components are proportional to the same eigenfunction of Eq. (3). As previously, these components are considered to be incoherent due to, for example, different carrier frequencies. Further, by analogy with [32], such components and solutions will be called "Manakov" ones [5]. Such solutions are degenerate because $\beta_{i,1}^{(n)}\equiv\beta_{i,2}^{(n)}\equiv\cdots\equiv\beta_{i,q}^{(n)}$ and $\Lambda_{i,l}^{(n)}\equiv\Lambda_{i,2}^{(n)}\equiv\cdots\equiv\Lambda_{i,q}^{(n)}$. Here the additional index $q=1,2,\ldots$ enumerates Manakov components. Degenerate solutions can be constructed from any nondegenerate self-consistent solution of arbitrary order and, therefore, must include not fewer than n independent components. To construct a degenerate solution, the ith component of any primary nondegenerate self-consistent nth order solution can be expanded in a sum of $s=1,2,\ldots$ Manakov components. Their amplitudes $A_{i,q}^{(n)}$ must satisfy the relationship

$$[A_i^{(n)}]^2=\sum_{q=1}^{s}[A_{i,q}^{(n)}]^2. \tag{12}$$

If $n=1$, one can construct a fully degenerate self-consistent solution.

VI. NONDEGENERATE CNOIDAL WAVES UP TO THE THIRD ORDER

We have constructed nondegenerate cnoidal waves consisting of up to three mutually incoherent components. For $n=1,2,3$, we have solved the set (10) and found all self-consistent combinations of nth-order modes, their amplitudes, and the ranges of allowable spatial periods. The nonlinear phase shift velocities $\beta_i^{(n)}$ have been determined from Eq. (9) by the expressions

$$\Delta\beta^{(1)}=[A_i^{(2)}]^2/k^2-[A_2^{(2)}]^2(1-k^2)/k^2, \tag{13}$$

$$\Delta\beta^{(2)}=\sum_{i=1,5}[A_i^{(2)}]^2[\gamma_i^{(2)}]^2-[A_2^{(2)}]^2(1-k^2)/k^4,$$

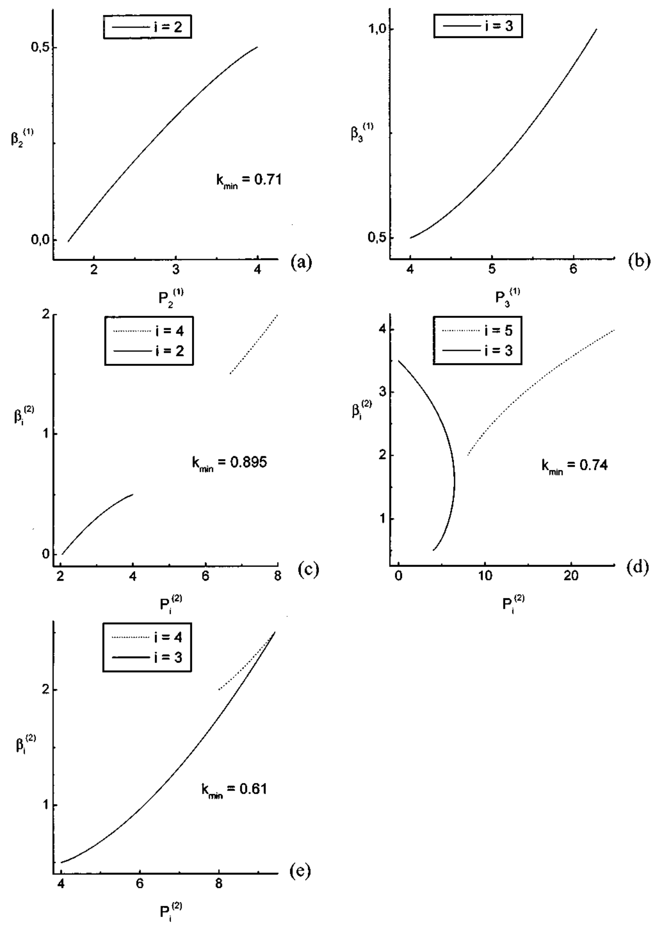

FIG. 1. β-P diagrams for cnoidal waves composed of $\rho_2^{(1)}$ (a), $\rho_3^{(1)}$ (b), $\rho_4^{(2)}$ and $\rho_2^{(2)}$ (c), $\rho_5^{(2)}$ and $\rho_3^{(2)}$ (d), and $\rho_4^{(2)}$ and $\rho_3^{(2)}$ (e); $\beta_i^{(n)}$ and $P_i^{(n)}$ are the dimensionless nonlinear phase shift velocity and averaged power of the ith component; $k_{\min}$ determines the range of cnoidal wave stability.

