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Turbulent decay of a passive scalar in the Batchelor limit: Exact results from a quantum-mechanical approach

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We show that the decay of a passive scalar θ advected by a random incompressible flow with zero correlation time in the Batchelor limit can be mapped exactly to a certain quantum-mechanical system with a finite number of degrees of freedom. The Schrödinger equation is derived and its solution is analyzed for the case where, at the beginning, the scalar has Gaussian statistics with correlation function of the form $e^{-|x-y|^2}$. Any equal-time correlation function of the scalar can be expressed via the solution to the Schrödinger equation in a closed algebraic form. We find that the scalar is intermittent during its decay and the average of $|\theta|^\alpha$ (assuming zero mean value of θ) falls as $e^{-\gamma_\alpha Dt}$ at large t, where D is a parameter of the flow, $\gamma_\alpha=\frac{1}{4}\alpha(6-\alpha)$ for $0<\alpha<3$, and $\gamma_\alpha=\frac{9}{4}$ for $\alpha\geqslant 3$, independent of α. [S1063-651X(99)51004-0]

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Kolmogorov theory (K41) [1] remains the cornerstone of our understanding of fully developed turbulence. This simple theory predicts a scaling law (the famous Kolmogorov-Obukhov $k^{-5/3}$ law) of the energy spectrum that is in remarkable agreement with experimental data. Since the 1980's, however, data gathered have consistently pointed out the failure of K41 in predicting the scaling law of high-order correlation functions [2,3]. The breakdown of K41 is closely related to the non-Gaussianity of the distribution of velocity increments. The phenomenon, dubbed intermittency, has become one of the central issues of theoretical works on turbulence. Recently, it has been found that the intermittency of a passive scalar advected by a turbulent flow might be even stronger than that for the velocity [4]. Such observations have led to the hope that the study of simple models, such as the Kraichnan model of scalar advection (see Refs. [5–10] and below), may provide clues to understanding the much more complex Navier-Stokes intermittency.

In this Rapid Communication, we consider the problem of turbulent decay of a passive scalar. In other words, we want to find statistical properties of a scalar θ satisfying the equation

$$\partial_t\theta+v_i\partial_i\theta=\kappa\Delta\theta, \tag{1}$$

where κ is a small diffusivity, v_i is a Gaussian random field, which is white in time,

$$\langle v_i(t,\mathbf{x})v_j(t',\mathbf{y})\rangle=\delta(t-t')f_{ij}(\mathbf{r}), \tag{2}$$

and

$$f_{ij}(\mathbf{r})=V\delta_{ij}-D\left(\frac{\xi+2}{\xi}\delta_{ij}r^\xi-r^{\xi-2}r_ir_j\right), \tag{3}$$

where $\mathbf{r}=\mathbf{x}-\mathbf{y}$, and ξ is some real number. The Kraichnan model usually contains a random external scalar source on the right-hand side (RHS) of Eq. (1). Such a source would make the steady state possible, but since we are interested in the decay, it is assumed that the source is absent. We will, furthermore, turn our attention to the Batchelor limit $\xi=2$, which corresponds to smooth flows with very large velocity correlation lengths (for comparison, the inertial range of real turbulence corresponds to $\xi=\frac{2}{3}$.) This limit has attracted recent interest due to its good analytical features [11,12].

Our result is that the scalar becomes more and more intermittent during the decay. Specifically, we found that the average of $\langle|\phi(x)|^\alpha\rangle$, where α is an arbitrary positive number, decays as $e^{-\gamma_\alpha Dt}$ at asymptotically large t, where $\gamma_\alpha=\frac{1}{4}\alpha(6-\alpha)$ if $\alpha<3$, and $\gamma_\alpha=\frac{9}{4}$ when $\alpha\geqslant 3$. The flatness $\langle\theta^4\rangle/\langle\theta^2\rangle\sim e^{7Dt/4}$ goes to ∞ as t grows. This is in sharp contrast with the steady-state case, where the scalar statistics is largely Gaussian [12].

