PHYSICAL REVIEW E VOLUME 59, NUMBER 2 FEBRUARY 1999

Linear and nonlinear refraction and Bragg scattering of water waves

Y. Agnon
Department of Civil Engineering, Technion, Haifa 32000, Israel
(Received 15 October 1998)

The mild slope equation (MSE) is widely used in water wave refraction-diffraction studies. An augmented extension of MSE is derived using operational calculus. It accounts for all the terms that are linear in the derivatives (to any order) of the depth. Consequently, partial previous extensions of MSE are rigorously derived. They are accurate, but only at exact Bragg resonance. This explains their success in scattering problems. Then, a nonlinear augmented MSE is derived. It extends the accuracy and the range of validity of existing models for nonlinear shoaling and scattering. [S1063-651X(99)51702-9]

PACS number(s): 47.35.+i, 41.20.Jb, 91.50.Cw, 92.10.Sx

I. BACKGROUND

The study of refraction and diffraction of irrotational water waves by an uneven bottom topography has received considerable attention. When the bottom slope is mild, the vertical coordinate, z, can be eliminated, reducing the dimension of the original three-dimensional (3D) elliptic formulation, thus yielding a more tractable problem. The dispersive nature of water waves complicates the solution and requires consideration of high order derivatives of the depth. We start with the linear time periodic case. The mild slope equation (MSE) [1] is derived by assuming propagating (not evanescent) waves and by approximating the velocity potential in the form

$$\tilde{\phi}=\mathrm{Re}[\varphi(x,y)Z(h,z)\exp(i\omega t)], \quad -h<z<0, \quad (1.1)$$

$$Z(h,z)\equiv \mathrm{sech}(kh)\cosh[k(z+h)], \quad (1.2)$$

where h is the variable water depth and k is the ''local wave number'' satisfying the linear dispersion relation:

$$\omega^2=gk\tanh(kh), \quad (1.3)$$

where g is the gravity's acceleration, the angular frequency ω is constant, and t is the time; $i=\sqrt{-1}$; φ is the complex amplitude of the velocity potential at $z=0$. $\|\nabla h\|\ll 1$, where $\nabla\equiv(\partial_x,\partial_y)$ is the horizontal gradient.

The mild slope equation [1] has the form

$$(\nabla^2+k^2)\varphi=f_1(k,h)\nabla h\cdot\nabla\varphi, \quad (1.4)$$

$$f_1\nabla h=-\nabla(cc_g)/(cc_g), \quad (1.5)$$

where cc_g is the product of the wave celerity $c=\omega/k$ and its group velocity $c_g=\omega_k$. Terms that are $O((\nabla h)^2,\nabla^2 h)$ are neglected in this equation. Nonlinear terms $[O(\epsilon^2)]$ are neglected in all the linear mild slope equations [ϵ is the characteristic wave steepness, $\omega\nabla\varphi/g=O(\epsilon)$]. The assumed vertical structure (1.2) of the velocity potential involves an $O(\nabla h)$ error in $\tilde{\phi}$. Averaging over the depth, this error gives rise to $O((\nabla h)^2,\nabla^2 h)$ errors in MSE. MSE has been derived in several different ways, using a Galerkin method, a Hamiltonian formulation, and a variational principle [2–4]. The feature of doubling the order of the error in the final equation compared to the error in the trial function is a benefit of the variational formulation.

Mei [5] gave a perturbation solution for the problem of nearly resonant Bragg scattering due to small oscillatory depth variation, $\delta\equiv h-h_0$, around a mildly sloped reference depth, h_0. He expanded the velocity potential, using δ/h as a small perturbation parameter. Addressing the same problem, Kirby [6] has obtained the extended mild slope equation (EMSE), which includes, in addition to $f_1\nabla h_0\cdot\nabla\varphi$, a term in $\nabla\delta$. For simplicity, we consider the case of a horizontal mean bottom (constant h_0), so $\nabla\delta$ becomes ∇h. EMSE is then

$$(\nabla^2+k^2)\varphi=f_0(k,h)\nabla h\cdot\nabla\varphi, \quad (1.6)$$

$$f_0(k,h)=-\mathrm{sech}^2(kh)g/(cc_g). \quad (1.7)$$

A term in δ that is usually written on the right-hand side is accounted for on the left-hand side. Terms of $O(\delta^2/h^2)$ are neglected. We see that the coefficient of ∇h in EMSE, f_0, is different from the corresponding coefficient in MSE [and modified mild slope equation (MMSE)], f_1. Dingemans [3] had used a Hamiltonian formulation to derive another equation (including a term in $\nabla^2 h$), which differs from EMSE in its coefficients. All of this calls for an explanation.

