Variational principle for the Navier-Stokes equations

R. R. Kerswell
Department of Mathematics, University of Bristol, Bristol BS8 1TW, England, United Kingdom

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A variational principle is presented for the Navier-Stokes equations in the case of a contained boundary-driven, homogeneous, incompressible, viscous fluid. Based upon making the fluid's total viscous dissipation over a given time interval stationary subject to the constraint of the Navier-Stokes equations, the variational problem looks overconstrained and intractable. However, introducing a nonunique velocity decomposition, $\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{\phi}(\boldsymbol{x},t)+\boldsymbol{\nu}(\boldsymbol{x},t)$, ''opens up'' the variational problem so that what is presumed a single allowable point over the velocity domain $\boldsymbol{u}$ corresponding to the unique solution of the Navier-Stokes equations becomes a surface with a saddle point over the extended domain $(\boldsymbol{\phi},\boldsymbol{\nu})$. Complementary or *dual* variational problems can then be constructed to estimate this saddle point value strictly from above as part of a minimization process or below via a maximization procedure. One of these reduced variational principles is the natural and ultimate generalization of the upper bounding problem developed by Doering and Constantin. The other corresponds to the ultimate Busse problem which now acts to lower bound the true dissipation. Crucially, these reduced variational problems require only the solution of a series of *linear* problems to produce bounds even though their unique intersection is conjectured to correspond to a solution of the nonlinear Navier-Stokes equations. [S1063-651X(99)08105-2]

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I. INTRODUCTION

Variational methods represent a unique theoretical tool for producing rigorous inequality results relevant to fluid turbulence. Although turbulent solutions to the Navier-Stokes equations are not currently available, larger velocity field sets in which they are embedded can be. Rigorous bounds on flow quantities are then derivable through optimization over such an extended velocity domain which is designed to satisfy as many dynamical consequences of the governing equations as possible yet still retain its tractability. Theoretically, results can always be improved by imposing further constraints until eventually only realizable velocity solutions are considered.

The underlying philosophy of a variational approach variously christened ''upper bound theory'' by Malkus [1,2] or the ''optimum theory of turbulence'' by Busse [3–6] is that a fluid field becomes turbulent for a purpose which manifests itself in the maximization of some flow functional. The grand objective is to identify what this functional is through comparing the observed turbulent field with the optimizing flow deduced from the associated variational problem for the functional. The search for this functional naturally began with the relevant global transport of the turbulent flow — for example, heat flux in convection or momentum transport in shear flow — since this is directly observable in experiments. Work by Howard [7] and Busse [3,5] has led to seminal upper bounding results for global energy dissipation rates in shear and convective turbulence (for reviews, see [6,8]). These bounds typically overestimate actual data by an order of magnitude and it remains unclear whether the correct asymptotic scalings with a Reynolds or Rayleigh number have been captured. Efforts to improve these results have so far faltered due to the mathematical complexity of the ensuing Euler-Lagrange equations, which almost immediately becomes unmanageable under the addition of further constraints [9,10]. Subsequent work has been redirected to examining new functionals [11–14] or developing novel applications [15–22].

Recently, a new ''background'' variational formulation has been discovered [23–26] which differs so fundamentally from the Howard-Busse approach that any relationship between them was fascinatingly unclear. It is now evident in the plane Couette flow problem that this new background approach furnishes the dual or complementary problem to that proposed by Busse [5] (see [27,28]). The key step in this new formulation is the use of a nonunique velocity decomposition consisting of a steady, scalar background field $\phi(z)$ which carries the inhomogeneous boundary conditions and a homogeneous fluctuation field $\boldsymbol{\nu}(\boldsymbol{x},t)$ [so that $\boldsymbol{u}(\boldsymbol{x},t)=\phi(z)\hat{\boldsymbol{x}}+\boldsymbol{\nu}(\boldsymbol{x},t)$, where $\hat{\boldsymbol{x}}$ is the direction of imposed shear and $\hat{\boldsymbol{z}}$ the normal to the plates], an idea that can be traced back to Hopf [29]. This extends the variational problem over an enlarged set of competitor fields $(\phi,\boldsymbol{\nu})$ in such a way that the required energy dissipation maximum now becomes a saddle point. The Doering-Constantin background variational problem can then be recognized as a minimizing procedure in ϕ to estimate this saddle point value strictly from above, whereas the Howard-Busse maximization problem in $\boldsymbol{\nu}$ provides estimates entirely from below. Practically, this complementary or dual relationship implies that the true saddle point value can be bracketed between trial function estimates derived within each procedure.