To attack the problem, we will reduce it to a certain problem of quantum mechanics, which can then be solved (for another attempt to apply quantum mechanics to turbulence, see [13].) We first note that the probability distribution functional of the scalar, which will be denoted $\Psi[t,\theta]$, can be expressed in term of a path integral [14]

$$\Psi[t,\theta]=\int\mathcal{D}\pi(t,\mathbf{x})\mathcal{D}\theta(t,\mathbf{x})\mathcal{D}v_i(t,\mathbf{x})\rho[v]\times\exp\left[i\int dt\,d\mathbf{x}\,\pi(\partial_t\theta+v_i\partial_i\theta-\kappa\Delta\theta)\right], \tag{4}$$

where the Gaussian measure for the velocity $\rho[v]$ is chosen to satisfy Eq. (2). The auxiliary variable π enforces Eq. (1). Integrating over v, one obtains

$$\Psi(t,\theta)=\int\mathcal{D}\pi\,\mathcal{D}\theta\exp\left[i\int dx\,\pi\partial_t\theta-\frac{1}{2}\int dt\,d\mathbf{x}\,d\mathbf{y}\,\pi(t,\mathbf{x})\partial_i\theta(t,\mathbf{x})\times f_{ij}(\mathbf{x}-\mathbf{y})\pi(t,\mathbf{y})\partial_j\theta(t,\mathbf{y})-i\kappa\int dx\,\pi\Delta\theta\right].$$

The path integral describes the evolution in Euclidean time of a quantum field theory with the Hamiltonian [15]

$$H=\frac{1}{2}\int d\mathbf{x}\,d\mathbf{y}\,\pi(\mathbf{x})\partial_i\theta(\mathbf{x})f_{ij}(\mathbf{x}-\mathbf{y})\pi(\mathbf{y})\partial_j\theta(\mathbf{y}) + i\kappa\int dx\,\pi\Delta\theta, \tag{5}$$

where θ and π are conjugate variables satisfying the usual commutation relation $[\theta(\mathbf{x}),\pi(\mathbf{y})]=i\delta(\mathbf{x}-\mathbf{y})$. The operator ordering in Eq. (5) corresponds to the physical regularization of the path integral (4). The evolution of the distribution functional $\Psi[\theta]$ is described by the Euclidean version of the Schrödinger equation, $\partial_t\Psi=-H\Psi$. Note that the functional Ψ itself, not its square, determines the probability distribution of θ. The average of, e.g., $|\theta|^\alpha$ is defined as $\langle|\theta|^\alpha\rangle=\int\mathcal{D}\theta|\theta|^\alpha\Psi[\theta]$. In further discussion, we will use the quantum-mechanical terminology, so the terms "probability distribution functional" (PDF) and "wave function" are used interchangeably.

In the Batchelor limit (3), the Hamiltonian can be simplified considerably. We will concentrate our attention on the homogeneous case, i.e., when the system is invariant under spatial translations. In the quantum language, this means that we restrict ourselves to the states $|\Psi\rangle$ having zero total momentum, $P_i|\Psi\rangle=0$, where $P_i=\int d\mathbf{x}\,\pi(\mathbf{x})\partial_i\theta(\mathbf{x})$ [16]. With this restriction, the Hamiltonian (5) can be rewritten in the following form:

$$H=\frac{D}{2}(4L_{ij}L_{ij}-L_{ii}L_{jj}-L_{ij}L_{ji})+i\kappa D_{ii}, \tag{6}$$

where the operators L_{ij} and D_{ij} are defined as

$$L_{ij}=\int d\mathbf{x}\,x_i\pi(\mathbf{x})\partial_j\theta(\mathbf{x}),\quad D_{ij}=\int d\mathbf{x}\,\pi(\mathbf{x})\partial_i\partial_j\theta(\mathbf{x}).$$

It is straightforward to check that L_{ij} and D_{ij} form a closed algebra with the commutation relations,

$$\begin{aligned}[L_{ij},L_{kl}]&=i(\delta_{jk}L_{il}-\delta_{li}L_{kj}),\\ [L_{ij},D_{kl}]&=-i(\delta_{il}D_{jk}+\delta_{ik}D_{jl}),\\ [D_{ij},D_{kl}]&=0.\end{aligned} \tag{7}$$

The fact that the algebra is closed implies that the system is actually one with a finite number of degrees of freedom. The quantum field theory thus degenerates to quantum mechanics. Notice that the L_{ij} form a closed subalgebra. Indeed, they are the operators of linear coordinate transformations. In fact, only the SL(3,R) generators enter into the Hamiltonian (6) (cf. [10].) H is invariant under the SO(3) algebra formed by the antisymmetric part of L_{ij}.