Chamberlain and Porter [2] have included $O((\nabla h)^2,\nabla^2 h)$ terms in their MMSE [terms in $(\nabla h)^2$ give rise to class II Bragg resonance and are not considered here]:

$$(\nabla^2+k^2)\varphi=f_1\nabla h\cdot\nabla\varphi+g_2\nabla^2 h\varphi, \quad (1.8)$$

$$g_2(k,h)=-\frac{g}{(cc_g)}\int_{-h}^{0}ZZ_h dz. \quad (1.9)$$

The vertical structure Z of Eq. (1.2) is assumed in the derivation of both EMSE and MMSE. With this assumption they find that the two model equations agree. MMSE is presented as a general model for wave propagation over variable topography. EMSE was also used to study Bragg scattering away from resonance.

Miles and Chamberlain [4] have noted that MMSE does not keep consistently $O((\nabla h)^2,\nabla^2 h)$ terms, arising from the neglect of $O(\nabla h)$ terms that are missing in the vertical structure of Eq. (1.2). Using a variational approach and expanding the vertical structure one order further, they derived a fourth order consistent equation. However, they use MMSE instead, since it gives very good results for scattering.

We saw that the coefficient of ∇h in MMSE (and in MSE) differed from the corresponding coefficient in EMSE. Thus, there appears to be a contradiction in accepting the two (and the equation in [3]) as general models. The picture will become clear by deriving the augmented mild slope equation (AMSE), which includes all the terms that are linear in the derivatives of h to any order:

$$(\nabla^2+k^2)\varphi=(f_1\nabla h+f_3\nabla^3 h+\cdots)\cdot\nabla\varphi + (f_2\nabla^2 h+f_4\nabla^4 h+\cdots)\varphi, \qquad (1.10)$$

where $f_n(k,h)$ will be determined.

Indeed, even the *geometry* of a periodic bathymetry cannot be approximately described by a finite number of derivatives, if its slope is not mild, unless we assume that it is purely harmonic. With this assumption and, moreover, assuming Bragg resonance, the vertical structure is that of Eq. (1.2), and EMSE and MMSE hold. MMSE cannot be used to describe wave shoaling, just as the term in ∇h in EMSE is only appropriate to describe the scattering of waves by Bragg resonance, and not their shoaling. Even on periodic bathymetries, both EMSE and MMSE deviate from the correct solution, away from Bragg resonance. We shall provide a rigorous derivation and justification for the use of MMSE (as well as EMSE) when this use is appropriate.

The linear augmented mild slope equation (AMSE) will be derived in Sec. II. It is illuminating to introduce two distinct expansions of AMSE. For the regime of shoaling on a gentle slope (''adiabatic''), the small parameter is related to the wave number of the bathymetry. Scattering, on the other hand, is dominated by Bragg resonance. Hence the expansion is in terms of the detuning from resonance. Only at exact Bragg resonance AMSE reduces to EMSE or to MMSE. The new coefficient of $\nabla^2 h$ in AMSE is up to three times bigger than the corresponding coefficient in MMSE.

Liu and Yue [7] expanded the free surface boundary condition to third order to get a regular perturbation analysis for class III (nonlinear) Bragg resonance, and applied the high order spectral method to the problem (they also study the linear problem). They state that their perturbation analysis is limited to idealized geometries. In [8], nonlinear Bragg resonance was studied using the Korteweg–de Vries (KdV) model and the nonlinear MSE [9,10].

In Sec. III we derive a set of coupled nonlinear AMSE for frequency sets that participate in triad interaction. A new form of the quadratic wave-wave interaction can be obtained. It accurately describes both free waves and bound waves (which contribute to cubic wave-wave interaction). It may also yield a nonlinear EMSE valid for class III Bragg resonance in the case of steep bottom oscillations. It proved sufficient to expand the free surface boundary condition just to second order.

