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Effect of correlation between additive and multiplicative noises on the activation from a double well

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We study the thermal activation problem of a bistable system driven by correlated additive and multiplicative noises by means of numerical simulation. We find that in the colored noise case the suppression effect of the positive correlation decreases and finally is released as the autocorrelation time τ of the colored noise grows. When τ is large enough, negative correlation becomes more suppressive than the positive correlation. Such phenomena are ascribed to the collaboration between noises as well as the memory effect of the multiplicative colored noise. [S1063-651X(99)51106-9]

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Recently the conventional problem of thermally activated escape from a potential well [1] received a surge of fresh interest in the context where the potential itself is no longer static, but disturbed by a random fluctuation [2–6]. The center of the problem is a stochastic system simultaneously driven by additive noise (AN) and multiplicative noise (MN), whose most attractive properties are relations between the mean first passage time (MFPT) and the potential fluctuation. One example is the studies on the MFPT versus the autocorrelation time τ of the potential fluctuation, which involve the well known work of resonant activation [3] and other investigations triggered by this seminal paper [4–6]. However, to our knowledge, most of the related works concentrate on the case where the MN, which accounts for potential fluctuation (PF), is independent of the AN that is responsible for thermal fluctuation (TF), and the authors did not consider the possible effects imposed by the correlation between the two noises. In recent years, it has been discovered that in systems driven by both MN and AN, the two noises can be correlated [7–9], and the correlation is able to change the steady properties of the systems greatly [10–13]. Nevertheless, how the correlation between PF and TF alters the activation process is still an interesting and unexposed problem. In Ref. [14], Madureira, Hänggi, and Wio reported their investigation into this problem, where MN is also of Gaussian white noise but correlated with the AN. They found that the transition rate of a double well system can be suppressed by the positive correlation and show a minimum as the function of the two noises' strengths ratio, which they named as ''giant suppression of the activation rate.'' In this Rapid Communication, we extend the above research work to the case that MN is of Ornstein-Uhlenbeck (OU) noise by means of numerical simulation. We focus on how the correlation strength and the auto correlation time of MN affect the relation between MFPT and MN strength.

Before presenting the model we consider, it is necessary to clarify the possibility of introducing correlation between AN and MN. In some situations, the two noises may have the same physical origin. One example is addressed in detail in Ref. [14]. Their basic point is that the parameters of the system can be affected by the identical environmental elements (e.g., temperature), and in this context noises driving the system may be coherent. For another example, in molecular biological systems or complex systems, barriers that a Brownian particle encounters come from the motion of another Brownian particle or particles [2]. In fact, in this original paper, the authors have implied that correlation between TF and PF can influence the activation process. In both of the cases, it is plausible and necessary to introduce correlation between noises and to investigate effects of the correlation. In general, the characteristic time scale of MN is different from that of AN and it is natural to consider MN as colored noise. Then the key point is how to introduce correlation between colored noise and white noise.

We study the Ginzburg-Landau bistable system simultaneously driven by MN and AN,

$$\dot{x}=x-x^3+x\epsilon(t)+\xi(t), \tag{1}$$

where $\xi(t)$ is a Gaussian white noise with correlation $\langle\xi(t)\xi(t')\rangle=2D\delta(t-t')$ and zero mean. $x\varepsilon(t)$accounts for potential disturbance by zero-mean Gaussian colored noise *ε(t)* with correlation given by $\langle\varepsilon(t)\varepsilon(t')\rangle =(Q/\tau)e^{(|t-t'|/\tau)}$. This colored noise is equivalent to the OU process depicted by

$$\dot{\varepsilon}=-\frac{\varepsilon(t)}{\tau}+\frac{\eta(t)}{\tau}, \tag{2}$$

in which $\eta(t)i$ is a Gaussian white noise with zero mean and the correlation given by $\langle\eta(t)\eta(t')\rangle=2D\delta(t-t')$. The ini-

