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Statistics of persistent events: An exactly soluble model

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It was recently realized that the persistence exponent appearing in the dynamics of nonequilibrium systems is a special member of a continuously varying family of exponents, describing generalized persistence properties. We propose and solve a simple stochastic spin model, where time intervals between spin flips are independent, and distributed according to a Lévy law. Both the limit distribution of the mean magnetization and the generalized persistence exponents are obtained exactly. We discuss the relevance of this model for phase ordering, spin glasses, and random walks. [S1063-651X(99)51301-9]

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The question of persistence for coarsening systems is to determine the fraction of space $R(t)$, which remained in the same phase up to time t [1,2]. Equivalently, in the Ising model at zero temperature, $R(t)$ is defined as the fraction of spins that did not flip up to time t [3]. In the scaling regime, $R(t)$ decays as $t^{-\theta}$, which defines the persistence exponent θ. The surprise caused by the discovery of new nontrivial exponents in the dynamics of simple nonequilibrium systems motivated a long series of works, mainly devoted to the search of simple models or experimental situations, where the persistence exponents could be computed or measured [4–9]. The existence of persistence exponents is now well established, and their nature is recognized as that of first passage exponents, probing the past history of the system.

More recently, two new aspects of persistence have been introduced [10]. Both involve the consideration of the local mean magnetization $M_t = t^{-1}\int_0^t dt'\ \sigma(t')$ of the spin at a given site for the Ising model, or of the sign of the field at a given point in space for the simple diffusion equation, both evolved from a random initial condition (see also [11]). This quantity is simply related to the fraction of time that the spin spent in the positive direction. Surprisingly, it turns out that the distribution of M_t over the entire system does not peak around zero for $t\to\infty$, but tends to a nontrivial limit distribution on $[-1,1]$, singular at both ends as $(1\mp x)^{\theta-1}$. The existence of a limit distribution is based on analytical arguments and numerical measurements for the one-dimensional (1D) Ising model at zero temperature [10], or for the diffusion equation [10,11], and can also be demonstrated in the independent interval approximation for the latter [10]. It was realized very recently that the same holds for the 2D Ising model at finite temperature, the limit distribution of M_t being now singular at $\pm m_0(T)$, the Onsager spontaneous magnetization, providing therefore a *stationary* definition of persistence at finite temperature [12].

The second new aspect is concerned with the probability of *persistent large deviations* above the level x, where $-1\leq x\leq 1$, defined as the probability that M_t remained greater than x, for all times $t'\leq t$ [10]. For the Glauber-Ising chain at zero temperature, or for the diffusion equation, this quantity was observed to decay algebraically at large times, with an exponent $\theta(x)$ continuously varying with x [10]. When $x=1$, this probability is thus the usual persistence probability, and $\theta(1)=\theta$. The existence of families of new nontrivial persistence exponents requires an explanation and poses some fundamental questions. For instance, are these exponents independent of the usual persistence exponent? Do they depend on temperature [12]? Unfortunately, computing the exact value of θ turns out to be a hard problem, so one does not expect the computation of $\theta(x)$, or even of the distribution of the mean magnetization, to be easily reachable. The origin of the difficulty is that spins at different sites are strongly correlated.

The aim of this Rapid Communication is to present a simple stochastic process that allows an exact analytical determination of both the limit distribution of the local mean magnetization M_t, and of the probability of persistent large deviations [hence of the family of exponents $\theta(x)$], the latter turning out surprisingly to be simply related to the former. One fundamental aspect of this work is therefore that it establishes the existence of the family of exponents $\theta(x)$. As we shall see, despite its simplicity, this process possesses a number of the essential features of actual coarsening models, in particular their nonstationary properties; hence, it can be seen as a very simplified model of coarsening. Finally, as will be discussed below, it brings out new views on disordered systems and random walks.