II. DERIVATION OF THE LINEAR AUGMENTED MILD SLOPE EQUATION

The velocity potential of an irrotational flow is governed by the Laplace equation:

$$\tilde{\phi}_{zz}=-\nabla^2\tilde{\phi}, \quad -h<z<0. \qquad (2.1)$$

In order to account (in Sec. III) for nonlinear, triad interaction, we now write the velocity potential as

$$\tilde{\phi}=\mathrm{Re}[\phi\exp(i\omega t)+\phi_1\exp(i\omega_1 t)+\phi_2\exp(i\omega_2 t)]$$

where

$$\omega=\omega_1+\omega_2$$

The subscripts 1 and 2 will denote quantities that relate to ϕ_1 and ϕ_2, respectively. The general case is obtained simply by integration of the quadratic term over all such triads.

The combined kinematic and dynamic free surface boundary condition, after eliminating the free surface elevation (and to second order in the wave steepness, ϵ), gives the vertical velocity, $w\equiv\phi_z$ $(z=0)$ by

$$gw-\omega^2\varphi=-i(\omega(2\nabla\varphi_1\cdot\nabla\varphi_2)-[\omega/g^2(\omega_1^2\omega_2^2-\omega^2\omega_1\omega_2) + k_1^2\omega_2+k_2^2\omega_1]\varphi_1\varphi_2)\equiv F^{(2)}. \qquad (2.2)$$

$F^{(2)}$ is the leading order nonlinear free surface term (e.g. [10]). It gives rise to the quadratic coupling among frequency triads. Neglecting terms that are $O(\epsilon^3)$, ϕ_{1z} and ϕ_{2z} were approximated by $\omega_1^2\varphi_1$ and $\omega_2^2\varphi_2$, respectively. The bottom kinematic boundary condition [to $O(\epsilon\delta/h_0)$] is

$$\phi_z=-\nabla\cdot(\delta\nabla\phi)\equiv B^{(2)}, \quad z=-h_0. \qquad (2.3)$$

$B^{(2)}$ is the leading order bottom forcing term (cf. [5]). Higher order terms are found through a standard Taylor expansion of the free surface and bottom boundary conditions, but are not required in our work. Similar conditions are written for ϕ_1 and for ϕ_2.

Following [11] we use the compact notation:

$$\cos(z\nabla)\equiv\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{2n}\nabla^{2n}, \qquad (2.4)$$

$$\sin(z\nabla)\equiv\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\nabla^{2n+1}, \qquad (2.5)$$

and similar expressions for other trigonometric functions, including $\mathrm{sinc}(z\nabla)\equiv\sin(z\nabla)/(z\nabla)$. These are representations of integral operators in the form of pseudodifferential operators. They allow an accurate description of the waves without restriction to weak dispersion (such restrictions apply to the cubic Schrödinger equation and Boussinesq-type models).

The potential ϕ, which solves the Laplace Eq. (2.1), is given by a Taylor series in terms of the still water level values φ and w (the vertical velocity ϕ_z) as follows:

$$\phi=\cos(z\nabla)\varphi+z\,\mathrm{sinc}(z\nabla)w \qquad (2.6)$$

(cf. [4]). Substituting this expression into Eq. (2.3), we obtain

$$\sin(h_0\nabla)\nabla\varphi+\cos(h_0\nabla)w=-\nabla\cdot(\delta(\cos(h_0\nabla)\nabla\varphi - \sin(h_0\nabla)w)). \qquad (2.7)$$

The manipulation of operator functions is essentially the same as that of real functions and can be checked by applying the addition and multiplication properties of ∇ to the Taylor series, term by term (cf. [12,13] for a simple account of operational calculus).