Given this dual relationship, an outstanding issue is then whether this new background formulation offers a new and tractable way forward in producing better bounds through the addition of further constraints. The purpose of this paper is to suggest that this is so by revealing a natural path for incorporating additional constraints which ultimately leads to a variational principle for the Navier-Stokes equations. Crucially, the saddle point structure discovered in the upper bound problem [28] is preserved by this procedure. The key idea is as before the degeneracy built into a nonunique representation of the velocity field. By successively relaxing the restricted form used for the background field until it too is eventually a three-dimensional time-dependent vector field depending on all three spatial variables, i.e., the velocity representation is fully degenerate $\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{\phi}(\boldsymbol{x},t)+\boldsymbol{\nu}(\boldsymbol{x},t)$, progressively more information is incorporated until finally the full Navier-Stokes equations become constraints. In this scenario, the variational problem is then one for the averaged energy dissipation rate over a fixed time interval subject to the full constraint of the Navier-Stokes equations: in other words, the *ultimate* upper bound problem for the energy dissipation rate. Of course, for prescribed initial and boundary conditions, the Navier-Stokes equations are presently presumed to have a unique solution rendering this variational problem hopelessly overconstrained for the velocity field $\boldsymbol{u}$. However, viewed over the extended function domain $(\boldsymbol{\phi},\boldsymbol{\nu})$, there is a saddle point structure and complementary variational principles are available to estimate the saddle point from either side. Within this context, it is clear that the ''fluctuation'' field $\boldsymbol{\nu}$ is precisely the Lagrange multiplier vector field imposing the Navier-Stokes equations as constraints. The complementary variational principles for estimating the presumed unique saddle point value and associated solution intriguingly require only the solution of linear problems to make progress and therefore appear eminently tractable. Armed with these two dual pieces of machinery, a feasible variational principle for the Navier-Stokes equations seems to arise out of an intractable-looking upper bound problem which has as its constraints the full Navier-Stokes equations.

The detailed presentation of these ideas begins by formulating this ''ultimate'' upper bound problem in Sec. II. The full background decomposition of the velocity field is then introduced and exploited to reveal the inherent saddle point structure which exists when the functional of interest is the dissipation rate. Complementary variational principles are then formally developed for estimating the saddle point value and realized velocity solution in Sec. III. A discussion of whether these variational principles actually touch at the saddle point follows in Sec. IV, before Secs. V and VI focus on their practical implementation. Generalizations of the Howard-Busse and Doering-Constantin upper bounding variational principles are discussed here and examined as practical algorithms for approximating solutions of the Navier-Stokes equations. Finally, Sec. VII contains a discussion of the paper's findings and their implications.

II. THE ULTIMATE UPPER BOUND PROBLEM FOR THE DISSIPATION RATE

Consider a homogeneous, incompressible fluid of kinematic viscosity ν in a volume $\mathcal{V}$ whose boundary $\partial\mathcal{V}$ is moving with some prescribed velocity $\boldsymbol{V}$ in a frame rotating at $\boldsymbol{\omega}$. Taking a typical length scale d of $\mathcal{V}$ and the viscous diffusion time scale d^2/ν to nondimensionalize the system leads to the Navier-Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t}+2\boldsymbol{\omega}\times\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}p=\nabla^2\boldsymbol{u}, \tag{2.1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \tag{2.2}$$

with boundary condition

$$\boldsymbol{u}=\mathrm{Re}\,\boldsymbol{V}(\boldsymbol{x},t) \quad \text{for } \boldsymbol{x}\in\partial\mathcal{V}, \tag{2.3}$$

where $\mathrm{Re}:=V_0 d/\nu$ is the Reynolds number and V_0 is a typical speed of the boundary. A fundamental issue for such a system is how large the viscous dissipation rate can become on average over a certain period $[0,T]$. The long-time limiting case $T\to\infty$ has been considered previously within the canonical context of plane Couette flow [5,6,10,23,24,27, 28,30–33]. There the approach has been to develop tractable variational problems built upon only the very first power integral and mean momentum constraints imposed by the full system (2.1)–(2.3) to produce upper bounds on the realizable long-time averaged (or equivalently statistically steady) energy dissipation rates. Eventually, of course, the goal has always been to add more and more constraints to bring these bounds closer to the observed values. Here, we consider the *ultimate* upper bound problem for the energy dissipation rate by imposing the full Navier-Stokes equations as our constraints.