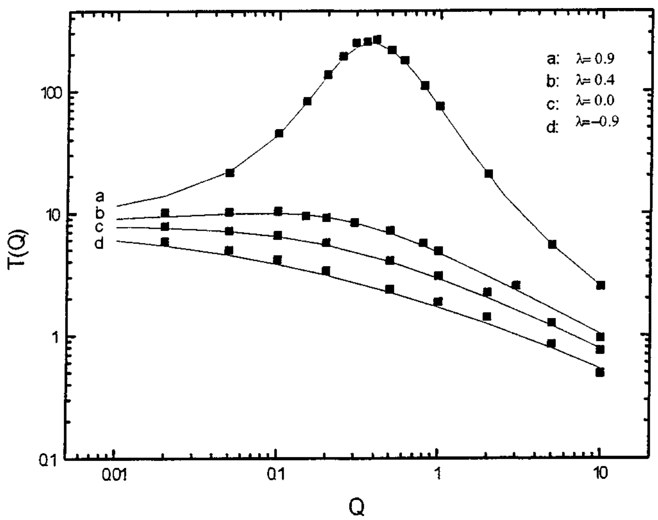

FIG. 1. White multiplicative noise case: the mean first passage time $T(Q)$ vs multiplicative noise strength Q for various values of correlation strength λ: (a) $\lambda=0.9$, (b) $\lambda=0.4$, (c) $\lambda=0.0$, and (d) $\lambda=-0.9$. The white noise strength D is 0.2. Solid lines are the results of numerical integration of Eq. (5). Solid squares stand for numerical simulation results. Note that the curve for $\lambda=0.9$ shows an abrupt peak and $\lambda=0.4$ a flat peak, while curves for $\lambda=0.0$ and $\lambda=-0.9$ fall off monotonically.

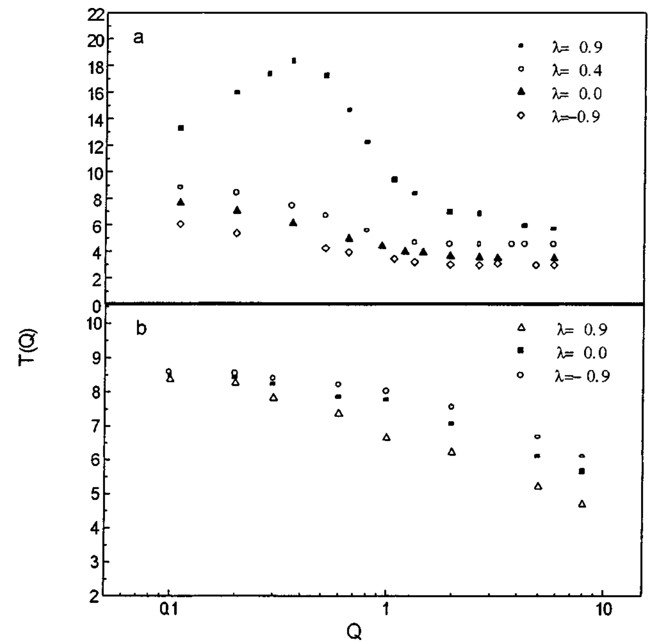

FIG. 2. Colored multiplicative noise: $T(Q)$ vs Q with different τ: (a) $\tau=0.05$ and (b) $\tau=2.0$. The white noise strength D is 0.2. Values of the parameter λ are listed in Figs. 2(a) and 2(b).

tial value of $\varepsilon(t)$ is required to be a zero-mean Gaussian random number with variance Q/τ.

We introduce the correlation between TF and PF

$$\langle \xi(t)\eta(t')\rangle = 2\lambda\sqrt{QD}\,\delta(t-t') \tag{3}$$

in which λ denotes correlation strength between noises.

We numerically simulated the double well system described by Eqs. (1) and (2). Several numerical simulation schemes have been put forward, such as the second-order stochastic Runger-Kutta (SRK) algorithm presented in Ref. [15], and the one-and-a-half order algorithm in Ref. [16]. We choose the former algorithm and extend it to our model. This algorithm is an iteration algorithm and can simplify programming. However, other algorithms are also employed to assure the correctness of results from SRK schemes. At the initial time, the sample particle starts from $x(0)=-1$ for the white noise case; for the colored noise case, the particle starts from $x(0)=-1$, $\varepsilon(0)=\zeta$ (ζ is a Gaussian random number with proper distribution). In each run, whenever the particle crosses the boundary ($x=0$), we take out this particle and record the time spent to build the MFPT. We repeat the cycle until all of the samples are used. During simulation the time step Δt is adjusted to keep $\Delta t \ll \tau$ and $\Delta t \ll 1$.The uncertainty of the values of MFPT is a few percent.