Let us operate on both sides of Eq. (2.7) with sec($h_0\nabla$). We get, at the leading order,

$$\nabla \tan(h_0\nabla)\varphi + w = O(\epsilon\delta/h_0). \tag{2.8}$$

We now consider the linearized problem. From Eq. (2.2), we can replace w on the left-hand side of Eq. (2.7) by $(\omega^2/g)\varphi$, with an $O(\epsilon^2)$ error. Since the right-hand side has δ as a factor, we may approximate w there, this time by $\nabla \tan(h_0\nabla)\varphi$ [from Eq. (2.8)], maintaining an $O(\delta^2/h_0^2)$ error. Thus, the operation of sec($h_0\nabla$) on Eq. (2.7) finally gives

$$\left(\nabla \tan(h_0\nabla) + \frac{\omega^2}{g}\right)\varphi = -\sec(h_0\nabla)\boldsymbol{\nabla}\cdot(\delta(\sec(h_0\nabla)\boldsymbol{\nabla}\varphi)) \tag{2.9}$$

(since $\cos\alpha + \sin\alpha\tan\alpha \equiv \sec\alpha$).

It is important to note that due to the choice of a constant reference depth, h_0 and ∇ commute (making the analysis a great deal easier). However, δ and ∇ do not commute, since δ varies in space. On the left-hand side we now have the operator

$$[\nabla \tan h_0\nabla/(k_0 \tanh k_0h_0) + 1]\omega^2/g, \tag{2.10}$$

where k_0 is the real root of the dispersion relation (1.3) at $h = h_0$. In addition, Eq. (2.10) has a series of imaginary roots related to ik_n, which stand for the evanescent modes:

$$-k_n \tan k_nh_0 = \omega^2/g, \quad n = 1,2,\ldots . \tag{2.11}$$

As with an algebraic polynomial, the ''dispersion operator'' (2.10) can be factored into an infinite product:

$$(\nabla^2/k_0^2 + 1)\prod_{n=1}^{\infty}(1 - \nabla^2/k_n^2)\frac{\omega^2}{g} \equiv (\nabla^2 + k_0^2)/G(h_0\nabla).$$

In considering propagating waves (i.e., nonevanescent), only the first factor, $(\nabla^2/k_0^2 + 1)$, is small [it is $O(\nabla h)$ and vanishes on a flat bottom]. The other factors were collected in $1/G$. $G(h_0\nabla)$ is an operator that is even in $h_0\nabla$ and in $\kappa \equiv h_0k_0$:

$$G(p) \equiv h_0^{-1}(p^2 + \kappa^2)/[p \tan(p) + \kappa \tanh(\kappa)]. \tag{2.12}$$

In order to get an MSE-type equation, we operate on Eq. (2.9) by the nonsingular component, $G(h_0\nabla)$. This yields a left-hand side in the form of the Helmholtz equation (and of MSE), appropriate for the restriction to propagating modes, but without further approximations of the vertical structure (as those of MSE, EMSE, and MMSE):

$$(\nabla^2 + k_0^2)\varphi = -G(h_0\nabla)\sec(h_0\nabla)\boldsymbol{\nabla}\cdot(\delta(\sec(h_0\nabla)\boldsymbol{\nabla}\varphi)).$$

Let us introduce operator notation. At the leading order in δ/h_0, the operation of $h_0^2\nabla^2$ on φ is seen from the above equation to be equivalent to multiplication of φ by $-\kappa^2$. Thus, to $O(\delta/h_0)$, we may approximate the operation of $h_0\nabla$ on φ on the right-hand side of the above equation by the symbol $i\kappa$. Even powers, $(i\kappa)^{2n}$, correspond to multiplication by $(-\kappa^2)^n$; these contribute to the terms $f_2, f_4, \ldots$ in Eq. (1.10). Odd powers, $(i\kappa)^{2n+1}$, correspond to $(-\kappa^2)^n h_0\boldsymbol{\nabla}$, and contribute to the terms $f_1, f_3, \ldots$. The error is $O(\delta^2/h_0^2)$. The operation of $h_0\boldsymbol{\nabla}$ on δ is denoted by $i\vec{K}$, and the operation of $h_0\boldsymbol{\nabla}$ on $\delta\phi$ has the symbol $i(\vec{K} + \vec{\kappa})$, since

$$\boldsymbol{\nabla}(\delta\varphi) = (\boldsymbol{\nabla}\delta)\varphi + \delta(\boldsymbol{\nabla}\varphi). \tag{2.13}$$