Technically, the viscous dissipation rate is

$$\frac{1}{2}\int\int\int|\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}^T\boldsymbol{u}|^2 d\mathcal{V} = \int\int\int|\boldsymbol{\nabla}\boldsymbol{u}|^2 d\mathcal{V}+\oint n_i u_j u_{i,j}\,dS \tag{2.4}$$

with the latter term uniquely determined for an incompressible velocity field specified on the boundary. In most situations, this term is zero because either the velocity vanishes on the boundary, the boundary conditions are periodic, or the boundary is planar and only moves tangentially to itself. However, regardless of whether this term vanishes or not, it plays no role in any variational analysis since it is invariant for any incompressible velocity field which satisfies the boundary conditions. As a result this term is suppressed in what follows, although of course ultimately it should be reintroduced to produce any total dissipation rate. We therefore look to determine stationary values of the dissipation functional

$$D_T:=\frac{1}{T}\int_0^T\langle\,|\boldsymbol{\nabla}\boldsymbol{u}|^2\,\rangle\,dt -\frac{a}{T}\int_0^T\left\langle\,\boldsymbol{\nu}\cdot\left\{\frac{\partial\boldsymbol{u}}{\partial t}+2\boldsymbol{\omega}\times\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}p-\nabla^2\boldsymbol{u}\right\}\right\rangle dt \tag{2.5}$$

with

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}=0, \quad \boldsymbol{u}=\mathrm{Re}\,\boldsymbol{V}|_{\partial\mathcal{V}} \tag{2.6}$$

and some initial condition $\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})$ where the bulk integral is defined as follows:

$$\langle A\rangle(t):=\int\int\int d\mathcal{V}A(\boldsymbol{x},t).$$

(In the case of an unbounded domain such as the plane layer, $\boldsymbol{x}\in\mathbb{R}^2\times[-\frac{1}{2},\frac{1}{2}]$, this can be defined as

$$\langle A\rangle(t):=\lim_{L_x,L_y\to\infty}\frac{1}{4L_xL_y}\int_{-L_x}^{+L_x}dx\int_{-L_y}^{+L_y}dy\int_{-1/2}^{+1/2}dz\,A(\boldsymbol{x},t)$$

for example.) Here $\boldsymbol{\nu}=\boldsymbol{\nu}(\boldsymbol{x},t)$ clearly plays the role of a Lagrange multiplier which imposes the Navier-Stokes equations as a constraint. We have also included an extra constant a for comparison with earlier upper bounding work but here it is evidently redundant and can be absorbed into the Lagrange multiplier $\boldsymbol{\nu}$. For flows in which absolute pressure is not important, p is merely a Lagrange multiplier associated with the condition of fluid incompressibility and can naturally be ''absorbed'' into $\boldsymbol{\nu}$ by insisting that $\boldsymbol{\nu}$ is divergence-free. Natural boundary conditions for $\boldsymbol{\nu}$ are homogeneous and appropriate ''initial'' conditions emerge to be the *final* conditions that $\boldsymbol{\nu}$ vanishes everywhere at $t=T$. This ensures that the one boundary term $[\boldsymbol{\delta u}\cdot\boldsymbol{\nu}]_0^T$ (where $\boldsymbol{\delta u}$ is a variation in the velocity field) produced by the variational procedure drops. In summary, we take

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nu}=0,\quad \boldsymbol{\nu}=\boldsymbol{0}|_{\partial\mathcal{V}},\quad \boldsymbol{\nu}(\boldsymbol{x},T)=\boldsymbol{0}. \tag{2.7}$$

The variational problem as formulated thus far is not immediately useful. The functional D_T is stationary when the variational derivatives of D_T with respect to $\boldsymbol{u}$ and $\boldsymbol{\nu}$ vanish, that is,

$$\left.\frac{\delta D_T}{\delta\boldsymbol{\nu}}\right|_{\boldsymbol{u}}=\left.\frac{\delta D_T}{\delta\boldsymbol{u}}\right|_{\boldsymbol{\nu}}=\boldsymbol{0}, \tag{2.8}$$