$$\Delta\beta^{(3)}=\sum_{i=1,5}[A_i^{(3)}]^2[\gamma_i^{(3)}]^2\bigg/k^2-\sum_{j=2,6}[A_j^{(3)}]^2[\gamma_j^{(3)}]^2(1-k^2)\bigg/k^2.$$

Depending on the concrete solution, some squared amplitudes $[A_i^{(n)}]^2$ must be zeroed here. The eigenvalues $\beta_i^{(n)}(k)$ are self-consistent in the meaning of their dependence on the concrete mode combination. Therefore, $\beta_i^{(n)}(k)$ can be different for the same mode in different solutions. This fact will be very important in further analysis of the stability of the obtained solutions (Sec. VII).

For short, p-component cnoidal waves, composed of the i_1th, i_2th,...,i_pth modes of the nth-order problem (3), will be called further the nth-order solution of "$i_1i_2\cdots i_p$" type. It is easy to check that a full number of possible one-component first-order solutions might be equal to $C_3^1=3$, but one of them ($i=1$) is realizable only in the defocusing case. One-component second- and third-order solutions can never be self-consistent (Sec. IV). All two-component first-order solutions ($C_3^2=3$) are realizable. As far as we know, such a solution of 32 type with both even (on ξ) components has never been presented before. Only four such second-order solutions (of 53, 52, 43, and 42 types) are realizable. Solutions of 31 and 21 types exist only in the defocusing case, whereas combinations of 54, 51, 41, and 32 types are never self-consistent. A second-order solution of 53 type has also never been presented before. Two-component third-order cnoidal waves do not exist for the same reason (see above). All three-component cnoidal waves have never been presented before. Only a first-order solution of the 321 type exists. A full number of three-component second-order solutions might be equal to $C_5^3=10$. However, only eight of them (of 543, 542, 532, 531, 521, 432, 431, and 421 types) are realizable. A solution of the 321 type exists only in the defocusing case whereas a combination of the 541 type can never be self-consistent. A full number of such third-order solutions might be equal to $C_7^3=35$, but only eight of them (of 753, 653, 743, 643, 752, 652, 742, and 642 types) are realizable. Solutions of 531, 521, 431, and 421 types exist in the defocusing case, whereas all other combinations can never be self-consistent.

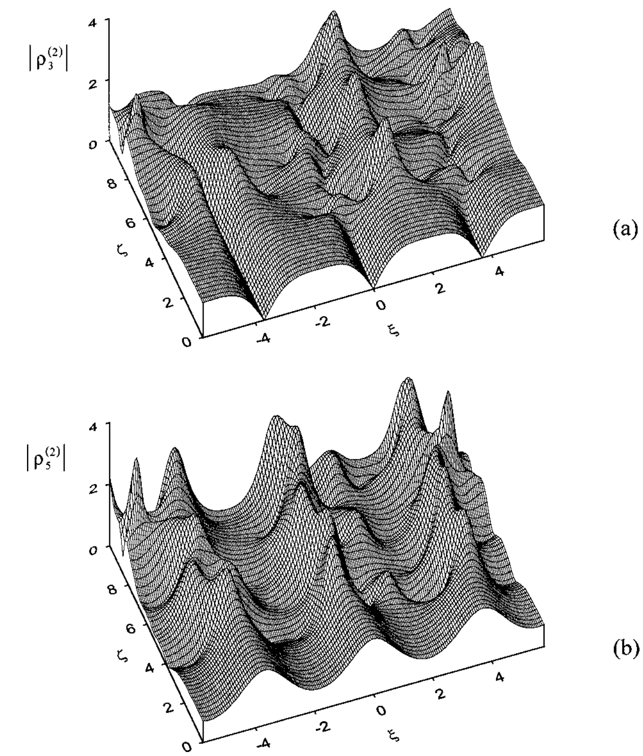

FIG. 2. Instability of cnoidal wave composed of $\rho_3^{(2)}$ and $\rho_5^{(2)}$: $|\rho_3^{(2)}(\xi,\zeta)|$ (a) and $|\rho_5^{(2)}(\xi,\zeta)|$ (b); $\rho_i^{(n)}$, ξ, and ζ are the dimensionless light field of the ith component and transverse and longitudinal coordinates; $k=0.7$; noise level $\sim$1%.