In principle, the Schrödinger equation with H defined in Eq. (6) can be solved (at least numerically.) In this paper, we will choose a representation of the algebra (7) where H has a relatively simple form, but the physics is nontrivial. Our choice is inspired by the observation by Townsend [17] that a Gaussian-shaped hot spot preserves its Gaussianity when advected by Batchelor-limit velocity flow (for a somewhat similar discussion without quantum mechanics, see [18].) Let us for a moment concentrate on the states in which θ has Gaussian statistics. This corresponds to the wave functions of the form $\Psi[\theta]\sim\exp(-\frac{1}{2}\theta K^{-1}\theta)$, where $K(x-y)=\langle\theta(x)\theta(y)\rangle$. We will further restrict ourselves to functions K that have the Gaussian shape, $K(x-y)\sim\exp[-\frac{1}{2}b_{ij}(x-y)_i(x-y)_j]$. More strictly, we require that, in Fourier components, the spectrum of θ has the form

$$\langle\theta^*(\mathbf{k})\theta(\mathbf{k}')\rangle=\theta_0\exp\left(-\frac{1}{2}a_{ij}k_ik_j\right)\delta(\mathbf{k}-\mathbf{k}'),$$

where θ_0 is a constant independent of $a_{ij}=(b_{ij})^{-1}$ (one can choose $\theta_0=1$.) Denote such states as $|a_{ij}\rangle$. The group elements act on $|a_{ij}\rangle$ as follows:

$$\begin{aligned}e^{-i\beta_{ij}L_{ij}}|a_{ij}\rangle&=|e^{-\beta}a(e^{-\beta})^T\rangle\quad\text{if}\quad\beta_{ii}=0,\\ e^{-i\beta_{ij}D_{ij}}|a_{ij}\rangle&=|a_{ij}+4\beta_{ij}\rangle.\end{aligned} \tag{8}$$

We now choose our representation to be the one acting on the subspace of the Hilbert space that contains all linear combinations of $|a_{ij}\rangle$ (although the latter do not form an orthogonal basis.) A vector in this subspace is characterized by the function $\psi(a_{ij})$, which is the coefficient of the expansion $|\Psi\rangle=\int da_{ij}\,\psi(a_{ij})|a_{ij}\rangle$. In general, the scalar statistics in $|\Psi\rangle$ is not Gaussian. The operators L_{ij} and D_{ij} can be written as first-order differential operators with respect to a_{ij}, and the Schrödinger equation becomes a second-order PDE on ψ.

Moreover, if the initial condition is isotropic, i.e., invariant under SO(3) rotations $\epsilon_{ijk}L_{jk}$, the wave function depends only on the eigenvalues of the matrix a_{ij}, not on the Eulerian angles characterizing the orientation of the eigenvectors. The wave function is now a function of three variables, $\psi(u_1,u_2,u_3)$, where we have denoted the eigenvalues of a_{ij} as e^{2u_i}. We rescale ψ so that the state $|\Psi\rangle$ is expressed via $\psi(u)$ as

$$|\Psi\rangle=\int du_i\,dU\,\psi(u)|a(u,U)\rangle, \tag{9}$$

where $a(u,U)=U\,\mathrm{diag}(e^{2u_i})U^{-1}$, U belongs to SO(3), and the integration over U is performed using the invariant measure on the SO(3) group manifold.