Our simulation aim to study how the mean first passage time $T(Q)$ depends on the MN strength Q and how this dependence varies with different values of τ and λ. During simulation we fix the AN strength $D=0.2$ for convenience. Some results are plotted in Figs. 1–4. The main phenomena are summarized as follows:

(i) Figure 1 is the white noise case ($\tau\to 0$). One can see that the $T(Q)$ curve for positive correlation exhibits a peak while curves for noncorrelation and negative correlation do not, and the larger the correlation strength is, the higher the peak becomes. That means positive correlation becomes more suppressive on the activation as λ grows, which is exactly the main conclusion of Ref. [14]. It should be noted that the definition of MFPT here $(-1\to 0)$ is different from that of Ref. [14] $(-1\to 1)$. However, our results show that the definition difference does not affect the phenomenon of ''suppression of activation.''

(ii) From Fig. 2(a), it can be seen that in the small τ regime, the peak of the $T(Q)$ curve for positive correlation still exists. However, it is apparent in this situation that the peak becomes lower and flatter, which denotes albeit in the small τ regime the suppression effect of the positive correlation does exist, yet this effect is weakened as τ increases. Another feature one should note is that with this small τ the curve for positive correlation descends by magnitude in contrary to the little change of those for non-correlation and

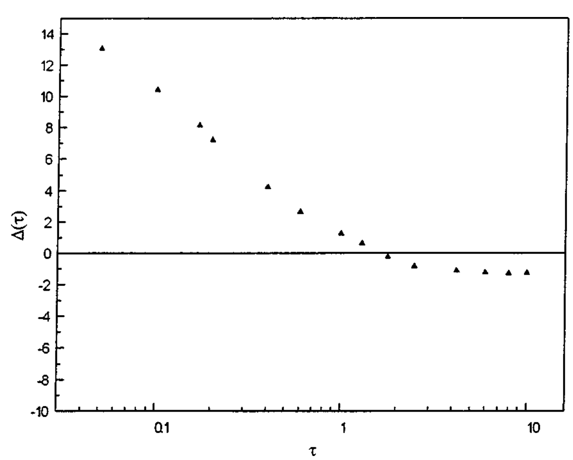

FIG. 3. $\Delta_Q(\tau)$ describing the relative effect of positive correlation of $\lambda=0.9$ to negative correlation of $\lambda=-0.9$ vs τ. The other parameters are $Q=D=0.2$.

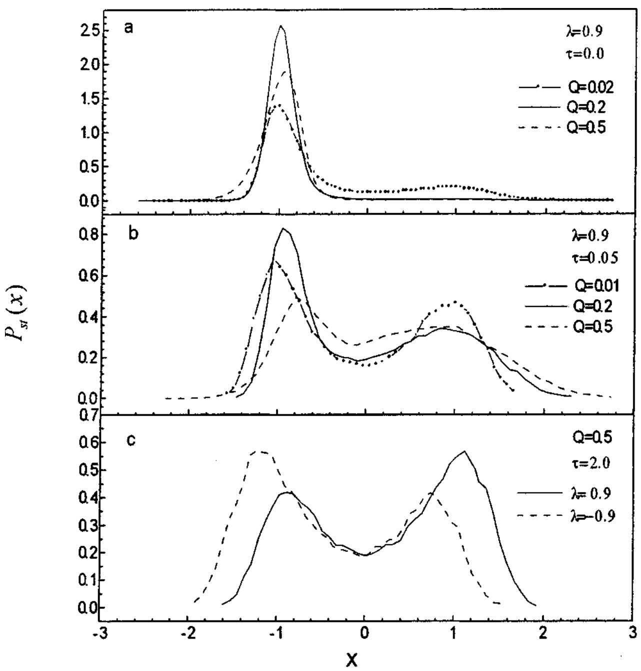

FIG. 4. Steady probability distributions for the white noise case (a) as well as the colored noise case (b) τ=0.05 and (c) τ=2.0. White noise strength D is 0.2. Other parameters are as listed in Figs. 4(a)–4(c).

negative correlation. That means MFPT difference between positive correlation and negative correlation decreases when τ grows.