This model describes the dynamics of a single spin, where the time intervals between spin flips are independent and distributed according to a Lévy law. Such a model is actually rather natural. Consider a coarsening system at zero temperature, the Ising model for definiteness. Because of the ever growing size of domains, a spin at a given site can remain in the same direction for a very long time before a domain wall crosses this particular point and flips the spin in the reversed direction. By definition of the persistence exponent θ, the time τ before a spin is flipped is very broadly distributed, with a power-law tail decaying as $\tau^{-1-\theta}$ for large τ. The simplest approximation is therefore to neglect the correlations between the different time intervals between flips, all assumed to be distributed with the same density $p(\tau)$, decaying as $\tau^{-1-\theta}$ [13]. For simplicity, the distribution of time intervals $p(\tau)$ is chosen hereafter to be a positive stable Lévy distribution of index $0<\theta<1$ denoted by $L_\theta^b(\tau)$. (The case $\theta>1$ will be discussed below.) Its Laplace transform reads $\hat{L}_\theta^b(s)=\exp(-bs^\theta)$, where b is the scale factor of the distribution, i.e., the typical values of τ are of order $b^{1/\theta}$ [14]. As is well known, $L_\theta^b(\tau)$ decays asymptotically as $\tau^{-1-\theta}$ [14].

We have investigated the statistics of the process, both *after n sign changes*, or *at time t*, with very similar results in the asymptotic regime. After n sign changes, the time elapsed and the magnetization of the spin read

$$t_n = t_{n-1} + \tau_n, \quad S_n \equiv t_n M_n = S_{n-1} + (-)^{n-1}\tau_n, \tag{1}$$

while, at time t, they are given by

$$t = t_{N_t} + \lambda, \quad S_t \equiv t M_t = S_{N_t} + (-)^{N_t}\lambda. \tag{2}$$

In the first case, n is given and t_n is a random variable, while in the second one, t is given and N_t is the random variable equal to the largest n for which $t_n \leqslant t$. Finally λ is the length of time measured backwards from t to the last event. The corresponding distributions are defined as

$$P(n,x) = \mathcal{P}(M_n = S_n/t_n \geqslant x), \tag{3}$$

$$P(t,x) = \mathcal{P}(M_t = S_t/t \geqslant x). \tag{4}$$

For distributions which are peaked around their means at large times, these quantities are referred to as the probabilities of large deviations and are exponentially decreasing with n or t, respectively. In the present case, where $p(\tau) = L_\theta^b(\tau)$ is a positive Lévy distribution, we find the limit distribution

$$P(x) = \lim_{n\to\infty} P(n,x) = \lim_{t\to\infty} P(t,x), \tag{5}$$

$$= \frac{1}{\pi\theta}\left[\frac{\pi}{2} - \arctan\left(\frac{r\omega^{-\theta} + \cos\pi\theta}{\sin\pi\theta}\right)\right], \tag{6}$$

where $\omega = (1-x)/(1+x)$ and $r=1$ (see below).

Let us sketch the proof of Eq. (6) for $P(n,x)$, leaving the calculation of $P(t,x)$ to a longer publication [15]. We introduce T_n^+ and T_n^-, which are the lengths of time spent by the spin, respectively in the positive or negative direction, such that $t_n = T_n^+ + T_n^-$ and $S_n = T_n^+ - T_n^-$, with $T_n^+ = \tau_1 + \tau_3 + \cdots + \tau_{2k+1}$, if $n = 2k+1$, and $T_n^+ = \tau_1 + \tau_3 + \cdots + \tau_{2k-1}$, if $n = 2k$, and $T_n^- = \tau_2 + \tau_4 + \cdots + \tau_{2k}$, in both cases. Then $\mathcal{P}(S_n/t_n \geqslant x) = \mathcal{P}(T_n^-/T_n^+ \leqslant \omega)$ with $\omega = (1-x)/(1+x)$. Since T_n^+ and T_n^- are sums of stable Lévy random variables L_θ^b, they are themselves stable Lévy random variables $L_\theta^{b^\pm}$, where, using the addition rule of the scale parameters, $b^- = kb$, and $b^+ = kb$ (if $n = 2k$), or $b^+ = (k+1)b$ (if $n = 2k+1$). The determination of $P(n,x)$ therefore amounts to computing the distribution of the ratio of two Lévy laws with parameters b^- and b^+. Denoting by H the Heaviside function, and using its Laplace representation along the Bromwich contour, one finds

