Resistance of Feynman diagrams and the percolation backbone dimension

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We present an alternative view of Feynman diagrams for the field theory of random resistor networks, in which the diagrams are interpreted as being resistor networks themselves. This simplifies the field theory considerably as we demonstrate by calculating the fractal dimension D_B of the percolation backbone to three loop order. Using renormalization group methods we obtain $D_B=2+\epsilon/21-172\epsilon^2/9261+2\epsilon^3[-74\,639+22\,680\zeta(3)]/4\,084\,101$, where $\epsilon=6-d$ with d being the spatial dimension and $\zeta(3)=1.202\,057\ldots$. [S1063-651X(99)51206-3]

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Percolation has gained a vast amount of interest over the last decades (for a review, see, e.g., [1,2]). Though it represents the simplest model of a disordered system it has many applications, e.g., polymerization, porous and amorphous materials, thin films, spreading of epidemics, etc. Consider a d-dimensional lattice where each bond is randomly occupied with probability p or empty with probability $1-p$. Occupied and empty bonds may stand for different physical properties. Assume that occupied bonds are electrical conductors whereas empty sites are insulators and that currents can flow only between nearest neighbors. Suppose a potential difference is applied between two sites x and x' located on the same cluster. In general not all bonds do carry nonzero current, since there may be dangling ends. This gives rise to the notion of the backbone. It is defined as the set of bonds that are connected to both x and x' by mutually nonintersecting paths. Except for Wheatstone bridge type configurations these are the bonds that carry nonzero current. The fractal dimension D_B of the backbone is defined near the critical concentration p_c by $M_B\sim|x-x'|^{D_B}$, where M_B denotes the average number of bonds (the mass) of the backbone.

In this Rapid Communication we evaluate D_B by renormalized field theory. Our approach is based on a field theoretic formulation of the randomly diluted nonlinear resistor network by Harris [3], which itself was based on work by Stephen [4] and Harris and Lubensky [5]. The aim of this Rapid Communication is to present our interpretation of Feynman diagrams as being resistor networks themselves [6] and to employ this interpretation to derive D_B up to third order in $\epsilon=6-d$.

Consider a nonlinear generalization of the random resistor network as proposed by Kenkel and Straley [7]. The bonds between sites i and j obey a generalized Ohm's law

$$V_j-V_i=\rho_{i,j}I_{i,j}|I_{i,j}|^{r-1} \tag{1}$$

or equivalently

$$\sigma_{i,j}(V_j-V_i)|V_j-V_i|^{s-1}=I_{i,j}, \tag{2}$$

where $\sigma_{i,j}$ ($\rho_{i,j}$) is the nonlinear conductance (resistance) of the bond, $I_{i,j}$ is the current flowing through the bond, and V_i is the potential at site i. The exponents r and s are describing the nonlinearity with $r=s^{-1}$.

The power P dissipated on the backbone between x and x' of this nonlinear network reads

$$P=UI=R_r(x,x')|I|^{r+1}, \tag{3}$$

where U denotes the voltage between the two ports, I the resulting current, and $R_r(x,x')$ the resistance of the backbone. On the other hand we may write

$$P=\sum_{i,j}|V_j-V_i|I_{i,j}=\sum_{i,j}\rho_{i,j}|I_{i,j}|^{r+1}, \tag{4}$$

where the sum is taken over all bonds on the cluster. The limit $r\to-1$, taken from above, provides for a convenient way of summing up all conductors carrying nonzero current,

$$R_{-1}(x,x')=\sum_{i,j}\rho_{i,j}. \tag{5}$$

We restrict ourselves to the case that all conductors have identical resistance ρ. Hence, R_{-1} is proportional to M_B and D_B is identical to $\lim_{r\to-1}\phi_r/\nu$, where ν is the correlation length exponent and ϕ_r is the resistance exponent defined by $M_r=\langle\chi(x,x')R_r(x,x')\rangle_C/\langle\chi(x,x')\rangle_C\sim|x-x'|^{\phi_r/\nu}$. $\langle\cdots\rangle_C$ denotes the average over all configurations of the diluted lattice and $\chi(x,x')$ is an indicator function that takes the value one if x and x' are on the same cluster and zero otherwise.