It is easy to check that nondegenerate n-component nth-order cnoidal waves are only those combinations which asymptotically go into nondegenerate n-component nth-order solitons when $k\rightarrow 1$. It means that n-component nondegenerate cnoidal waves never contain the lowest ($i=1$) mode of the nth-order problem (3) and two modes with the same asymptotic for $k\rightarrow 1$. Otherwise, the corresponding asymptotic would be a degenerate n-component Manakov soliton with $p<n$ independent components. As a result, the full number of nondegenerate n-component nth-order cnoidal waves is equal to 2^n. For $p>n$, self-consistent solutions can contain the lowest ($i=1$) mode of Eq. (3). When k

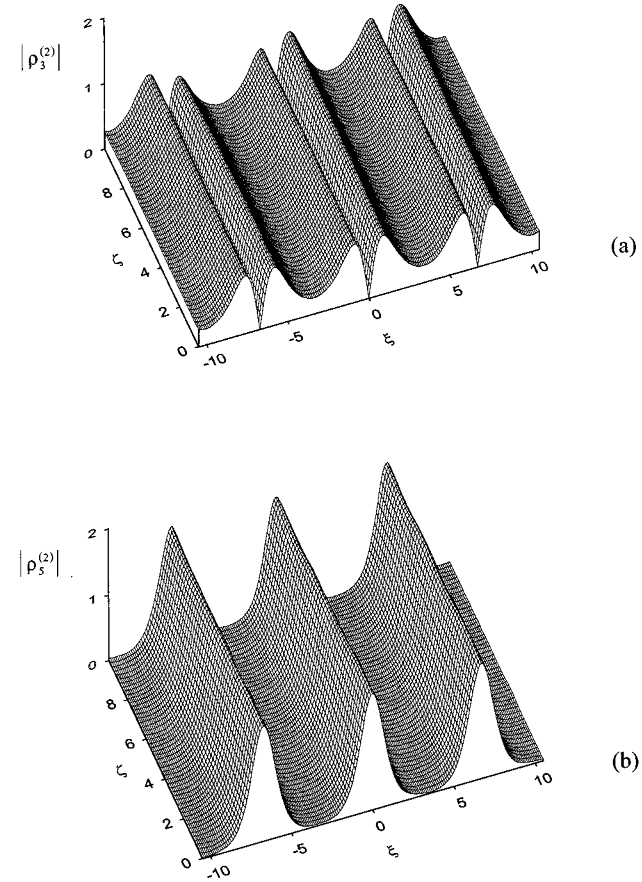

FIG. 3. Stability of cnoidal wave composed of $\rho_3^{(2)}$ and $\rho_5^{(2)}$: $|\rho_3^{(2)}(\xi,\zeta)|$ (a) and $|\rho_5^{(2)}(\xi,\zeta)|$ (b); $\rho_i^{(n)}$, ξ, and ζ are the dimensionless light field of the ith component and transverse and longitudinal coordinates; $k=0.99$; noise level $\sim$1%.

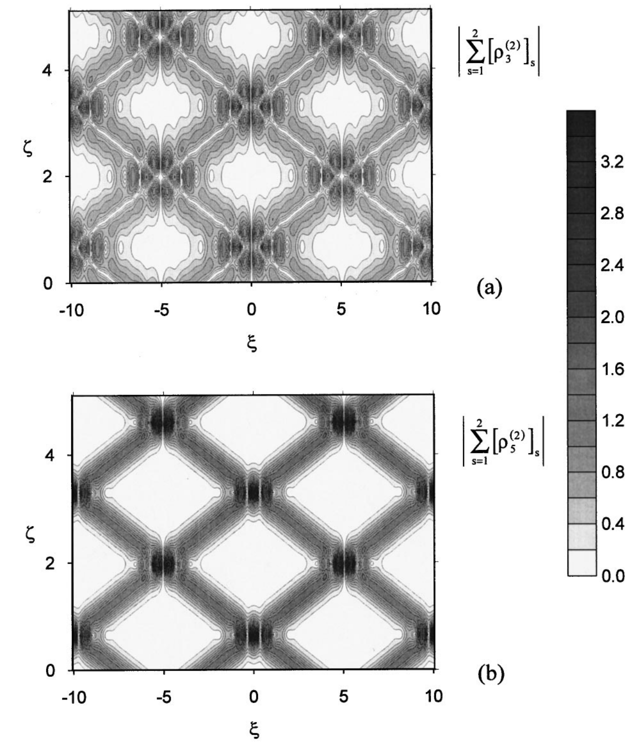

FIG. 4. Crossing of two ($s=1,2$) cnoidal waves composed of $[\rho_3^{(2)}]_s$ and $[\rho_5^{(2)}]_s$. Isolines of $|\Sigma_{s=1}^2[\rho_3^{(2)}(\xi,\zeta)]_s|$ (a) and $|\Sigma_{s=1}^2[\rho_5^{(2)}(\xi,\zeta)]_s|$ (b); $\rho_i^{(n)}$, ξ, and ζ are the dimensionless light field of the ith component and transverse and longitudinal coordinates; $k=0.99$.