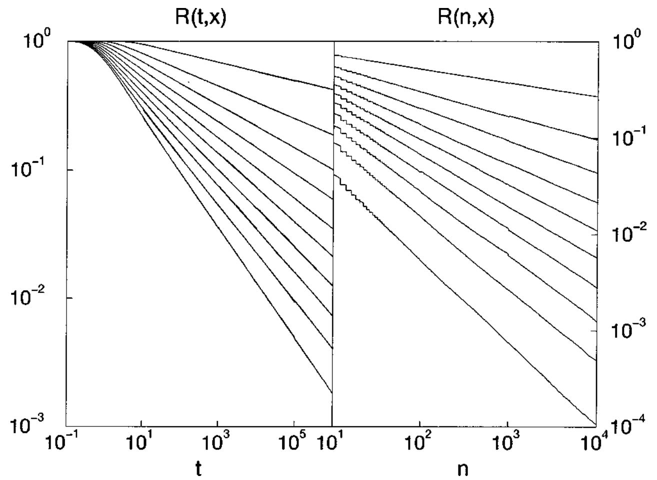

FIG. 1. Plot of $R(t,x)$ (left) and $R(n,x)$ (right) for $\theta = 1/2$ and various values of x, in log-log coordinates. The power-law behavior of both quantities for large times is clearly seen.

$$\mathcal{P}(T_n^-/T_n^+ > \omega) = \int_0^\infty d\tau_1 d\tau_2 L_\theta^{b^+}(\tau_1) L_\theta^{b^-}(\tau_2) H\left(\frac{\tau_2}{\tau_1} - \omega\right)$$
$$= \int \frac{ds}{2i\pi s}\exp[-b^+(s\omega)^\theta]$$
$$\times \exp[-b^-(-s)^\theta].$$

This integral leads to Eq. (6) with $r = b^-/b^+$. In the limit $n\to\infty$, $r\to 1$. This derivation also shows that whenever n is even, $P(n,x) = P(x)$.

The limit density $f(x) = -P'(x)$ of the mean magnetization reads

$$f(x) = \frac{\sin\pi\theta}{2\pi}\,\frac{2 + \omega + \omega^{-1}}{2\cos\pi\theta + \omega^\theta + \omega^{-\theta}}. \tag{7}$$

It is even, and diverges when $x \to \pm 1$ as $(1 \mp x)^{\theta-1}$. For $\theta < \theta_c = 0.5946\ldots$, where θ_c is the solution of $\theta_c = \cos(\pi\theta_c/2)$, $x=0$ corresponds to a minimum of $f(x)$, while for larger θ, it corresponds to a local maximum. This can be interpreted as a precursory sign of the fact that $f(x)$ tends to $\delta(x)$ for $\theta > 1$. [It also shows that $f(x)$ cannot be approximated by a β-distribution when θ is too large. In this respect, compare it to the discussion in [11].]

We now consider the probability of persistent large deviations, defined as the probability that the mean magnetization M was, for all previous times, greater than some level x. More precisely one defines the quantities $R(n,x) = \mathcal{P}(M_{n'} \geqslant x, \forall n' \leqslant n)$ and similarly $R(t,x) = \mathcal{P}(M_{t'} \geqslant x, \forall t' \leqslant t)$. Numerical computations show that both quantities decay algebraically in the asymptotic regime (see Fig. 1), respectively as

$$R(n,x) \sim n^{-\phi(x)}\ (n \gg 1), \quad R(t,x) \sim t^{-\theta(x)}\ (t \gg 1),$$

where the two families of exponents are related by $\theta(x) = \theta\phi(x)$ (see Fig. 2). This relation is indeed expected since for a given n, t_n scales as $n^{1/\theta}$. Note that by definition of the model, $\theta(1) = \theta$.