where

$$\left[\frac{d}{d\epsilon}D_T(\boldsymbol{u},\boldsymbol{\nu}+\epsilon\tilde{\boldsymbol{\nu}};a)\right]_{\epsilon=0}=\frac{1}{T}\int_0^T\left\langle\left.\frac{\delta D_T}{\delta\boldsymbol{\nu}}\right|_{\boldsymbol{u}}\cdot\tilde{\boldsymbol{\nu}}\right\rangle dt, \tag{2.9}$$

$$\left[\frac{d}{d\epsilon}D_T(\boldsymbol{u}+\epsilon\tilde{\boldsymbol{u}},\boldsymbol{\nu};a)\right]_{\epsilon=0}=\frac{1}{T}\int_0^T\left\langle\left.\frac{\delta D_T}{\delta\boldsymbol{u}}\right|_{\boldsymbol{\nu}}\cdot\tilde{\boldsymbol{u}}\right\rangle dt. \tag{2.10}$$

The first of these variational vector equations is of course the Navier-Stokes equations, which for given initial conditions and boundary conditions are presently presumed to have a unique solution. The full variational problem then amounts to finding the dissipation of this unique solution, which then is both a maximum and minimum at once.

A reduced variational problem of some interest can be found, however, although it is not clear how effective it may be. If the latter of the two variational criteria for stationarity,

$$\left.\frac{\delta D_T}{\delta\boldsymbol{u}}\right|_{\boldsymbol{\nu}}=\boldsymbol{0}\Rightarrow a(\boldsymbol{\nabla\nu}+\boldsymbol{\nabla}^T\boldsymbol{\nu})\cdot\boldsymbol{u}+\boldsymbol{\nabla}p-2\nabla^2\boldsymbol{u}$$
$$=-a\frac{\partial\boldsymbol{\nu}}{\partial t}-2a\boldsymbol{\omega}\times\boldsymbol{\nu}-a\nabla^2\boldsymbol{\nu}, \tag{2.11}$$

is considered as an equation defining $\boldsymbol{u}$ *given* $\boldsymbol{\nu}$, we can produce a reduced variational problem for $D_T(\boldsymbol{u}(\boldsymbol{\nu}),\boldsymbol{\nu};a)$ over just the field $\boldsymbol{\nu}$. Comparing the dissipation values associated with two trial fields, $\tilde{\boldsymbol{\nu}}$ and $\boldsymbol{\nu}^*$, then straightforwardly leads to the difference expression

$$D_T(\tilde{\boldsymbol{u}}(\tilde{\boldsymbol{\nu}}),\tilde{\boldsymbol{\nu}};a)-D_T(\boldsymbol{u}^*(\boldsymbol{\nu}^*),\boldsymbol{\nu}^*;a)$$
$$=\frac{1}{T}\int_0^T\left\langle(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^*)\cdot\frac{\delta D_T}{\delta\boldsymbol{\nu}}(\boldsymbol{u}^*,\boldsymbol{\nu}^*)-|\boldsymbol{\nabla}(\tilde{\boldsymbol{u}}-\boldsymbol{u}^*)|^2-a(\tilde{\boldsymbol{u}}-\boldsymbol{u}^*)\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{\nu}}\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{u}^*)\right\rangle dt$$
$$-\frac{a}{T}[\langle(\tilde{\boldsymbol{u}}-\boldsymbol{u}^*)\cdot\tilde{\boldsymbol{\nu}}\rangle]_0^T. \tag{2.12}$$

If $\boldsymbol{\nu}^*$ is the presumably unique solution to the full variational problem, the first (linear) term on the right-hand side vanishes. The second and third terms are assured negative semidefinite *if* we only select trial fields $\boldsymbol{\nu}(\boldsymbol{x},t)$ which satisfy the *spectral constraint*:

$$\inf_{\boldsymbol{\nabla}\cdot\boldsymbol{v}=0,\boldsymbol{v}=\boldsymbol{0}|_{\partial\mathcal{V}}}\langle|\boldsymbol{\nabla v}|^2+a\boldsymbol{v}\cdot\boldsymbol{\nabla\nu}\cdot\boldsymbol{v}\rangle\geq 0 \tag{2.13}$$

at every instant in time over $[0,T]$. This is accurately termed a spectral constraint *for* $\boldsymbol{\nu}$ (borrowing some terminology from [24]) because it is the requirement that the eigenspectrum of the linear self-adjoint operator