The resistance $R_r(x,x')$ can be obtained by solving the circuit equations

$$\sum_j\sigma_{i,j}(V_i-V_j)|V_i-V_j|^{s-1}=I_i, \tag{6}$$

where $I_i=I(\delta_{i,x}-\delta_{i,x'})$. The circuit equations may be viewed as a consequence of the variation principle

$$\frac{\partial}{\partial V_i}\left[\frac{1}{s+1}P(\{V\})+\sum_j I_jV_j\right]=0, \tag{7}$$

where $\{V\}$ denotes the set of voltages belonging to the sites of the backbone. Obviously the backbone may contain closed loops as subnetworks. Suppose there are currents $\{I^{(l)}\}$ circulating independently around these closed loops. Then the

power is not only a function of I but also of the set of loop currents. Conservation of charge holds for every ramification of the backbone, and this gives rise to another variation principle,

$$\frac{\partial}{\partial I^{(l)}} P(\{I^{(l)}\},I)=0. \tag{8}$$

Equation (8) may be used to eliminate the loop currents and thus provides us with a method to determine the total resistance of the backbone via Eq. (3).

A field theory for the nonlinear random resistor network was set up by Harris [3] in analogy to the linear model [4,5]. In order to overcome difficulties associated with $\langle\cdots\rangle_C$ one employs the replica technique [8]. The network is replicated D fold: $V_x\to\vec{V}_x=(V_x^{(1)},\ldots,V_x^{(D)})$. One considers the correlation function $G(x,x';\vec{\lambda})=\langle\psi_{\vec{\lambda}}(x)\psi_{-\vec{\lambda}}(x')\rangle_{\rm rep}$ of $\psi_{\vec{\lambda}}(x)=\exp(i\vec{\lambda}\cdot\vec{V}_x)$, where $\vec{\lambda}\cdot\vec{V}_x=\Sigma_\alpha\lambda^{(\alpha)}V_x^{(\alpha)}$ and $\vec{\lambda}\neq\vec{0}$:

$$G(x,x';\vec{\lambda})=\left\langle Z^{-D}\int\prod_j\prod_{\alpha=1}^{D}dV_j^\alpha\exp\left(-\frac{1}{s+1}P(\{\vec{V}\})+i\vec{\lambda}\cdot(\vec{V}_x-\vec{V}_{x'})\right)\right\rangle_C. \tag{9}$$

Here $P(\{\vec{V}\})=\Sigma_{i,j,\alpha}\sigma_{i,j}|V_i^{(\alpha)}-V_j^{(\alpha)}|^{s+1}$ and Z is the usual normalization. In contrast to the linear network P is not quadratic, and hence the integration is not Gaussian. As a working hypothesis we assume that a saddle point approximation is justified. For details and conditions to be imposed on $\vec{\lambda}$, see [3]. The saddle point equation is nothing more than the variation principle stated in Eq. (7). Thus, the maximum of the integrand is determined by the solution of the circuit equations (6) and, up to an unimportant constant,

$$G(x,x';\vec{\lambda})=\left\langle\exp\left(\frac{\Lambda_r}{r+1}R_r(x,x')\right)\right\rangle_C=\langle\chi(x,x')\rangle_C\left(1+\frac{\Lambda_r}{r+1}M_r(x,x')+\cdots\right), \tag{10}$$

where $\Lambda_r=\Sigma_{\alpha=1}^{D}(-\lambda^{(\alpha)2})^{(r+1)/2}$. Note that the limit $D\to0$ has to be taken before $r\to-1$ for Eq. (10) to be well defined. Contact to the Potts model can be established by switching to voltage variables $\vec{\theta}=\Delta\theta\vec{k}$ taking discrete values on a D-dimensional torus, i.e. $\vec{k}$ is chosen to be a D-dimensional integer with $-M<k^{(\alpha)}\leq M$ and $k^{(\alpha)}=k^{(\alpha)}\bmod(2M)$. In this discrete picture there are $(2M)^D-1$ independent state variables per lattice site, and one introduces the Potts spins

$$\Phi_{\vec{\theta}}(x)=(2M)^{-D}\sum_{\vec{\lambda}\neq\vec{0}}\exp(i\vec{\lambda}\cdot\vec{\theta})\psi_{\vec{\lambda}}(x)=\delta_{\vec{\theta},\vec{\theta}_x}-(2M)^{-D} \tag{11}$$

subject to the condition $\Sigma_{\vec{\theta}}\Phi_{\vec{\theta}}(x)=0$.