$\rightarrow 1$, such a solution asymptotically goes into a soliton, consisting of the nth-order Legendre polynomial $P_n(\alpha\xi)$ (a "dark" component) and the nth Legendre associated polynomials $P_n^m(\alpha\xi)$ ("bright" components). For $n=1$ and $p=2$, such a solution is known as a self-consistent pair, composed of bright and dark solitons [7,47]. Moreover, when $p>n$, self-consistent solutions can also contain two modes with the same asymptotic for $k\rightarrow 1$. When $k\rightarrow 1$, such solutions go into partially degenerate Manakov solitons.

VII. STABILITY OF NONDEGENERATE MULTICOMPONENT CNOIDAL WAVES

Let us discuss the stability of the obtained multicomponent cnoidal waves. Though many papers concerning with the stability problem have been published [26,27,48–54], a universal approach to its solution has not been developed until now [55]. This is primarily connected with a variety of scenarios of transverse instability developments [49,51,53] and transitions of the systems considered to chaos [56]. To check the stability of the self-consistent solutions, some special techniques are mainly used. One can list, among them, a linearization technique [26,53], including some of its modifications connected with analysis of so-called modulation instability [57]; a phase portrait technique [50]; a technique connected with counting the number of negative eigenvalues of the Sturm-Liouville operator [50]; direct computer simulation of propagation of the obtained self-consistent solutions perturbed by a noise [32,58], etc.

The authors of [59] suggested a rather universal criterion of stability of fundamental nonlinear modes [60] for relatively weak nonlinearity [61]. The criterion is based on a topological analysis of the $\beta_i^{(n)}[P_i^{(n)}]$ dependence. Here $P_i^{(n)}=\int|\rho_i^{(n)}|^2 d\xi$ is the power, transported by the ith nonlinear mode of the nth order. For short, we will identify this dependence as a "β-P" diagram. The key points of analysis [59] are searching the β-P diagram for bifurcation points and clarifying the derivative $\partial\beta_i^{(n)}/\partial P_i^{(n)}$ sign. A necessary but not insufficient criterion of stability is the positive sign of the last quantity. Unfortunately, because we are interested in the stability of self-consistent combinations of some nonlinear modes, this is not the case. There are too many ways to perturb a multicomponent solution. One can add, for example, correlated or uncorrelated perturbations in the amplitudes of some components simultaneously, in the amplitude of the only component, in PRC parameters, etc. A loss of stability of the only component can destabilize a multicomponent solution as a whole due to cross-modulation. That is why, after analysis of the stability of the obtained solutions by means of the β-P diagram technique, we perform an additional checking of their stability in relation to two types of perturbations. First, by numerical integration of a shortened wave equation [3], we simulate the propagation of such waves, perturbed by an additive Gaussian noise with varying parameters: the correlation radius (along the ξ axis) and the amplitude variance. Second, in the same way, we check the stability of the solutions obtained regarding collisions with