The Schrödinger equation $\psi(u)$ can then be derived (details are found in [19]). It has the form

$$\partial_t\psi=D(\partial_1^2+\partial_2^2+\partial_3^2-\partial_1\partial_2-\partial_2\partial_3-\partial_3\partial_1)\psi - \sum_{i=1}^{3}[3D\partial_i(f_i\psi)+2\kappa\partial_i(e^{-2u_i}\psi)], \tag{10}$$

where $\partial_i\equiv\partial/\partial u_i$,

$$\begin{aligned}f_1&\equiv f(u_1;u_2,u_3)=\frac{e^{4u_1}-e^{2(u_2+u_3)}}{(e^{2u_1}-e^{2u_2})(e^{2u_1}-e^{2u_3})},\\ f_2&\equiv f(u_2;u_3,u_1),\quad f_3\equiv f(u_3;u_1,u_2).\end{aligned} \tag{11}$$

Special caution is required when two u_i are equal to each other; however, this will not affect our subsequent discussion.

To fully define the problem, the initial condition of $\psi(u)$ is needed. One can take as the initial state the vector $|a_{ij}\rangle$, where $a_{ij}=\mathrm{diag}(1,1,1)$. This corresponds to a scalar that has Gaussian statistics, zero mean value, and the correlation function $\langle\theta(x)\theta(0)\rangle$ proportional to $e^{-x^2/2}$ at $t=0$. The correlation length of θ is taken to be of order 1. In terms of ψ, the initial condition is $\psi(t=0,u)=\delta(u_1)\delta(u_2)\delta(u_3)$.

Equation (10) can be interpreted in an intuitive way by using a three-dimensional random walk that has the Fokker-Planck equation coinciding with Eq. (10) [20],

$$\dot{u}_i=3Df_i+2\kappa e^{-2u_i}+\xi_i, \tag{12}$$

where ξ_i are white noises that correlate as follows:

$$\begin{gathered}\xi_1+\xi_2+\xi_3=0,\\ \langle\xi_1(t)\xi_1(t')\rangle=\langle\xi_2(t)\xi_2(t')\rangle=\langle\xi_3(t)\xi_3(t')\rangle=2D\delta(t-t'),\\ \langle\xi_1(t)\xi_2(t')\rangle=\langle\xi_2(t)\xi_3(t')\rangle=\langle\xi_3(t)\xi_1(t')\rangle\\ =-D\delta(t-t').\end{gathered} \tag{13}$$

Let us discuss the physical meaning of Eq. (12). A point (u_1,u_2,u_3) corresponds to the configuration of θ having the spectrum $\langle|\theta(\mathbf{k})|^2\rangle\sim\exp(-\frac{1}{2}\Sigma e^{2u_i}k_i^2)$. In the configuration space, θ is approximately constant inside an ellipsoid with major axes proportional to e^{u_i}. When advected by the flow, this ellipsoid is subjected to random linear transformations. If the only transformations of the ellipsoids are those that stretch or compress the ellipsoid in the directions of its major axes, the results would be $\dot{u}_i=\xi_i$, where ξ_i are random. Equation (13) reflects the conservation of the volume of the ellipsoid during random stretching and compressing. However, the ellipsoid may be subjected to stretching or compressing in directions other than the major axes, as well as to shearing. These effects are accounted for by the term $3Df_i$ on the RHS of Eq. (12). The incompressibility is not violated, due to the identity $f_1+f_2+f_3=0$. The terms $2\kappa e^{-2u_i}$ are not important unless one major axis of the ellipsoid is as small as the diffusion scale. In the latter case, diffusion smears out the scalar and causes it to be correlated at a larger distance. This is exactly the effect of the $2\kappa e^{-2u_i}$ terms in the Langevin equation. Due to the sign of these terms, the volume of the ellipsoid and, hence, also $u_1+u_2+u_3$, always grows during the random walk.

Since any correlation function can be computed for $|a_{ij}\rangle$, where the scalar statistics is Gaussian, one can find any correlation function with respect to $|\Psi\rangle$ if one knows the solution to Eq. (10) (e.g., from numerical integration.) For example, the average of $|\theta|^\alpha$ ($\alpha>0$) over the state $|a(u,U)\rangle$ is proportional to $e^{-\alpha(u_1+u_2+u_3)/2}$; therefore, its average with respect to $|\Psi\rangle$ is

$$\langle|\theta|^\alpha\rangle=C_\alpha\langle\theta^2(t=0)\rangle^{\alpha/2}\int du\,\psi(u)\times\exp\left[-\frac{\alpha}{2}(u_1+u_2+u_3)\right],$$

where $C_\alpha=\pi^{-1/2}2^{\alpha/2}\Gamma[(\alpha+1)/2]$. This relation is exact.