The replication procedure induces the effective Hamiltonian

$$H_{\rm rep}=-\ln\left\langle\exp\left(-\frac{1}{s+1}P\right)\right\rangle_C, \tag{12}$$

which may be expanded in terms of ψ,

$$H_{\rm rep}=-\sum_{x,x'}\sum_{\vec{\lambda}\neq\vec{0}}K(\vec{\lambda})\psi_{\vec{\lambda}}(x)\psi_{-\vec{\lambda}}(x'). \tag{13}$$

Next the kernel is Taylor expanded in the limit of large σ,

$$K(\vec{\lambda})=\tau-w\Lambda_r, \tag{14}$$

with τ and $w\sim\sigma^{-1}$ being expansion coefficients and higher order terms are neglected since $H_{\rm rep}$ is decaying exponentially. By defining the discrete derivative $\partial/\partial\theta^{(\alpha)}$ through

$$-\sum_{\vec{\theta}}\Phi_{\vec{\theta}}(x)\frac{\partial^2}{\partial\theta^{(\alpha)2}}\Phi_{\vec{\theta}}(x)=\sum_{\vec{\lambda}\neq\vec{0}}\lambda^{(\alpha)2}\psi_{\vec{\lambda}}(x)\psi_{-\vec{\lambda}}(x), \tag{15}$$

one obtains upon Fourier transformation

$$K(\Delta_{\vec{\theta}})=\tau-w(\Delta_{\vec{\theta}})^{(r+1)/2}. \tag{16}$$

To set up a field theoretic Hamiltonian $\mathcal{H}$ we proceed with the usual coarse graining step and replace the Potts spins $\Phi_{\vec{\theta}}(x)$ by the order parameter $\varphi(\mathbf{x},\vec{\theta})$ defined on a d-dimensional spatial continuum. Constructing all possible invariants of the symmetries of the model from $\Sigma_{\vec{\theta}}\varphi(\mathbf{x},\vec{\theta})^p$ (p denotes some power >1) and gradients thereof leads to the following Hamiltonian in spirit of the Landau-Ginzburg-Wilson functional (for details see [6]):

$$\mathcal{H}=\int d^dx\sum_{\vec{\theta}}\left\{\frac{\tau}{2}\varphi^2+\frac{1}{2}(\nabla\varphi)^2-\frac{w}{2}\varphi(\Delta_{\vec{\theta}})^{(r+1)/2}\varphi+\frac{g}{6}\varphi^3\right\}, \tag{17}$$

where terms of higher order in the fields have been neglected since they turn out to be irrelevant in the renormalization group sense. Note that $\mathcal{H}$ reduces to the usual Potts–model Hamiltonian by setting $w=0$.

Now we set up a diagrammatic expansion. Contributing elements are the vertex $-g$ and the propagator

$$\frac{1-\delta_{\vec{\lambda},\vec{0}}}{\mathbf{p}^2+\tau-w\Lambda_r}=\frac{1}{\mathbf{p}^2+\tau-w\Lambda_r}-\frac{\delta_{\vec{\lambda},\vec{0}}}{\mathbf{p}^2+\tau}. \tag{18}$$

Equation (18) shows that the principal propagator decomposes into a propagator carrying $\vec{\lambda}$'s (conducting) and one not carrying $\vec{\lambda}$'s (insulating). This allows for a schematic decomposition of principal diagrams into sums of diagrams consisting of conducting and insulating propagators. Here a new interpretation of the Feynman diagrams emerges [6]. They may be viewed as resistor networks themselves with conducting propagators corresponding to conductors and insulating propagators to open bonds. Schwinger parameters s_i of conducting propagators correspond to resistances σ_i^{-1} and the replica variables $i\vec{\lambda}_i$ to currents. The replica currents are conserved in each vertex and we may write $\vec{\lambda}_i=\vec{\lambda}_i(\vec{\lambda},\{\vec{\kappa}\})$, where $\vec{\lambda}$ is an external current and $\{\vec{\kappa}\}$ denotes the set of