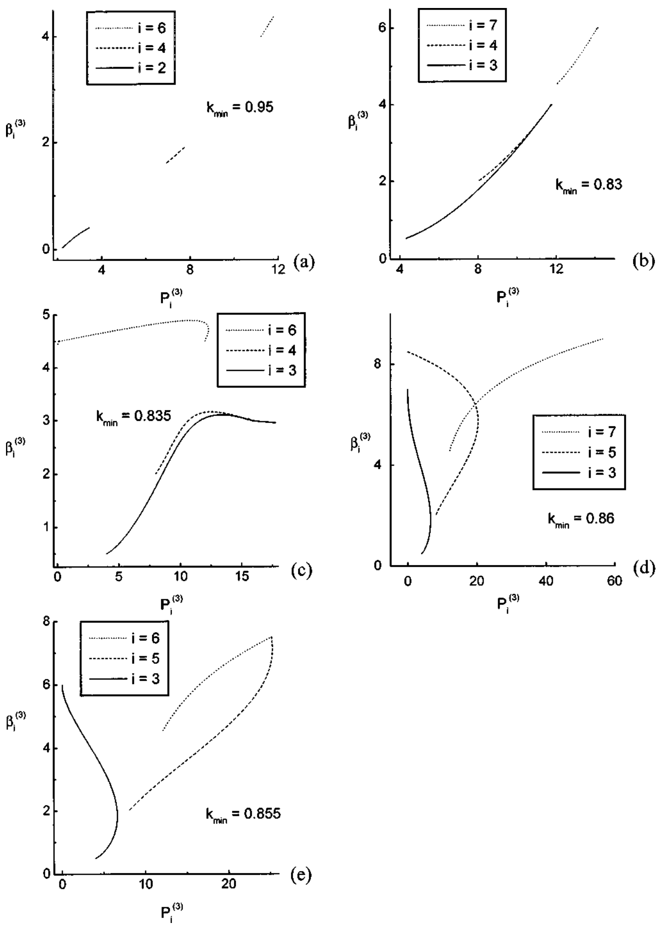

FIG. 5. β-P diagrams for cnoidal waves composed of $\rho_6^{(3)}$, $\rho_4^{(3)}$, and $\rho_2^{(3)}$ (a); $\rho_7^{(3)}$, $\rho_4^{(3)}$, and $\rho_3^{(3)}$ (b); $\rho_6^{(3)}$, $\rho_4^{(3)}$, and $\rho_3^{(3)}$ (c); $\rho_7^{(3)}$, $\rho_5^{(3)}$, and $\rho_3^{(3)}$ (d); and $\rho_6^{(3)}$, $\rho_5^{(3)}$, and $\rho_3^{(3)}$ (e); $\beta_i^{(n)}$ and $P_i^{(n)}$ are the dimensionless nonlinear phase shift velocity and averaged power of the ith component; $k_{\min}$ determines the range of cnoidal wave stability.

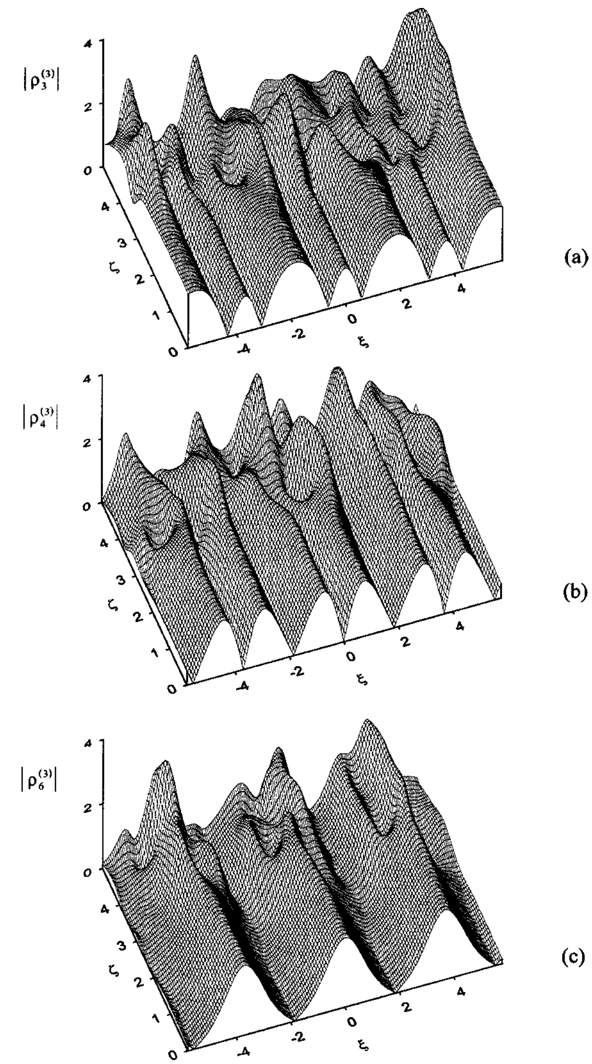

FIG. 6. Instability of cnoidal wave composed of $\rho_3^{(3)}$, $\rho_4^{(3)}$, and $\rho_6^{(3)}$: $|\rho_3^{(3)}(\xi,\zeta)|$ (a), $|\rho_4^{(3)}(\xi,\zeta)|$ (b), and $|\rho_6^{(3)}(\xi,\zeta)|$ (c); $\rho_i^{(n)}$, ξ, and ζ are the dimensionless light field of the ith component and transverse and longitudinal coordinates; $k=0.7$; noise level $\sim$1%.

each other. In this case, crossing cnoidal waves are directed into PRC's at an angle to each other and their spatial spectra do not initially ($\zeta=0$) overlap.