$$\mathcal{L}(\boldsymbol{\nu};a)\boldsymbol{v}:=a(\boldsymbol{\nabla\nu}+\boldsymbol{\nabla}^T\boldsymbol{\nu})\cdot\boldsymbol{v}+\boldsymbol{\nabla}p-2\nabla^2\boldsymbol{v} \tag{2.14}$$

is positive semidefinite over the space of incompressible $\boldsymbol{v}$ which vanishes on the boundary. The last (boundary) term can be made to vanish identically by our ''final'' condition $\boldsymbol{\nu}(\boldsymbol{x},T)=\boldsymbol{0}$ and carefully arranging $\partial\boldsymbol{\nu}/\partial t$ at $t=0$ such that the velocity field always satisfies the correct initial condition, $\boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})$. With these restrictions on $\tilde{\boldsymbol{\nu}}$, we can conclude that

$$D_T(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{\nu}})\leq D_T(\boldsymbol{u}^*,\boldsymbol{\nu}^*)=D_T^{\mathrm{NS}}, \tag{2.15}$$

or in other words we can estimate the true dissipation rate *from below*. Unless $\boldsymbol{\nu}^*$ satisfies the spectral constraint or more exceptionally there exists a $\tilde{\boldsymbol{\nu}}\neq\boldsymbol{\nu}^*$ with $\langle|\boldsymbol{\nabla}(\tilde{\boldsymbol{u}}-\boldsymbol{u}^*)|^2+a(\tilde{\boldsymbol{u}}-\boldsymbol{u}^*)\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{\nu}}\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{u}^*)\rangle=0$, there is strict inequality in Eq. (2.15) and it is unclear how close to D_T^{NS} one can get using this scheme since there is no way to estimate D_T^{NS} from above.

A. Towards a variational principle

The problem with the variational formulation given above is that for $D_T=D_T(\boldsymbol{u},\boldsymbol{\nu};a)$, one of the natural variational equations to be solved, $\delta D_T/\delta\boldsymbol{\nu}|_{\boldsymbol{u}}=\boldsymbol{0}$, leads directly to the Navier-Stokes equations. This can be avoided by restructuring the underlying independent function fields $(\boldsymbol{u},\boldsymbol{\nu})$. Replacing $\boldsymbol{u}$ by a new vector (background) field $\boldsymbol{\phi}(\boldsymbol{x},t)=\boldsymbol{u}(\boldsymbol{x},t)-\boldsymbol{\nu}(\boldsymbol{x},t)$, which is therefore incompressible and satisfies the same boundary conditions as $\boldsymbol{u}$, achieves this in a perfectly general way. The functional $D_T=D_T(\boldsymbol{\phi},\boldsymbol{\nu};a)$ is now expressible as

$$\frac{a}{T}[\langle \tfrac{1}{2}\boldsymbol{\nu}^2\rangle]_0^T + D_T = \frac{1}{T}\int_0^T dt\,\langle|\nabla\boldsymbol{\phi}|^2\rangle - \mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a)$$

$$= \mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) - \frac{(a-1)}{T}\int_0^T dt\,\langle|\nabla\boldsymbol{\nu}|^2\rangle, \tag{2.16}$$

where

$$\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) := \frac{1}{T}\int_0^T dt\Bigg\{\Bigg\langle a\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\phi}}{\partial t} + 2a\boldsymbol{\nu}\cdot\boldsymbol{\omega}\times\boldsymbol{\phi} + a\boldsymbol{\nu}\cdot\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi} + (a-1)|\nabla\boldsymbol{\nu}|^2 + a\boldsymbol{\nu}\cdot\nabla\boldsymbol{\phi}\cdot\boldsymbol{\nu} - (a-2)\boldsymbol{\nu}\cdot\nabla^2\boldsymbol{\phi}\Bigg\rangle\Bigg\}, \tag{2.17}$$

$$\mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) := \frac{1}{T}\int_0^T dt\Bigg\{\Bigg\langle |\nabla\boldsymbol{\phi}|^2 - a\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\phi}}{\partial t} + 2a\boldsymbol{\phi}\cdot\boldsymbol{\omega}\times\boldsymbol{\nu} + a\boldsymbol{\phi}\cdot\nabla\boldsymbol{\nu}\cdot\boldsymbol{\phi} + a\boldsymbol{\nu}\cdot\nabla\boldsymbol{\nu}\cdot\boldsymbol{\phi} + (a-2)\boldsymbol{\nu}\cdot\nabla^2\boldsymbol{\phi}\Bigg\rangle\Bigg\}. \tag{2.18}$$