(iii) Figure 2(b) shows that when τ is large enough all of the $T(Q)$ curves fall off monotonically, which means the effect of *giant suppression of activation* vanishes with large enough τ and MFPTs for all values of correlation between noises decrease monotonically versus Q.

(iv) Combining Fig. 2(b) with Fig. 2(a), one can see another feature unique for colored noise case. In Fig. 2(a), the curve for positive correlation is above those for noncorrelation and negative correlation, whereas Fig. 2(b) exhibits a contrary tendency: the curve for negative correlation is above that for positive correlation. In other words, as τ grows curves for positive and negative correlation are turned over; negative correlation becomes more suppressive. To exhibit this feature more clearly we introduce a quantity

$$\Delta_Q(\tau)=T_Q(\tau;\lambda_0)-T_Q(\tau;-\lambda_0), \qquad (4)$$

in which $T_Q(\tau;\lambda)$ is the MFPT versus τ for fixed values of Q and λ. This quantity describes the relative effect of positive correlation to that of negative correlation. When $\Delta_Q(\tau)$ is positive, correlation of λ_0 is more suppressive; when $\Delta_Q(\tau)$ is negative, correlation of λ_0 enhances activation, relatively. Figure 3 is the curve of $\Delta_Q(\tau)$ versus τ with $Q=0.2$ and $\kappa_0=0.9$. The curve decreases monotonically from positive and crosses zero approximately at τ=1.75, which means that correlation of $\lambda_0=0.9$ changes its relative effect of activation when τ crosses the approximate value of 1.75.

To comprehend the above phenomena requires some quantitative or qualitative explanations. First, let us take the white noise case. When $\tau\to 0$, our model reverts to that of Ref. [14]. The steady probability distribution $P_{st}(x)$ of state variable x is given as

$$P_{st}(x)=N^{-1}h(x)\exp\left[\int^x f(z)h^2(z)dz\right], \qquad (5)$$

in which $h(x)=[D+2\lambda\sqrt{DQ}g(x)+Qg^2(x)]^{1/2}$ and $f(x)=x-x^3$, $g(x)=x$. The integration in e exponent can be obtained by fraction integration. According to Karmer's escape theory, MFPT is given as the following:

$$T(Q)=\int_{-1}^{0}dxh(x)\times\exp\left[-\int^x dzf(z)h^2(z)\right]\int_{-\infty}^{x}dx'h(x')\times\exp\left[\int^{x'}dzf(z)h^2(z)\right]. \qquad (6)$$

An expression of MFPT can be obtained by steepest descent approximation; however, here we compute the equation by numerical integration and the results are plotted in Fig. 1 as solid lines. The difference between theory and simulation is less than 5%. Now that an analytical explanation for the white noise case has been obtained, we now turn to another qualitative and more heuristic explanation that need not consider the detailed shape of bistable potential and can be applied to the colored noise case. First, positive correlation between TF and PF means that when TF is positive, PF is statistically positive with a major probability. Because now TF and PF are both of white noise, they vary with the same time scale. Consequently, when the Brownian particle moves to a positive region due to thermal activation, the instantaneous barrier is lifted up (or equivalently the potential well is lowered) with major probability and thus the activation is suppressed. The negative correlation case is just the contrary: correlation enhances activation. Therefore, a reasonable conclusion is that MFPT of positive correlation is larger than that of negative correlation. Second, consider the suppression effect of the positive correlation. When PF strength increases, the suppression is enhanced. However, when strength of PF is larger enough than that of TF, the system is approximately driven solely by the MN. For a symmetrical bistable potential, the Brownian particles of such a system distribute mainly around the barrier, which make MFPT very small. In this way, as Q grows MFPT undergoes a maximum. Such a pattern can also be seen from the variation of $P_{st}(x)$with Q. The curves in Fig. 4(a) are obtained from Eq. (5). From this figure, one can see that when Q increases from zero, the left peak of $P_{st}(x)$ ascends. However, when Q becomes larger, two other influences of increasing Q emerge: to move the left peak of $P_{st}(x)$ to the barrier $x=0$ and make the probability diffuse to the positive region, both of which lead to the decrease of MFPT. Therefore, MFPT is certain to show a maximum as Q grows from zero. As for the negative correlation case, in a mechanism likewise, anticorrelation boosts activation and this effect is enhanced by the increase of Q, which give rise to the monotonical descent of $T(Q)$ curve.