When κ is small, the exponential behavior of $\langle|\theta|^\alpha\rangle$ can be found analytically. This can be done by using the path-integral description of the random walk (12) and finding the saddle-point trajectories that dominate $|\theta|^\alpha$ [19]. In this Rapid Communication, we use a heuristic, yet more physical, method to find the large time behavior of $\langle|\theta|^\alpha\rangle$.

Let us assume that after letting the system (12) evolve for a while, the values of u_1, u_2, and u_3 become widely separated. We assume $u_1<u_2<u_3$, and wide separation means $u_2-u_1\gg1$, $u_3-u_2\gg1$. From Eq. (11) one sees immediately that in this regime; $f_1=-1$, $f_2=0$, and $f_3=1$ (in fact, these asymptotic values of f_i are related to the Lyapunov exponents, see, e.g., Ref. [21]).

Let us first ignore the term proportional to diffusivity in Eq. (12). The velocity $\dot{u}_i$ has two contributions: one from f_i and another from the noise ξ_i. The first contribution implies that the mean values of u_i drift with constant velocities, $u_1(t)=-3Dt$, $u_2(t)=0$, and $u_3(t)=3Dt$, while the noises make u_i fluctuate around these mean values. The condition of wide separation of u's is satisfied when $t\gg D^{-1}$. The advection, on average, compresses a fluid element in one direction by a factor of e^{3Dt} and stretches it in another direction by the same factor. The remaining third direction is not substantially compressed or stretched. In this regime, the diffusion is still not operative, and $\langle|\theta|^\alpha\rangle$ remains constant.

At $t=(6D)^{-1}\ln\kappa^{-1}$ ($\gg D^{-1}$ if κ is very small), the mean value of u_1 becomes $\frac{1}{2}\ln\kappa$. The term κe^{-2u_1} in the Langevin equation (12) cannot be ignored anymore. Physically, regions of different θ have been brought this close together so that diffusion is no longer negligible. Let us consider the equation for u_1, $\dot{u}_1=-3D+2\kappa e^{-2u_1}+\xi_1$, near $u_{\min}=\frac{1}{2}\ln\kappa$. The first term on the RHS pushes u_1 toward smaller values, while the second term prevents u_1 from becoming substantially smaller than $u_{\min}$. The variable u_1 thus fluctuates around $u_{\min}$. Therefore, the random walk becomes effectively two dimensional:

$$\begin{gathered}\dot{u}_2=\xi_2,\quad \dot{u}_3=3D+\xi_3,\\ \langle\xi_2(t)\xi_2(t')\rangle=\langle\xi_3(t)\xi_3(t')\rangle=2D\delta(t-t'),\\ \langle\xi_2(t)\xi_3(t')\rangle=-D\delta(t-t').\end{gathered} \tag{14}$$

Additionally, it is required that u_2+u_3 not decrease with time, due to the previously found fact that $u_1+u_2+u_3$ can only increase (if u_2+u_3 decreases, this means that u_1 steps away from the value $u_1=u_{\min}$.) Now there is a possibility for $|\theta|^\alpha$ to decay, since it is proportional to $e^{-\alpha(u_1+u_2+u_3)/2}$, but $u_1+u_2+u_3$ is no longer a constant. Assuming that the random walk (14) starts at $u_2=u_2^0$ and $u_3=u_3^0$, the distribution of u_2 and u_3 at large times is Gaussian:

$$\rho(u_2,u_3)\sim\exp\left\{-\frac{1}{3Dt}[(u_2-u_2^0)^2+(u_3-u_3^0-3Dt)^2+(u_2-u_2^0)(u_3-u_3^0-3Dt)]\right\}. \tag{15}$$

The mean value of $|\theta|^\alpha$ can be computed by taking the average of $e^{-\alpha(u_2+u_3)/2}$ over the distribution (15). Consider the case of $0<\alpha\leqslant3$ first. The integral

$\int du_2\, du_3\, \rho(u_2,u_3) e^{-\alpha(u_2+u_3)/2}$ is dominated by the region near $u_2 - u_2^0 = -\frac{1}{2}\alpha Dt$, $u_3 - u_3^0 = (3-\alpha/2)Dt$. The value of the average is proportional to $e^{-\gamma_\alpha Dt}$, where $\gamma_\alpha = \frac{1}{4}\alpha(6-\alpha)$.