The expression (2.16) is more familiarly derived within the framework of Doering and Constantin's background upper bounding method as follows. Taking $(1/T)\int_0^T\langle\boldsymbol{\nu}\cdot(2.1)\rangle dt$ gives

$$\frac{1}{T}[\langle \tfrac{1}{2}\boldsymbol{\nu}^2\rangle]_0^T + \frac{1}{T}\int_0^T dt\Bigg\{\Bigg\langle \boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\phi}}{\partial t} + 2\boldsymbol{\nu}\cdot\boldsymbol{\omega}\times\boldsymbol{\phi} + \boldsymbol{\nu}\cdot\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi} + \boldsymbol{\nu}\cdot\nabla\boldsymbol{\phi}\cdot\boldsymbol{\nu} + |\nabla\boldsymbol{\nu}|^2 - \boldsymbol{\nu}\cdot\nabla^2\boldsymbol{\phi}\Bigg\rangle\Bigg\} = 0. \tag{2.19}$$

Adding an unspecified multiple, a ([30]), of Eq. (2.19) to the identity

$$\langle|\nabla\boldsymbol{u}|^2\rangle = \langle|\nabla\boldsymbol{\phi}|^2\rangle - 2\langle\boldsymbol{\nu}\cdot\nabla^2\boldsymbol{\phi}\rangle + \langle|\nabla\boldsymbol{\nu}|^2\rangle, \tag{2.20}$$

also time-averaged over the interval $[0,T]$, then leads back to Eq. (2.16).

The new variational equations are derived by insisting that the first variation δD_T produced by variations $\boldsymbol{\delta\nu}$ in $\boldsymbol{\nu}$ and $\boldsymbol{\delta\phi}$ in $\boldsymbol{\phi}$ vanishes. With respect to variations in $\boldsymbol{\nu}$, the variational statement is

$$\delta D_T = -\frac{a}{T}[\langle\boldsymbol{\delta\nu}\cdot\boldsymbol{\nu}\rangle]_0^T + \frac{1}{T}\int_0^T\Bigg\langle\boldsymbol{\delta\nu}\cdot\frac{\delta D_T}{\delta\boldsymbol{\nu}}\Bigg|_{\boldsymbol{\phi}}\Bigg\rangle dt = 0 \tag{2.21}$$

for all allowable variations $\boldsymbol{\delta\nu}$, where

$$\frac{\delta D_T}{\delta\boldsymbol{\nu}}\Bigg|_{\boldsymbol{\phi}} := -\Bigg\{a\frac{\partial\boldsymbol{\phi}}{\partial t} + 2a\boldsymbol{\omega}\times\boldsymbol{\phi} + a\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi} - 2(a-1)\nabla^2\boldsymbol{\nu} + a(\nabla\boldsymbol{\phi} + \nabla^T\boldsymbol{\phi})\cdot\boldsymbol{\nu} - (a-2)\nabla^2\boldsymbol{\phi} + \nabla p_\nu\Bigg\} = \mathbf{0}. \tag{2.22}$$

With respect to variations in $\boldsymbol{\phi}$, the variational statement is

$$\delta D_T = -\frac{a}{T}[\langle\boldsymbol{\delta\phi}\cdot\boldsymbol{\nu}\rangle]_0^T + \frac{1}{T}\int_0^T\Bigg\langle\boldsymbol{\delta\phi}\cdot\frac{\delta D_T}{\delta\boldsymbol{\phi}}\Bigg|_{\boldsymbol{\nu}}\Bigg\rangle dt = 0 \tag{2.23}$$

for all allowable variations $\boldsymbol{\delta\phi}$, where

$$\frac{\delta D_T}{\delta\boldsymbol{\phi}}\Bigg|_{\boldsymbol{\nu}} := -2\nabla^2\boldsymbol{\phi} + a\frac{\partial\boldsymbol{\nu}}{\partial t} + 2a\boldsymbol{\omega}\times\boldsymbol{\nu} + a(\nabla\boldsymbol{\nu} + \nabla^T\boldsymbol{\nu})\cdot\boldsymbol{\phi} + a\boldsymbol{\nu}\cdot\nabla\boldsymbol{\nu} + (a-2)\nabla^2\boldsymbol{\nu} + \nabla p_\phi = \mathbf{0}. \tag{2.24}$$

Note that now the difference

$$\frac{\delta D_T}{\delta\boldsymbol{\nu}}\Bigg|_{\boldsymbol{\phi}} - \frac{\delta D_T}{\delta\boldsymbol{\phi}}\Bigg|_{\boldsymbol{\nu}} = \frac{\delta D_T}{\delta\boldsymbol{\nu}}\Bigg|_{\boldsymbol{u}} \tag{2.25}$$

gives the Navier-Stokes equations.