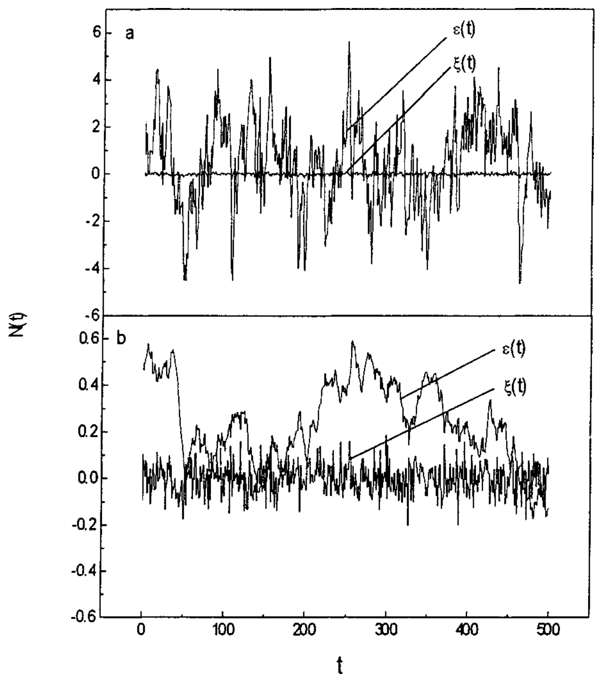

FIG. 5. Two sample trajectories *N(t)* of white noise and colored noise for (a) $\tau=0.05$ and (b) $\tau=2.0$. The other parameters are $Q=D=0.2$ and $\lambda=0.9$. Note that in the former case the time scale of colored noise approaches that of white noise, while in the latter case the colored noise becomes smoother and is no longer synchronous with the white noise.

At present, for the analytical solvable white noise case we have a heuristic explanation of the mechanism that is also applied to the colored case that cannot be solved exactly in general. In the regime of $\tau\neq 0$ but $\tau\ll 1$, the time scale of MN approaches that of AN and the mechanism of suppression due to collaboration between noises still takes effect [see the variations of $P_{st}(x)$ against Q in Fig. 4(b)]. However, as τ increases, novel features ascribed to memory of colored MN arise gradually. When τ grows the auto correlation time region of PF increases and the variance of PF Q/τ decreases, the potential fluctuates more smoothly (as shown in Fig. 5), which makes the PF lag behind TF as a consequence. For a better understanding, consider the positive correlation case. Supposing TF is negative at time *t*, then PF is also negative with a major probability. When TF changes from negative to positive at time $t\ (>t)$, PF still remains negative due to the memory effect of colored noise. Then statistically the suppression of activation is weakened, which is equivalent to the effect of decreasing correlation strength λ. In this situation, the effect of increasing Q on the suppression is lessened, which leads to the phenomenon that the peak of $T(Q)$ curve becomes lower and flatter. When Q is large enough, the increase of Q will boost activation, thus the peak of the $T(Q)$ curve will vanish. As for the negative correlation case, the mechanism of colored MN weakens the anticorrelation effect in a mechanism identical to that for positive correlation. That means the enhancement of activation is statistically weakened, which leads to the increase of MFPT of negative correlation as τ grows. When τ is large enough, negative correlation becomes more suppressive than positive correlation. This is just the ''turn over'' phenomenon shown in Fig. 3, which can also be verified by the variations of $P_{st}(x)$ against τ [see Fig. 4(c)]. In summary, the phenomena we observed can all be attributed to the collaboration between multiplicative noise and additive noise as well as the memory effect of multiplicative colored noise that destroys the statistical synchronization of TF and PF. How to quantitatively explain our discovery will be presented elsewhere.

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