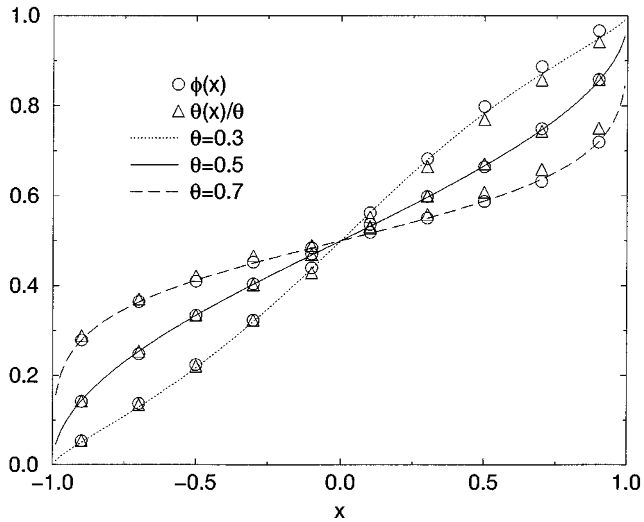

FIG. 2. Plot of the exponents $\phi(x)$ and $\theta(x)/\theta$ for $\theta=0.3,0.5,0.7$, showing that the relation $\theta(x)=\theta\phi(x)$ holds. The lines correspond to the exact result $\phi(x)=1-P(x)$.

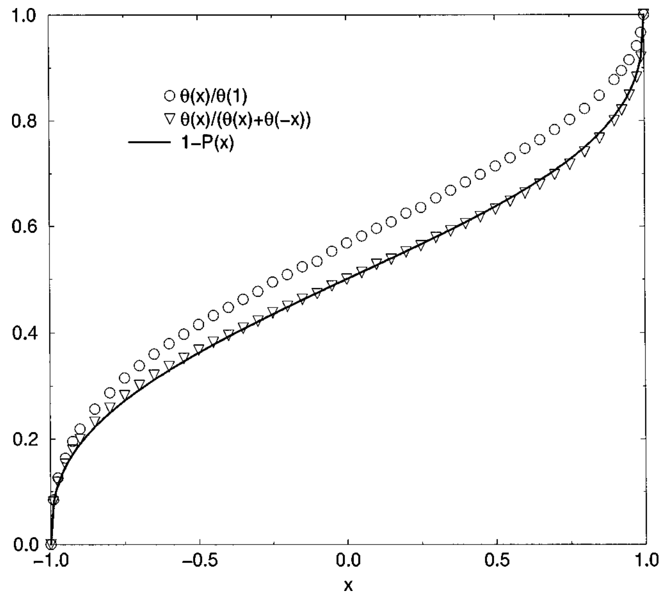

FIG. 3. Comparison between the function $\phi(x)$ for the Glauber-Ising chain, $1-P(x)$, and $\theta(x)/[\theta(x)+\theta(-x)]$ (see text).

We also observe with very good accuracy (see Fig. 2) the relation

$$\phi(x)=1-P(x)=\int_{-1}^{x} du\, f(u), \tag{8}$$

which we now establish exactly. For this, we note that $R(n,x)$ is the joint probability that $S_{n'}\geq xt_{n'}$ for all $1\leq n'\leq n$. Since clearly $R(2k,x)=R(2k+1,x)$, we assume that n is even, and write

$$R(n=2k,x)=\mathcal{P}(\xi_1\geq 0,\xi_1+\xi_2\geq 0,\ldots,\xi_1+\xi_2+\cdots+\xi_k\geq 0), \tag{9}$$

where $\xi_i=(1-x)\tau_{2i-1}-(1+x)\tau_{2i}$. Since the τ_i are positive Lévy variables of index θ, the ξ_i are also Lévy variables of index θ, with an asymmetry parameter $\beta=(\omega^\theta-1)/(\omega^\theta+1)$, which measures the relative weight of the negative and positive tails [14]. The solution to Eq. (9) for general stable Lévy variables is known [20,16].

It reads

$$R(n=2k,x)=\frac{\Gamma(k+1-q)}{\Gamma(k+1)\Gamma(1-q)}, \tag{10}$$

where $1-q$ is the probability that ξ is positive. This probability is precisely the quantity $P(n=2,x)$ introduced above, itself equal to $P(x)$. Hence $q=1-P(x)$. Finally, the large k behavior of the right-hand side (rhs) of Eq. (10) is $\propto k^{-q}$, i.e., $\phi(x)=q$, which completes the proof of Eq. (8). We checked that the plot of Eq. (10) was indistinguishable from that obtained numerically for $R(n,x)$. Equations (6), (8), and (10) are the main results of this work.

In the rest of this paper we discuss the relevance of these results to phase ordering, random walks and, unexpectedly, to some aspects of the statistical mechanics of spin glasses.