Maintaining an $O(\delta^2/h_0^2)$ error, we have a compact explicit form of the augmented mild slope equation (AMSE) (1.10):

$$(\nabla^2 + k_0^2)\varphi = -G(i\vec{K} + i\vec{\kappa})\text{sech}(\vec{K} + \vec{\kappa})\text{sech}(\kappa)\boldsymbol{\nabla}\cdot(\delta\boldsymbol{\nabla}\varphi). \tag{2.14}$$

AMSE retains all the terms that are linear in δ and its derivatives, to any order. If δ is sinusoidal, G is given in closed form. For a *general bathymetry* we may choose between two distinct approximations:

(1) The mild slope equation is known to hold under the assumption, $\|\boldsymbol{\nabla}h\| \ll \kappa$, i.e., the depth variation is small over a wavelength. Expanding AMSE in powers of $\vec{K}$, we get MSE by neglecting terms of order K^2 and higher. Keeping *all* $O(K^2)$ terms, which represent the second derivatives of the depth, we get the new explicit equation

$$(\nabla^2 + k^2)\varphi = f_1\boldsymbol{\nabla}h\cdot\boldsymbol{\nabla}\varphi + f_2\nabla^2h\varphi + O(K^3),$$

$$\begin{aligned}2hf_2 &\equiv \partial^2[G(i\vec{K} + i\vec{\kappa})\text{sech}(\vec{K} + \vec{\kappa})\vec{k}(\vec{K} + \vec{\kappa})]/\partial K^2\\ &= 2\kappa[3 - 9\kappa^2 + 8\kappa^4 + 12\kappa^2\cosh(2\kappa) - 3\cosh(4\kappa)\\ &\quad - 3\kappa^2\cosh(4\kappa) - 6\kappa\sinh(2\kappa) + 16\kappa^3\sinh(2\kappa)\\ &\quad + 9\kappa\sinh(4\kappa)]/\{3[2\kappa + \sinh(2\kappa)]^3\},\end{aligned}$$

which is equivalent [to $O(\delta^2/h_0^2)$] to the fourth order equation derived by Miles and Chamberlain [4]. The differences are (i) the present equation is second order. (ii) The expressions are obtained simply. (iii) Here we have only included terms in ∇^2h and not in $(\boldsymbol{\nabla}h)^2$. The extension is easy, but outside the present scope. MMSE is successful in predicting scattering. Is it possible that the difference between MMSE and AMSE is small? f_2, the *full* coefficient of ∇^2h in AMSE, was compared with the corresponding g_2 of MMSE. We found that f_2/g_2 tends to 3 for small κ, which is the domain of interest. Keeping all orders of $\vec{K}$ produces AMSE [Eq. (1.10)]. After the expansion, we have set $h_0 = h$. The term in δ itself is thus canceled, balancing the replacement of k_0 by k. In f_2, $\kappa = kh$.

Truncating the expansion in AMSE [Eq. (1.10)] is appropriate for variations that are large scale compared with the wavelength. This is the case of gentle shoaling. It does not account for strong reflection, which is dominated by Bragg resonance, when high order terms in $\vec{K}$ (i.e., high order derivatives of h) are also important.

(2) A different set of approximations is required for the study of scattering. It is obtained by expanding in a new parameter: the detuning from Bragg resonance,

$$\mu = (\vec{K} + \vec{\kappa})^2 - \kappa^2. \tag{2.15}$$

We may expand AMSE in powers of μ and discard all terms of order μ and higher to obtain EMSE (this is done by setting $\mu = 0$; all orders of $\vec{K}$ are effectively maintained). This is the case of exact resonance. Note that at resonance

$$G(i\kappa) = \lim_{\kappa' \to \kappa} G(i\kappa') = g\partial(k^2)/\partial(\omega^2) = g/(cc_g). \tag{2.16}$$

MMSE and Dingemans's [3] extended mild slope equation also agrees with AMSE at exact resonance. MMSE can be obtained by expanding AMSE around MSE and then discarding terms of order μ and higher. It has the advantage that it uniformly treats changes in the mean depth and depth perturbations (so the bathymetry need not be decomposed into a mean component and an oscillatory one). It also provides a way to treat slope discontinuities [2,4].

