Critical behavior of a random diode network

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We study the percolation properties of a random diode network (RDN) which contains two kinds of directed bonds on a square lattice. This network is a special case of the random insulation-resistor-diode network. Both Monte Carlo simulations and series expansion for the percolation probability show that an estimated critical exponent, $\beta=0.1794\pm0.008$, is different from known values for a conventional insulation-resistor-diode network. RDN belongs to neither the isotropic percolation universality class nor to the directed percolation universality, which we attribute to a difference of symmetry breakdown around the critical point.
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I. INTRODUCTION

A random insulation-resistor-diode network (IRD) is a kind of a generalized percolation model [1–4]. In IRD, each bond is occupied by one of the following components: a positive diode (conducting either upward or to the right), a negative diode (conducting in the opposite direction), a resistor (conducting in both directions), and an isolation (disconnecting) with probabilities p, q, r and $s=1-p-q-r$, respectively. By setting $r=s=0$ and $p=1-q$, a random diode network on a square lattice is obtained as a special case of IRD. The isotropic bond percolation (IP) [5,6] and the directed bond percolation (DP) [7–9] are also obtained as special cases of IRD by setting $p=q=0$ and $p=r=0$, respectively. By using the position-space renormalization group [3,4] and ''*planar lattice duality*'' relation [10,11], the phase diagram for IRD and geometrical properties such as correlation length exponents were studied by Redner *et al.* It is known that the critical point for IP is exactly $\frac{1}{2}$ and that the percolation probability critical exponent must be $\beta_{\mathrm{IP}}=5/36$. On the other hand, no exact values are known for DP. However, a long series expansion for the percolation probability obtained by Jensen and Guttmann gives the critical exponent $\beta_{\mathrm{DP}}=0.276\,43(10)$ [12–15].

The critical point for RDN corresponds to an intersection of the four boundaries in Fig. 1. One of the natural questions is whether the critical exponent for RDN is unique or it is the same as β_{IP} or β_{DP}. In this paper, we report that the percolation probability for RDN is characterized by a β value which is different from both β_{IP} and β_{DP}.

Consider a finite square region on a square lattice, defined as

$$V_n^0=\{(x,y)\in Z^2;-n\leqslant x\leqslant n,-n\leqslant y\leqslant n\}. \tag{1}$$

We assume that each bond is occupied by a positive diode or a negative diode with probabilities $1-q$ and q, respectively. If there is at least one current path between two sites, we say that the sites are connected. Let $P_n(q)$ be the probability that the origin (0,0) is connected to at least one site on the border of V_n^0, B_n, defined by

$$B_n=\{(x,y)\in Z^2,|x|=n\}\cup\{(x,y)\in Z^2,|y|=n\}. \tag{2}$$

Then, the percolation probability $P_\infty(q)$ is defined as a limit, $P_\infty(q)=\lim_{n\to\infty}P_n(q)$.

II. MONTE CARLO SIMULATIONS

In the case of $q=1/2$, one finds that the percolation probability for RDN is equal to that for the isotropic bond percolation, thus the percolation probability is exactly zero. Furthermore, if $q\neq\frac{1}{2}$, the percolation probability is strictly greater than zero. It implies that the critical point for RDN is $q=1/2$. Although this value is the same with the critical point of the isotropic percolation, the critical behavior is quite different. The most significant difference is that for RDN the infinite cluster is directed if $p\neq q$. Consider the connection probability between the origin and a site (ξ,ξ), $P(\xi,q)$. If $q<\frac{1}{2}$, then as ξ approaches minus infinity, $P(\xi,q)$ converges to zero. Since $P_\infty(q)=P_\infty(1-q)$, ''*polarization*'' of the infinite cluster reverses at $q=\frac{1}{2}$. To ob-