In our version of β-P diagrams (Figs. 1 and 5), $P_i^{(n)}$ is the averaged power, transported by the ith component of an nth-order multicomponent cnoidal wave, because, in contrast to [59], $|\rho_i^{(n)}|^2$ is integrated within the spatial period of each solution. Self-consistent values of $\beta_i^{(n)}$ are calculated by Eq. (13). Dependencies $\beta_i^{(n)}[P_i^{(n)}]$ for all the components of each multicomponent solution are plotted on its overall β-P diagram, and these dependences can be quite different for the same modes of the Lame equation in different solutions.

It is easy to check that $\partial\beta_{3,2}^{(1)}/\partial P_{3,2}^{(1)}>0$ [Figs. 1(a) and 1(b)] and both one-component cnoidal waves are potentially stable in their existence ranges ($1\geqslant k\geqslant 0$ for $\rho_3^{(1)}$ and $1\geqslant k\geqslant k_{\min}\cong 0.71$ for $\rho_2^{(1)}$). However, our numerical integration with a small additive stochastic Gaussian noise (see above) has shown that both the stability margins and propagation dynamics of $\rho_{3,2}^{(1)}$ are sharply dependent on k. When $k\rightarrow 1$ (solutions of soliton type) and $k\rightarrow 0$ (this case is realizable for $\rho_3^{(1)}$), stability margins significantly increase and decrease, because in these two cases we deal with stable bright solitons and unstable (due to modulation instability [22,57]) plane waves.

Let us discuss the stability of two-component cnoidal waves. Figure 1(d) shows the β-P diagram for the solution, composed of $\rho_5^{(2)}$ and $\rho_3^{(2)}$. The former component is potentially stable when $1\geqslant k\geqslant k_{\min}\cong 0.74$, whereas $\rho_3^{(2)}$ is potentially stable for any k. Figure 2 illustrates the development of modulation instability in both components when $k=0.7$. Figure 3 shows stable propagation of such cnoidal waves of soliton type when $k=0.99$. Figures 3(a) and 3(b) represent 2D spatial distributions of the corresponding components $|\rho_3^{(2)}|$ and $|\rho_5^{(2)}|$ of the cnoidal wave on the (ξ,ζ) plane. In both cases, the relative level of noise is the same ($\sim$1%). The collision of two such cnoidal waves of soliton type ($k=0.99$) is illustrated by Fig. 4. The corresponding interference patterns $|\Sigma_{s=1}^2[\rho_3^{(2)}(\xi,\zeta)]_s|$ and $|\Sigma_{s=1}^2[\rho_5^{(2)}(\xi,\zeta)]_s|$ are shown separately in Figs. 4(a) and 4(b) for each coherent pair of the components of crossing cnoidal waves. It is easy to see that spatial profiles of both components are robust. A quite different character is inherent to the β-P diagram of