Prior to the derivation of AMSE, a naive analysis might have attributed the good performance of MMSE in scattering problems to an assumed smallness of the neglected terms in $\nabla^2 h$. Comparing the coefficients of $\nabla^2 h$ between MMSE (g_2) and AMSE (f_2), f_2 is found to be up to three times larger than g_2. However, EMSE and MMSE also neglect terms in higher derivatives of h. Now we can identify the reason for the good performance of MMSE and EMSE near Bragg resonance: it is the further elimination of higher order derivatives of h that exactly evens out the (large) neglected terms in $\nabla^2 h$. These neglected terms arise from the $O(\boldsymbol{\nabla} h)$ correction (see [4]) to the vertical structure. Scattering is dominated by the components of the bathymetry that are near Bragg resonance. Hence, correctly modeling the scattering near resonance leads to faithful results.

III. NONLINEAR AUGMENTED MILD SLOPE EQUATIONS

In order to include nonlinear interaction, we repeat the procedure that was used in the derivation of the linear AMSE. This time we keep $F^{(2)}$, which was given in Eq. (2.2). We get nonlinear (NL) AMSE:

$$(\nabla^2 + k_0^2)\varphi = -G(h_0\nabla)[\sec(h_0\nabla)B^{(2)} - F^{(2)}]. \tag{3.1}$$

$B^{(2)}$ is given in Eq. (2.3) and G in Eq. (2.12). By keeping $F^{(2)}$ we get an additional term on the right-hand side of AMSE, yielding a quadratic AMSE.

As with the linear AMSE there are several alternative limiting forms of NL AMSE that may be derived, depending on the dominant interactions that are considered. We can derive a NL MSE which includes all orders of $\mu_{12} \equiv \vec{k} - \vec{k}_1 \mp \vec{k}_2$ in the nonlinear term (but no slope effect), and $O(\nabla h)$ in the linear term. The nonlinear term may be written in terms of φ_1, φ_2, and their gradients alone. The result of [9] is an approximation to this equation. The equation in [9] is less accurate than the present result. It corresponds to taking the limit of G in NL MSE as $(\vec{\kappa}_1 + \vec{\kappa}_2)^2$ tends to κ^2 [see Eq. (2.16)]. As noted in [9], their error in the bound waves is as large as the bound waves themselves $[O(\epsilon^2)]$. In contrast, the present equation represents the bound waves (subharmonic and superharmonic) in an exact way even for a wide spectrum.

In order to treat short scale depth variations, we require the terms that are linear in the bound waves and in all the derivatives of h. This leads to nonlinear EMSE and MMSE which describes class III Bragg resonance [7] for steep bottom oscillations, more accurately than was done in [8]. The detailed equations will be presented separately.

[1] J. C. W. Berkhoff, in *Proceedings of the 13th International Conference on Coastal Engineering* (ASCE, 1972), p. 471.

[2] P. G. Chamberlain and D. Porter, J. Fluid Mech. **291**, 333 (1995).

[3] M. W. Dingemans, *Water Wave Propagation Over Uneven Bottoms, Part 1* (World Scientific, Singapore, 1997).

[4] J. W. Miles and P. G. Chamberlain, J. Fluid Mech. **361**, 175 (1998).

[5] C. C. Mei, J. Fluid Mech. **152**, 315 (1985).

[6] J. T. Kirby, J. Fluid Mech. **162**, 171 (1986).

[7] Y. Liu and D. K. P. Yue, J. Fluid Mech. **356**, 297 (1998).

[8] Y. Agnon, E. N. Pelinovsky, and A. Sheremet, Stud. Appl. Math. **102**, 49 (1998).

[9] Y. Agnon, A. Sheremet, J. Gonsalves, and M. Stiassnie, Coastal Eng. **20**, 29 (1993).

[10] Y. Agnon and A. Sheremet, J. Fluid Mech. **345**, 79 (1997).

[11] Lord Rayleigh, Philos. Mag. **5** (1), 257 (1876).

[12] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Part II* (Wiley-Interscience, New York, 1953).

[13] L. Berg, *Introduction to the Operational Calculus* (North-Holland, Amsterdam, 1967).