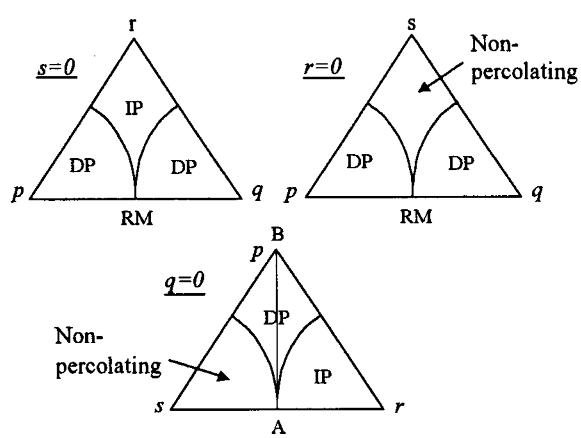

FIG. 1. Projections of the phase diagram of IRD to (a) $s=0$, (b) $r=0$, and (c) $q=0$. RM denotes the critical point. For $q<1/2$, there is a macroscopic current in the direction of (1,1).

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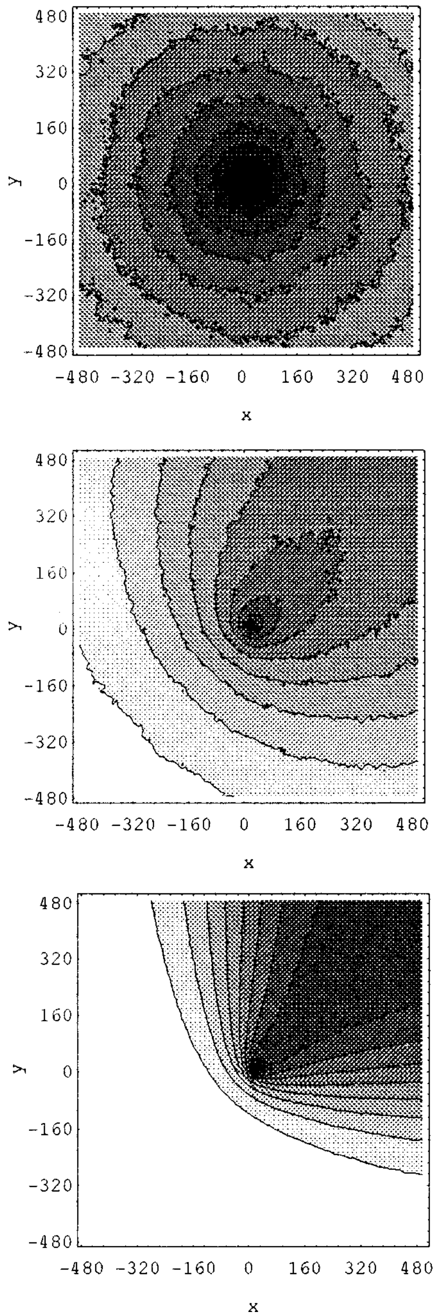

FIG. 2. Contour plots of pair correlation function between the origin and a site (x,y). Regions with higher values are shown in darker gray; $q=0.5$, 0.49, and 0.47 from top to bottom.

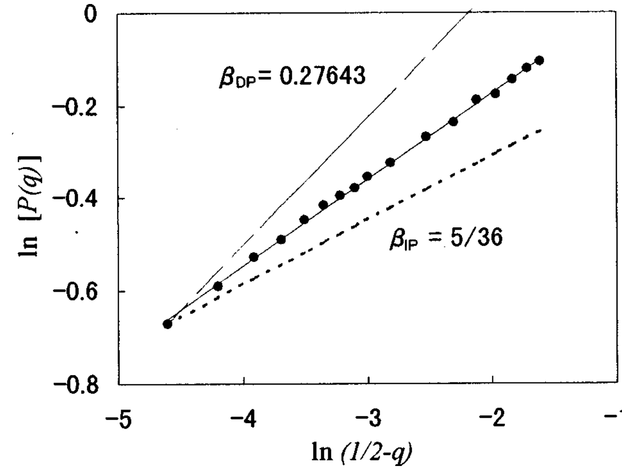

FIG. 3. $\ln P_\infty(q)$ for RDN against $\ln(1/2-q)$ (Monte Carlo simulation). Slopes give the critical exponent β values. Slopes of the upper (lower) dashed line are $\beta_{\mathrm{IP}}=5/36$ and $\beta_{\mathrm{DP}}=0.27643(10)$, respectively.

serve the reversion and to help our intuition, contour plots of the pair correlation function between the origin and a site located at (x,y) are shown in Fig. 2. The corresponding data were obtained by Monte Carlo simulations on a 2048×2048 square lattice, averaged over 5×10^4 realizations. The ''polarization'' quickly becomes rather sharp as parameter q separates from the critical point.