The fundamental observation to be made from Eq. (2.16) is that D_T appears to possess classic convex-concave saddle point structure over $(\boldsymbol{\phi},\boldsymbol{\nu})$; D_T is individually quadratic in either $\boldsymbol{\phi}$ or $\boldsymbol{\nu}$ and positive definite in the highest spatial derivative term involving $\boldsymbol{\phi}$ and negative definite in that of $\boldsymbol{\nu}(a>1)$. Although this proves a slight oversimplification, the overall conclusion that the presumably unique stationary point of D_T is a saddle point nevertheless appears justified. The new variational equations (2.22) and (2.24) individually form the basis of complementary variational principles to estimate this saddle point value strictly from above and below. Taken together, of course, they offer no advantage over the previous set (2.8), but treated separately they naturally dismantle the nonlinearity of the Navier-Stokes equations. This manifests itself in the fact that these complementary variational principles only require the solution of linear problems.

III. COMPLEMENTARY VARIATIONAL PRINCIPLES

For definiteness in what follows, we confine our attention to functions $\boldsymbol{\phi}\in\boldsymbol{\Omega}$ and $\boldsymbol{\nu}\in\boldsymbol{\Gamma}$, where

$$\boldsymbol{\Omega} := \{\boldsymbol{\phi}\,|\,\phi_i(\boldsymbol{x},t)\in C^2(\mathcal{V})\times C^1[0,T],\ i=1,2,3;\ \nabla\cdot\boldsymbol{\phi}=0,\ \boldsymbol{\phi}=\mathrm{Re}\boldsymbol{V}|_{\partial\mathcal{V}}\}, \tag{3.1}$$

$$\boldsymbol{\Gamma} := \{\boldsymbol{\nu}\,|\,\nu_i(\boldsymbol{x},t)\in C^2(\mathcal{V})\times C^1[0,T],\ i=1,2,3;\ \nabla\cdot\boldsymbol{\nu}=0,\ \boldsymbol{\nu}=\mathbf{0}|_{\partial\mathcal{V}}\}. \tag{3.2}$$

Given initial conditions on $\boldsymbol{u}=\boldsymbol{\phi}+\boldsymbol{\nu}$, D_T has a unique stationary point corresponding to the appropriate solution of the Navier-Stokes equation. We now construct two reduced variational principles which can be used to approach this point and its associated solution either strictly from above or below.

A. Minimization problem

The basic idea is to perform the optimization over $\boldsymbol{\nu}$ first by solving the variational equation (2.22) for a *given* trial background field $\boldsymbol{\phi}(\boldsymbol{x},t)$. Providing certain conditions are met, the subsequent dissipation rate estimate can be assured to exceed the saddle point value. These conditions revolve around the temporal boundary conditions and a pointwise-in-time spectral condition which determines whether the chosen background trial field leads to an overestimation of the saddle point value or not. The second optimization over $\boldsymbol{\phi}$ then seeks to minimize this upper estimate of the realized dissipation.

To force the boundary terms in both Eqs. (2.21) and (2.23) to drop, we impose initial conditions on $\boldsymbol{\nu}$ and $\boldsymbol{\phi}$ and insist that $\boldsymbol{\nu}(\boldsymbol{x},T)$ vanishes to be consistent with the approach taken in Sec. II. Most importantly in what follows, we must ensure that any possible velocity field $\boldsymbol{u}(\boldsymbol{x},t)$ can be represented as the sum of the trial field $\boldsymbol{\phi}$ and a fluctuation field $\boldsymbol{\nu}$ at *any* time and spatial position. Given that $\boldsymbol{\nu}(\boldsymbol{x},T)=\boldsymbol{0}$, this means, for example, that $\boldsymbol{\phi}(\boldsymbol{x},T)$ must equal the final realized velocity field, $\boldsymbol{u}(\boldsymbol{x},T)=\boldsymbol{u}^{\mathrm{NS}}(\boldsymbol{x},T)$, where $\boldsymbol{u}^{\mathrm{NS}}(\boldsymbol{x},t)$ is the Navier-Stokes solution given the initial condition $\boldsymbol{u}^{\mathrm{NS}}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})$. Additionally, since the true starting condition for $\boldsymbol{\phi}$ is also unknown, this must be chosen, for example, $\boldsymbol{\phi}(\boldsymbol{x},0)=\boldsymbol{\phi}_0(\boldsymbol{x})$, which means that $\boldsymbol{\nu}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})-\boldsymbol{\phi}_0(\boldsymbol{x})$. The dissipation rate functional is then expressible as