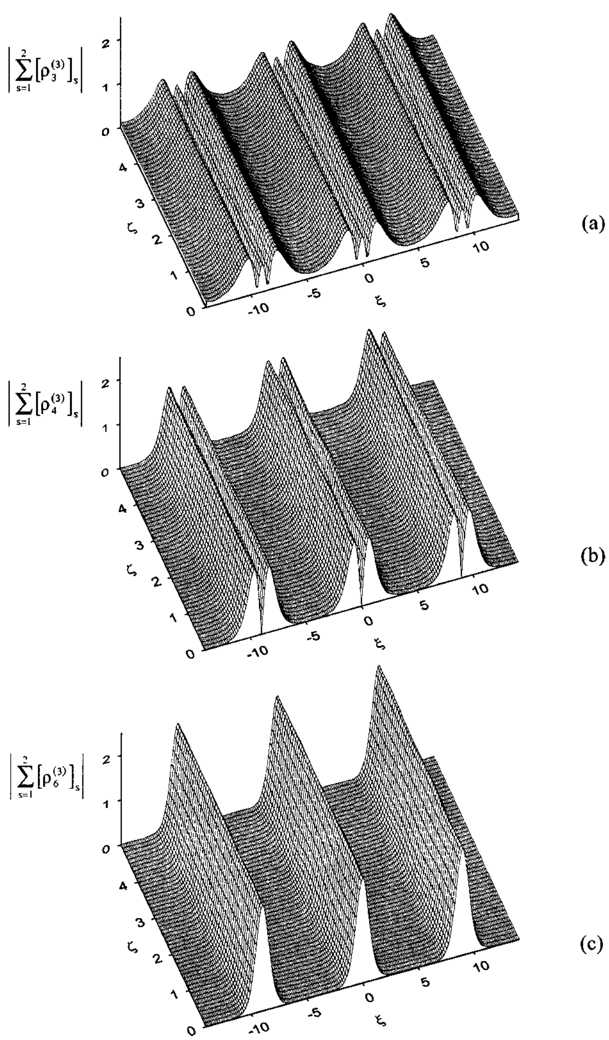

FIG. 7. Stability of cnoidal wave composed of $\rho_3^{(3)}$, $\rho_4^{(3)}$, and $\rho_6^{(3)}$: $|\rho_3^{(3)}(\xi,\zeta)|$ (a), $|\rho_4^{(3)}(\xi,\zeta)|$ (b), and $|\rho_6^{(3)}(\xi,\zeta)|$ (c); $\rho_i^{(n)}$, ξ, and ζ are the dimensionless light field of ith component, transverse, and longitudinal coordinates; $k=0.99$; noise level $\sim$1%;

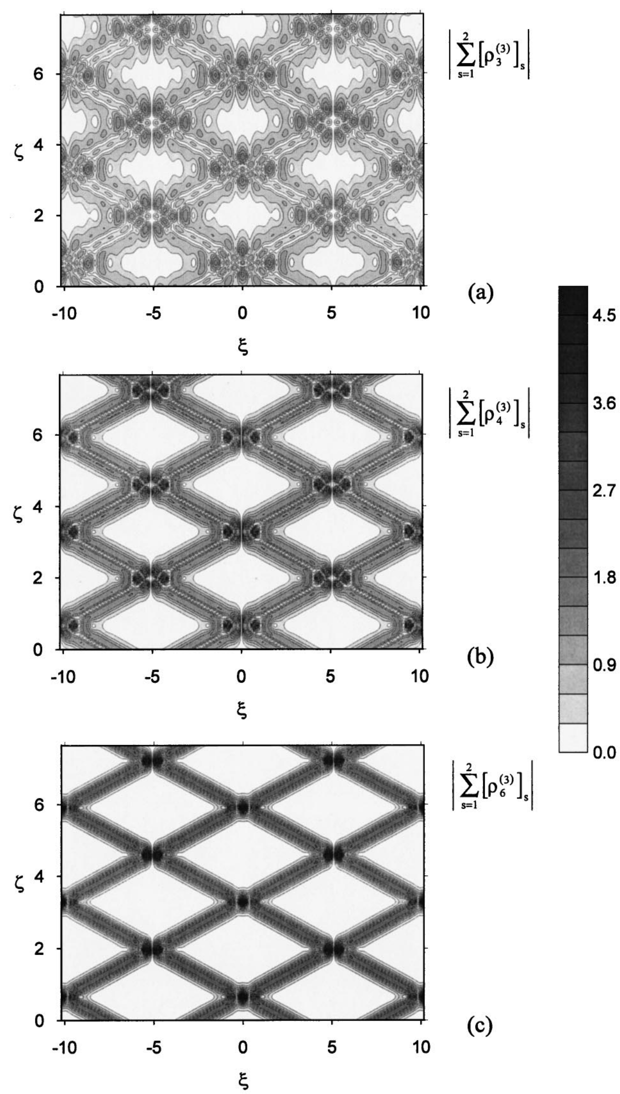

FIG. 8. Crossing of two ($s=1,2$) cnoidal waves composed of $[\rho_3^{(3)}]_s$, $[\rho_4^{(3)}]_s$, and $[\rho_6^{(3)}]_s$. Isolines of $|\Sigma_{s=1}^2[\rho_3^{(3)}(\xi,\zeta)]_s|$ (a), $|\Sigma_{s=1}^2[\rho_4^{(3)}(\xi,\zeta)]_s|$ (b), and $|\Sigma_{s=1}^2[\rho_6^{(3)}(\xi,\zeta)]_s|$ (c); $\rho_i^{(n)}$, ξ, and ζ are the dimensionless light field of the ith component and transverse and longitudinal coordinates; $k=0.99$.

the wave, composed of $\rho_4^{(2)}$ and $\rho_3^{(2)}$ [Fig. 1(e)]. Both curves have positive slopes $(\partial\beta_{4,3}^{(2)}/\partial P_{4,3}^{(2)}>0)$. However, $\rho_4^{(2)} \propto\cos(\xi)$ and $\rho_3^{(2)}\propto\sin(\xi)$ when $k\rightarrow 0$. It means that the overall intensity distribution becomes a constant and modulation instability must be observed. Our simulation has shown that such cnoidal waves can stably propagate if $1\geqslant k\geqslant k_{\min}\cong 0.61$. The waves of 52 and 42 types are well localized $(1\geqslant k\geqslant k_{\min}\cong 0.895)$ and have great stability margins. Figure 1(c) illustrates the character of the corresponding β-P diagrams.