We estimated the critical exponent by a Monte Carlo simulation. Clusters including the origin are generated by the following branching process, which is a kind of Markov process in $(2+1)$ dimensions. Here we call a site connected to the origin a particle. A particle is set on the origin at the initial step $n=0$. Particles created at the $(n-1)$th step produce new particles to their right-hand (left-hand) vacant neighbors with probability $1-q$ (q) at the next step n. In a similar way, particles are created upward (downward) with probability $1-q$ (q). If there are no new particles at a finite step, the origin belongs to a finite cluster.

Side lengths of square region L and the number of independent runs N_r were varied from $L=10\,420, N_r=5000$ close to q_c, to $L=1280, N_r=10^4$ away from q_c.

Figure 3 shows $\ln P_n(q)$ (for $N_r=5000$) against $\ln(1/2-q)$. Although it is well known that the percolation probability for both IP and DP exhibits power-law behavior near the critical point, this is not obvious in the case of RDN. However, we observe that simulation points fall nicely on a straight line. Thus, we conclude that the power law near the critical point holds for RDN as well. Assuming a power law, the percolation probability is expressed as $P_\infty(q)\sim|q_c-q|^\beta$, where q_c is the critical point and β is the critical exponent.

By comparing the straight line in the middle, which has a slope $\beta=0.187$ and the remaining straight lines with slopes $\beta_{\mathrm{IP}}=5/36$ and $\beta_{\mathrm{IP}}=0.276\,43(10)$ in Fig. 3, one can clearly see that the β value for RDN is different from those for IP and DP, and the difference cannot be accounted for by simulation errors.

III. SERIES EXPANSION OF THE PERCOLATION PROBABILITY

To estimate the critical exponent more precisely, we derive a series expansion of the percolation probability for RDN, which is represented as a polynomial $P_\infty(q)=\Sigma_{n=0}^{\infty}c_nq^n$ for small q. We used a program based primarily on the algorithm of Martin [16,17] for enumerating isolated connected clusters.

Clusters including the origin are generated in the same way as that introduced in the Monte Carlo simulation, however *all* possible clusters are generated in finite steps. It is useful to introduce l_x (l_y) defined as a projection length of a cluster to the x (y) axis, respectively. Since the number of nearest-neighbor sites for a cluster with l_x and l_y is greater

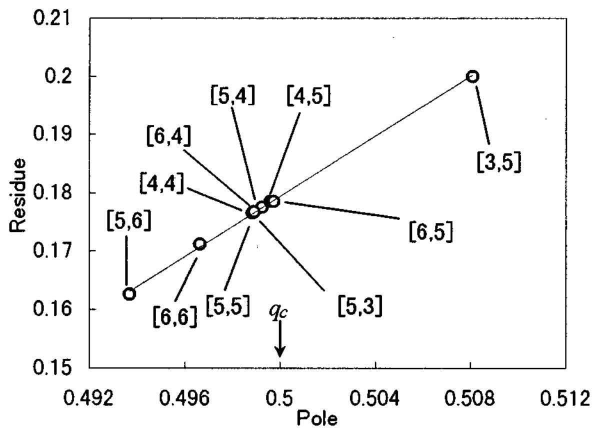

FIG. 4. Pole residues for Dlog Padé approximants to percolation probability.

than $k = l_x + l_y$; the degree in q for the cluster probability is also greater than k.