First, the stochastic process presented above, where time intervals between spin flips are independent and distributed according to a Lévy distribution, exhibits nontrivial temporal properties, both from mathematical [16,14], and physical [17,18] points of view. For example, although $p(\tau)$ is fixed in time, the probability distribution of the length of time $\tilde{\lambda}$ from some time origin (or waiting time) t_w to the next flip is nonstationary for $\theta<1$, i.e., it depends both on t_w and $\tilde{\lambda}$, while it is asymptotically independent of t_w for $\theta>1$. As a consequence, the probability that a given spin did not flip between times t_w and t_w+t is a function of t/t_w if $\theta<1$, while it is independent of t_w if $\theta>1$ [17]. Thus for $\theta<1$, this model captures the *aging* [19] nature of the persistence phenomenon. This property is deeply related to the fact that the largest τ_i in the sum $t_n=\Sigma_{i=1}^n\tau_i$ contributes to a finite fraction of t_n for $\theta<1$ even in the limit $n\to\infty$, while this fraction is asymptotically zero for $\theta\geq 1$ [18]. Correspondingly, this also ensures that the distribution of the mean magnetization does not peak around $x=0$, as was shown above.

Despite its simplicity, the model discussed here thus shares many features of more complex coarsening processes. As shown above, it leads to nontrivial predictions for the quantities $P(x)$ and $\theta(x)$. Also, the behavior of $R(t,x)$ observed in Fig. 1 strongly resembles that found in [10] for the Glauber-Ising chain or the diffusion equation. These predictions can be seen as approximations for these more general models. In Fig. 3, we compare, for the Glauber model at zero temperature, the function $\phi(x)=\theta(x)/\theta(1)$, as determined numerically in [10], with $1-P(x)$, the distribution of magnetization measured in [10]. Although there is qualitative agreement between these curves, Eq. (8) is clearly only approximate. A better approximation, following from considerations on ''exchangeable'' variables, suggests that $1-P(x)$ is actually equal to $\theta(x)/[\theta(x)+\theta(-x)]$ [15]. As can be seen in Fig. 3, this is well obeyed by the numerical results. However, the same approximation leads to much poorer results for the persistence exponents $\theta(x)$ of the diffusion equation.

The model presented here is actually, in some respects, similar to the random energy model (REM) for spin-glasses [18]. For example, the rhs of Eq. (10) is identical to the expression for the participation ratio Y_{k+1} in the REM, with a reduced temperature equal to q [18,15]. An interesting question would be to generalize this model to include some correlations between the time intervals τ_i.

We also studied the case $\theta > 1$, where $p(\tau)$ has a finite first moment. In this case, it is easy to check that t_n grows linearly with n, while S_n grows as $n^{1/\theta}$ for $1<\theta<2$ and as $\sqrt{n}$ for $\theta>2$ [14]. Hence, the quantity M_n tends to zero for large n, and $f(x)$ collapses to a δ function. However, the persistence exponents $\theta(x)$ remain well defined, and are found to be equal to $\theta(x>0)=\theta$, $\theta(x=0)=1/2$, and $\theta(x<0)=0$. This shows that the relation between $\theta(x)$ and $P(x)$ actually still holds in this degenerate case, except for $x=0$ where the value of $P(x)$ is ill defined. However, the nature of the persistence phenomenon in this model is quite different when $\theta>1$, where it becomes stationary (see above). It would be interesting to see if this is also true of more general models where $\theta>1$, such as the diffusion equation in high dimensions [5,10,11].

When $\theta=1/2$, $p(\tau)=L^b_{1/2}(\tau)$ is precisely the distribution of the time intervals between two returns to the origin of the binomial random walk with equal steps ± 1, in the regime of long times. The fraction of time spent by the walk on the positive half axis is $T_t^+/t=y$, the distribution of which is well known, and given at large times by the arcsine density $1/\pi\sqrt{y(1-y)}$, which is precisely Eq. (7), with $\theta=1/2$, and $x=S_t/t=2T_t^+/t-1$. In this respect, Eq. (7) can be considered as a generalization of the arcsine law to the case of the walk defined in this work. A striking consequence of the present work is the existence for the simple random walk of the families of exponents $\theta(x)$ and $\phi(x)$. This result brings an answer to a question raised in [10].

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