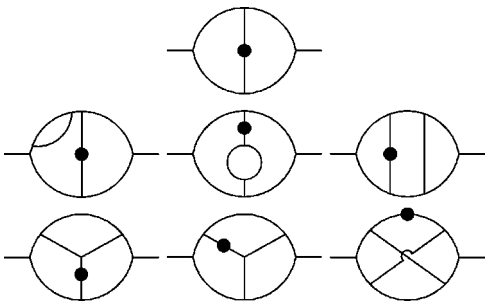

FIG. 1. The diagrams we computed to determine D_B. The lines stand for conducting propagators; the solid dots stand for $\frac{1}{2}\varphi^2$ insertions.

independent loop currents. The $\vec{\lambda}$-dependent part of a diagram can be expressed in terms of its power P,

$$\exp\left(w\sum_i s_i\Lambda_{ri}\right)=\exp[wP(\vec{\lambda},\{\vec{\kappa}\})]. \tag{19}$$

Our interpretation suggests an alternative way of computing the Feynman diagrams. To evaluate sums over independent loop currents

$$\sum_{\{\vec{\kappa}\}}\exp[wP(\vec{\lambda},\{\vec{\kappa}\})] \tag{20}$$

we employ the saddle point method. Note that the saddle point equation is nothing more than the variation principle stated in Eq. (8). Thus, solving the saddle point equations is equivalent to determining the total resistance $R(\{s_i\})$ of a diagram, and the saddle point evaluation of Eq. (20) yields

$$\exp(R_r(\{s_i\})w\Lambda_r). \tag{21}$$

A completion of squares in the momenta renders the momentum integrations straightforward. Thereafter all diagrams are of the form

$$\begin{aligned} I(\mathbf{p}^2,\vec{\lambda}^2)&=I_P(\mathbf{p}^2)+I_W(\mathbf{p}^2)w\Lambda_r+\cdots\\ &=\int_0^\infty\prod_i ds_i[1+R_r(\{s_i\})w\Lambda_r+\cdots]\\ &\quad\times D(\mathbf{p}^2,\{s_i\}), \end{aligned} \tag{22}$$

where $D(\mathbf{p}^2,\{s_i\})$ is a usual integrand of the ϕ^3 theory. The ϕ^3 theory was investigated to three loop order by de Alcantara Bonfim *et al.* [12], and hence the remaining task is to calculate the contributions proportional to w.

In order to check if our working hypothesis holds we performed two loop calculations for the cases $r\to 0$ and $r\to\infty$ and compared to known results. In the limit $r\to 0$ the resistance between two points becomes essentially equal to the length of the shortest paths between these points. We mapped our diagrams onto those studied by one of us (Janssen [9]) and obtained exactly the same diagrammatic expansion. Consequently, our result for the exponent governing the so-called chemical distance $d_{\min}=2-\epsilon/6-[937/588+45/49(\ln 2-9/10\ln 3)](\epsilon/6)^2+O(\epsilon^3)$, $\epsilon=6-d$, is the same as given in [9]. The limit $r\to\infty$ is related to the red (singly connected) bonds. Our calculation gives unity for the

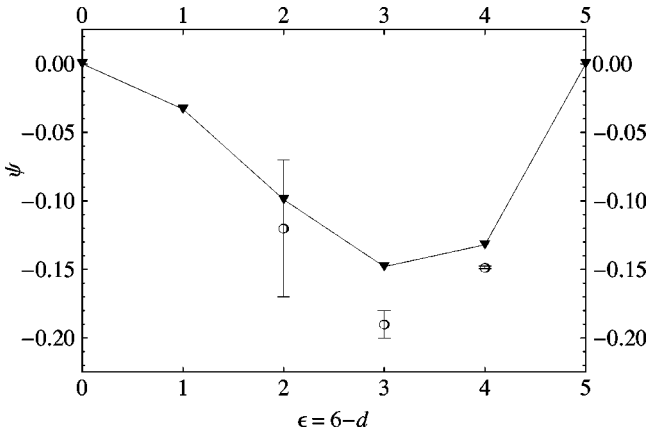

FIG. 2. Dependence of the exponent ψ on dimensionality. The rational approximation (triangles) is compared to numerical results (circles) by Grassberger ($d=2$) and Moukarzel ($d=3,4$). They determined $D_B=2-\eta+\psi=\gamma/\nu+\psi$ by simulations. For $d=2$ we insert the exact values [16,17] $\nu=4/3$ and $\gamma=43/18$. For $d=3$ we use Monte Carlo results by Ziff and Stell [18]: $\nu=0.875\pm0.008$, $\gamma=1.795\pm0.005$. For $d=4$ we take $\nu^{-1}=1.44\pm0.05$ [15] and $\gamma=1.44$ [2].

corresponding exponent in accordance with results by Blumenfeld and Aharony [10] and de Arcangelis *et al.* [11]. We rate these two loop results as a strong indication for the validity of the saddle point approach.