If the number k corresponding to a cluster is greater than or equal to the predetermined degree of the series, the cluster stops growing and a new configuration which never appeared in the former process is generated from the cluster created one step before. We note that the coefficients of $P_\infty(q)$ are determined as increasing the number of k. By calculating probabilities for all clusters with $k \leqslant 13$, we get the following series expansion:

$$P_\infty(q) = 1 - q^2 - q^4 - 2q^6 - 2q^7 - 2q^8 - 12q^9 + 2q^{10} - 54q^{11} + 38q^{12} - 198q^{13} + O(q^{14}). \quad (3)$$

This series was obtained by enumerating about eight hundred millions isolated clusters and it takes about two weeks on our personal computer (PentiumII 450 MHz). In order to evaluate the critical exponent β, we suppose that the percolation probability is governed by a simple power law and used Padé approximants to the series for $(d/dq)\ln P_\infty(q)$. The critical point q_c and the critical exponent β are given by the first pole on the positive q axis and the residue of the Padé approximant at this pole. The results are summarized in Fig. 4. The estimated critical point values agree with the exact value $q_c = 1/2$. The series obtained is rather short, so a noticeable deviation in β remains, however since the exact critical point is known, we can obtain a better estimation of $\beta^*(q)$ by forming Padé approximants to the series $(q_c - q)(d/dq)\ln P_\infty(q)$ in q [18]. The critical exponent is obtained by setting $q = q_c$. We summarize the results in Table I.

TABLE I. Estimations of the critical exponent β for RDN by evaluating the Padé approximants to the $(1/2 - q)(d/dq)\ln P_\infty(q)$.

N	$[N-1/N]$	$[N/N]$	$[N+1/N]$
4	0.182663	0.179487	0.179713
5	0.179627	0.179481	0.179001
6	0.181671	0.181569	

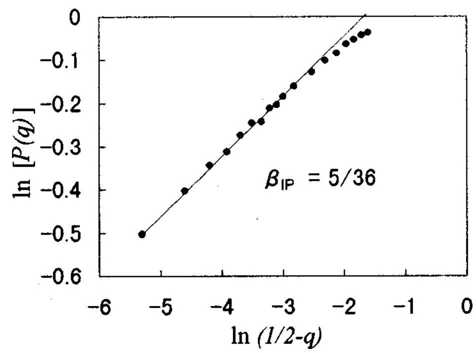

FIG. 5. $\ln P_\infty(q)$ of modified RDN against $\ln(1/2 - q)$ (Monte Carlo simulation). The solid line is drawn with slope $\beta_{IP} = 5/36$.

By combining series expansion and Monte Carlo values for β, we estimate the percolation probability critical exponent for RDN as $\beta = 0.1794 \pm 0.008$.

To consider a relation between the symmetry breakdown in RDN and the critical exponent, we introduce a modified random diode network. We denote the original model as a type (2:2) and the modified model as a type (1:3). In the type (1:3), a site is connected to a lower site with probability q [instead of $1 - q$ for the type (2:2)]. We performed Monte Carlo simulation in the same way as for RDN except for changing the maximum system size from $L = 10\,240$ to 2560. A plot of $\ln P_\infty(q)$ against $\ln(1/2 - q)$ is shown in Fig. 5. The slope obtained by the least-squares method is close to $\beta_{IP} = 5/36$. Here we note that while the infinite cluster is directional, the critical exponent is not β_{DP} but β_{IP}.

As mentioned in the beginning of this paper, for the (2:2) type model changing the parameter q across the critical point switches the ''polarization'' of the cluster. This property is peculiar to the type (2:2) and the new model does not have the $P_\infty(q) = P_\infty(1-q)$ symmetry.

IV. EXTENSION OF RDN

It is interesting to study the transition from the type (2:2) to the type (1:3). We extend RDN by setting the connection probability to a lower site as $\epsilon(1-q) + (1-\epsilon)q$. The type (2:2) and the type (1:3) are given by setting $\epsilon = 0$ and $\epsilon = 1$, respectively. In the case of $0 < \epsilon < 1$, the most significant difference is that the second critical point $q_c(\epsilon)$ exists between $q = 1/2$ and $q = 1$. The critical exponent β is extremely sensitive to errors for the critical point, thus we estimated the critical point by time-dependent simulations, which is an efficient method for determining critical points [19]. At the critical point, we assume that the percolation probability is governed by a power law for large n which is introduced in the branching process as follows:

$$P(n, \epsilon) \propto n^{-\delta(\epsilon)}. \quad (4)$$

We performed 10^4 independent runs up to 5000 steps for different values of $\epsilon = 0.5$ and $\epsilon = 0.8$. Nice straight lines are observed in plots for $\ln P(n,\epsilon)$ against $\ln(n)$ at $q_c(0.5) = 0.5680(5)$ and $q_c(0.8) = 0.8220(5)$, respectively. The inset in Fig. 6 shows local slopes of $\ln P(n,0.8)$, $\delta(n)$ against $1/n$, which give the good estimation of δ for large n at the