The stability of three-component third-order cnoidal waves is illustrated by Fig. 5 (β-P diagrams), Figs. 6 and 7 [unstable ($k=0.7$) and stable ($k=0.99$) propagation of the wave composed of $\rho_6^{(3)}$, $\rho_4^{(3)}$, and $\rho_3^{(3)}$], and Fig. 8 [crossing of two such cnoidal waves ($k=0.99$)]. Two-dimensional spatial distributions of the corresponding components $|\rho_3^{(3)}|$, $|\rho_4^{(3)}|$, and $|\rho_6^{(3)}|$ (Figs. 6 and 7), as well as corresponding interference patterns $|\Sigma_{s=1}^2[\rho_3^{(3)}]_s|$, $|\Sigma_{s=1}^2[\rho_4^{(3)}]_s|$, and $|\Sigma_{s=1}^2[\rho_6^{(3)}]_s|$ for each coherent pair of the components of crossing cnoidal waves (Fig. 8) are shown separately on the (ξ,ζ) plane. Four three-component third-order cnoidal waves are always well localized $(1\geqslant k\geqslant k_{\min}\cong 0.95)$ and have great stability margins. Figure 5(a) illustrates the character of the corresponding β-P diagrams. Though stability ranges vary from one solution to other (see Fig. 5), the general regularity noticed above is the same: the stability margin of any cnoidal wave gradually increases as $k\rightarrow 1$, i.e., with increase of its spatial localization. Notice that the limitations obtained on k values (see Figs. 1 and 5), restrain not too much acceptable spatial periods of cnoidal waves. For example, if $k_{\min}\cong 0.95$, the spatial period of the corresponding solution may vary in the range from the infinity to 10.4.

So our simulation has shown that the main trouble of the criterion [59] is connected with an abrupt demarcation line between stability and instability ranges. In the case of perturbations with small finite amplitudes, a much more realistic concept looks like the stability margin of each solution because the length of its stable propagation gradually decreases as the noise level increases. That is why the ranges (Figs. 1 and 5), determined by the criterion [59], are the ranges of potential (relative) stability and absolute instability. To finish the section, notice that saturation of PRC nonlinearity [39,62] should significantly extend the stability margins of multicomponent cnoidal waves.

VIII. CONCLUSIONS AND FINAL REMARKS

So relying on the simplest model of PRC's with drift nonlinear response, we have formulated an algorithm of building up of a different class of multicomponent photorefractive cnoidal waves. We have obtained a set of equations to construct such solutions of an arbitrary order and have shown that solutions, consisting of up to three mutually incoherent components, are stable. We have shown that the considered cnoidal waves asymptotically go into multicomponent solitons [31,32] when their spatial period tends to infinity. Most parts of known one- and two-component first- and second-order cnoidal waves [9,11,37,38,41,42] are particular cases of the considered class. Some known two-component cnoidal waves (see, for example, [42]) can be easily obtained by shifting our two-component solutions in ξ by a quarter of their period. This procedure enables us to transform the considered class to another one. Its main, specific feature is missing an asymptotic transition to the multicomponent solitons of [31,32] because both spatial periods and shift of such solutions tend to infinity when $k\rightarrow 1$. As a result, all bright components turn into zero whereas dark ones go into constants. The authors of [38] have constructed a two-component cnoidal wave looking like a sum of two Lame equation modes of different orders. Actually, because of the linearity of the auxiliary problem built, the sum of its solutions of some orders is a solution of an extended problem with a kernel function equal to the sum of kernel functions of the same orders. In such a manner we can construct much more new multicomponent solutions, but obviously, not all of them will be self-consistent.

In our opinion, self-consistent multicomponent periodical solutions of the nonlinear Schrödinger equation must be of rather general character because this equation takes into account the first (cubic) term in expansion of the nonlinear polarization in a standard wave equation. In many cases, it enables one to describe the propagation of stable wave packets, composed of electronic wave functions, taking into account, for example, the electron-phonon interaction. It means the multicomponent solutions considered could be important in physics of 1D chains or 2D atomic planes in ferromagnetics, HTSC's, and conjugated polymers. Here the concept of some incoherent but bounded and stable electronic wave packets—components of the multicomponent wave packet (excitons, biexcitons, superconductive pairs, etc., condensed in a kind of Bose condensate)—might be very fruitful. The required incoherence of the components could be supplied by a phase relaxation and different carrier frequencies.

ACKNOWLEDGMENT

This work has been performed with financial support from the Russian Foundation for Basic Research (Grant Nos. 98-02-17230 and 98-02-17231).

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