Note that the region where the integral is saturated has u_2 decreasing with time, $u_2 = u_2^0 - \frac{1}{2}\alpha Dt$. Eventually, u_2 will become as small as $u_{\min}$, and the term κe^{-2u_2} in the Langevin equation becomes important. Now, both u_1 and u_2 fluctuate around $u_{\min}$. However, as we will explain, the exponential decay law does not change. Indeed, when u_1 and u_2 remain approximately constant, the evolution of u_3 is described by the one-dimensional random walk,

$$\dot{u}_3 = 3D + \xi_3, \quad \langle \xi_3(t)\xi_3(t')\rangle = 2D\delta(t-t').$$

The distribution of u_3 is now $\rho(u_3) \sim \exp[-(4Dt)^{-1}(u_3 - u_3^0 - 3Dt)^2]$. Taking the average of $e^{-\alpha u_3/2}$ (which is proportional to $\langle|\theta|^\alpha\rangle$ since u_1 and u_2 are constant), one finds that the decay law is still $e^{-\gamma_\alpha Dt}$, where $\gamma_\alpha = \frac{1}{4}\alpha(6-\alpha)$.

For the particular case $\alpha = 2$, our result can be checked against the calculations based on the exact evolution equation for the scalar spectrum [5]. This comparison has been done; the results indeed agree.

When $\alpha > 3$, the solution $u_2 \sim -\frac{1}{2}\alpha Dt$, $u_3 \sim (3-\alpha/2)Dt$ is no longer realizable, since it has decreasing $u_2 + u_3$. The average of $|\theta|^\alpha$ is then determined by the edge of the distribution function, i.e., by $u_2 \sim -\frac{3}{2}Dt$ and $u_3 \sim \frac{3}{2}Dt$, or, after u_2 reaches $u_{\min}$, $u_2 \approx u_{\min}$ and $u_3 \sim$const. The expectation value decays as $e^{-9Dt/4}$. The reason the decay law does not contain α is the following: when $\alpha \geqslant 3$, the main contribution to $|\theta|^\alpha$ comes from the realizations in the statistical ensemble where θ is unaffected by diffusion (i.e., the ellipsoid in which θ is approximately constant has never been too thin during its evolution.) The average $\langle|\theta|^\alpha\rangle$ is thus determined by the probability of such realizations, which depends only on characteristics of the flow but not on α. This probability, as has been found, falls as $e^{-9Dt/4}$. This implies, in particular, that the flatness $\langle\theta^4\rangle/\langle\theta^2\rangle$ grows as $t^{7/4}$, meaning that the scalar becomes more and more intermittent during its decay.

More careful analysis shows that the decay law $e^{-\gamma_\alpha Dt}$ that we have found is valid only at large enough t. At intermediate t, there is a smooth transition from $\langle|\theta|^\alpha\rangle$=const to $\langle|\theta|^\alpha\rangle \sim e^{-\gamma_\alpha Dt}$ [19]. The full analysis does not change the long-time tail of $\langle|\theta|^\alpha\rangle$.

In conclusion, we have shown that by mapping to quantum mechanics, the problem of turbulent decay of a randomly advected scalar in the Batchelor limit can be made completely solvable. The power of the approach described in this paper is not limited to the calculations of $\langle|\theta|^\alpha\rangle$; analogous calculations can be done for any equal-time correlation function. For example, the long-time tail of $\langle|\partial_x\theta|^\alpha\rangle$ is also $e^{-\gamma_\alpha Dt}$ with the same γ_α. The situation here is not similar to the steady state, where the scalar and its derivatives have very different statistics, with the scalar being largely Gaussian and its derivatives being intermittent [12]. The relevance of the techniques presented and results to the general problem of intermittency is yet to be explored.

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