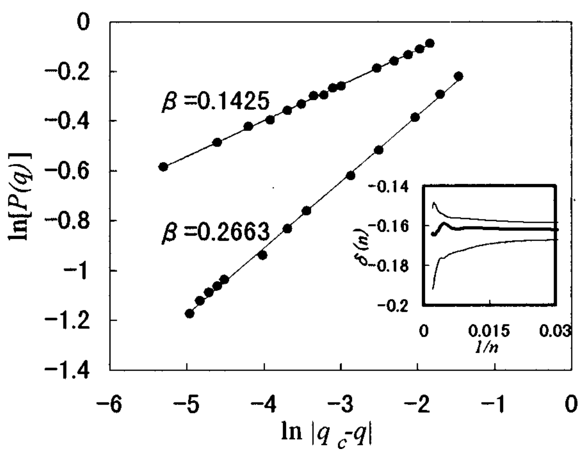

FIG. 6. Results given by Monte Carlo simulations for the extended RDN with $\epsilon=0.5$ and 0.8. The slope of fitted lines in plots of $\ln P(q)$ for $\epsilon=0.5$ against $\ln(|q_c-q|)$ gives the critical exponent β. The slope of the upper line shows β near the critical point $q_c=1/2$ and the lower line shows β near the critical point $q_c=0.5680$. Each of them is close to $\beta_{\rm IP}=5/36$ and $\beta_{\rm DP}=0.276\,43$, respectively. The inset shows $\delta(n)$ for $\epsilon=0.8$ against $1/n$ given by the local slope method with, from top to bottom, $q=0.8225$, 0.8220, and 0.8215.

critical point. Critical exponents which are obtained are $\beta=0.27(1)$ ($\epsilon=0.5$), $\delta=0.157(5)$ ($\epsilon=0.5$), and $\delta=0.161(5)$ ($\epsilon=0.8$), which agree with the DP exponent $\beta_{\rm DP}=0.276\,43$ and $\delta=0.159(1)$ (we cannot accurately estimate β in the case of $\epsilon=0.8$ due to the slow convergence against the system size near the critical point). We also measured the critical exponent for the percolation probability for $\epsilon=0.5$ near the $q_c=1/2$. As shown in Fig. 6, the estimated value $\beta=0.14(1)$ is close to $\beta_{\rm IP}=0.1388\ldots$. We conjecture that if $0<\epsilon<1$, the critical exponent $\beta(\epsilon)$ is independent of the value ϵ, however the proof of it is an open question.

V. SUMMARY AND DISCUSSION

We found new critical behavior for the percolation probability in a special case of random diode networks. Here, we discuss a relation between RDN and another special case of IRD, which is defined by setting $q=0$, $s=r$, and $p=1-2r$. Changing the parameter r from $r=0$ to $r=1/2$ corresponds to a movement in the phase diagram in Fig. 1(c) from the point A to the point B. Consider the connection probability between a site and its immediate neighbor to the right. The sites are connected when the bond between them is occupied by a resistor or a positive diode. Therefore, the connection probability is given by $1-r$. On the other hand, a site is connected to its immediate neighbor to the left only if the bond between them is occupied by a resistor, and so the connection probability is given by r. Consequently, the percolation probability $P_\infty(r)$ is the same with $P_\infty(q)$. The phase transition is characterized by $\beta_{\rm IP}$. Therefore, percolation probability near the multicritical point A is characterized by two different critical exponents. Thus, we are dealing with two models having the same critical point and different critical exponents for percolation probability. Contrasting them might provide a better understanding of RDN.

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