Now we turn to the calculation of D_B. In the limit $r\to-1$ only the nonplanar diagrams listed in Fig. 1 contribute to the diagrammatic expansion. We use dimensional regularization and renormalize $w\to Z^{-1}Z_w w$. By employing minimal subtraction to compensate ϵ poles we obtain

$$Z_w=1+\frac{u^2}{4\epsilon}+\frac{u^3}{\epsilon^2}\left[\frac{7}{12}-\frac{29}{144}\epsilon-\frac{2}{3}\zeta(3)\epsilon\right]+O(u^4), \tag{23}$$

where $u\propto g^2\mu^{-\epsilon}$, with μ being an inverse length scale. Note that all nonprimitive divergencies are cancelled as renormalizability of the perturbation expansion requires. The critical exponents are determined by the Wilson functions $\gamma_{\ldots}(u)=\mu(\partial/\partial\mu)\ln Z_{\ldots}$ evaluated at the infrared stable fixed point u^*. In particular we are interested in $\eta=\gamma(u^*)$ and $\psi=\gamma_w(u^*)$ governing the scaling relation

$$G(|\mathbf{x}-\mathbf{x}'|;w)=l^{d-2+\eta}G(l|\mathbf{x}-\mathbf{x}'|;w/l^{2-\eta+\psi}), \tag{24}$$

where l is a inverse length scale. η was calculated to order ϵ^3 in [12]. For ψ we find

$$\psi=-2\left(\frac{\epsilon}{7}\right)^2+\left[16\zeta(3)-\frac{2075}{126}\right]\left(\frac{\epsilon}{7}\right)^3+O(\epsilon^4). \tag{25}$$

The choice $l=l\mathbf{x}-\mathbf{x}'l^{-1}$ and a Taylor expansion of the right-hand side of Eq. (24) lead to

$$G(|\mathbf{x}-\mathbf{x}'|;w)=|\mathbf{x}-\mathbf{x}'|^{2-d-\eta}(1+w|\mathbf{x}-\mathbf{x}'|^{2-\eta+\psi}+\cdots). \tag{26}$$

From Eqs. (26) and (10) it follows that

$$D_B = 2 - \eta + \psi = 2 + \frac{1}{21}\epsilon - \frac{172}{9261}\epsilon^2 + 2\frac{-74\,639 + 22\,680\zeta(3)}{4\,084\,101}\epsilon^3 + O(\epsilon^4). \quad (27)$$

Note that our result agrees to second order in ϵ with calculations by Harris and Lubensky [13] based on another approach. This is again in favor of our working hypothesis.

We compare our result to numerical simulations by Grassberger [14] and Moukarzel [15]. Due to the rich structure of η in the percolation problem ψ might be better suited for such a comparison than D_B. It is known exactly that ψ vanishes in one dimension. This feature is incorporated by a rational approximation yielding

$$\psi \approx -\frac{2\epsilon^2}{49}\left(1 - \frac{\epsilon}{5}\right)\left(1 + 1.2625\frac{\epsilon}{500}\right), \quad (28)$$

which is compared to simulations in Fig. 2. For $d=4$ the results agree within the numerical errors. However, a higher accuracy of the numerical estimate is desirable. For $d=3$ and $d=2$ the analytic result looks less realistic and the numerical values are larger. The shape of the dependence of ψ on dimensionality is much the same.

We conclude with a few comments. Our interpretation of the Feynman diagrams simplifies calculations considerably. The technique used here can be applied to study other aspects of transport on percolating clusters. In $d=4$ our result for D_B agrees with recent numerical simulations. For dimensions close to the upper critical dimension 6, our result is the most accurate analytical estimate for D_B that we know of.

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