$$D_T=\frac{1}{T}\int_0^T\langle|\nabla\boldsymbol{\phi}|^2\rangle+\frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle-\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) \tag{3.3}$$

with $\boldsymbol{\phi}\in\boldsymbol{\Omega}_1$ and $\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1$, where

$$\boldsymbol{\Omega}_1(\boldsymbol{\phi}_0):=\{\boldsymbol{\phi}\in\boldsymbol{\Omega}\,|\,\boldsymbol{\phi}(\boldsymbol{x},0)=\boldsymbol{\phi}_0(\boldsymbol{x}),\boldsymbol{\phi}(\boldsymbol{x},T)=\boldsymbol{u}^{\mathrm{NS}}(\boldsymbol{x},T)\}, \tag{3.4}$$

$$\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0):=\{\boldsymbol{\nu}\in\boldsymbol{\Gamma}\,|\,\boldsymbol{\nu}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})-\boldsymbol{\phi}_0(\boldsymbol{x}),\boldsymbol{\nu}(\boldsymbol{x},T)=\boldsymbol{0}\}. \tag{3.5}$$

Now, if a trial background field $\boldsymbol{\phi}\in\boldsymbol{\Omega}_1$ can be chosen such that

$$\inf_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)}\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a)>-\infty, \tag{3.6}$$

then we have immediately the upper bound ([23,24]),

$$D_T\leqslant\frac{1}{T}\int_0^T\langle|\nabla\boldsymbol{\phi}|^2\rangle+\frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle-\inf_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)}\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a). \tag{3.7}$$

The crucial point is that we have this degeneracy in the velocity representation which can be used precisely at this point to advantage. *Given* a $\boldsymbol{\phi}(\boldsymbol{x},t)$ field, there is still always a fluctuation field $\boldsymbol{\nu}(\boldsymbol{x},t)$ which can ''reach'' any realizable velocity field $\boldsymbol{u}(\boldsymbol{x},t)$: as a result, Eq. (3.7) must hold for $\boldsymbol{u}^{\mathrm{NS}}$. The condition for $\mathcal{G}_T$ to have a stationary point is that $\delta D_T/\delta\boldsymbol{\nu}=\boldsymbol{0}$:

$$a(\nabla\boldsymbol{\phi}+\nabla^T\boldsymbol{\phi})\cdot\boldsymbol{\nu}+\nabla p_\nu-2(a-1)\nabla^2\boldsymbol{\nu}=-a\frac{\partial\boldsymbol{\phi}}{\partial t}-2a\boldsymbol{\omega}\times\boldsymbol{\phi}-a\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}+(a-2)\nabla^2\boldsymbol{\phi} \tag{3.8}$$

which is a *linear, spatial* problem for $\boldsymbol{\nu}$ which needs no temporal boundary conditions. The value of $\mathcal{G}_T$ at this stationary point $\boldsymbol{\nu}=\boldsymbol{\nu}^*$ is

$$\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu}^*;a)=\frac{1}{2T}\int_0^T\left\langle\boldsymbol{\nu}^*\cdot\left\{a\frac{\partial\boldsymbol{\phi}}{\partial t}+2a\boldsymbol{\omega}\times\boldsymbol{\phi}+a\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}-(a-2)\nabla^2\boldsymbol{\phi}\right\}\right\rangle dt. \tag{3.9}$$

The fact that there are conditions on $\boldsymbol{\nu}(\boldsymbol{x},0)$ and $\boldsymbol{\nu}(\boldsymbol{x},T)$ must be reinterpreted as, in fact, conditions on $\partial\boldsymbol{\phi}/\partial t$ at either end of the time interval. For this stationary point to be a minimum over $\boldsymbol{\Gamma}$, there is the spectral condition on the background field that

$$\langle(a-1)|\nabla\boldsymbol{\nu}|^2+a\boldsymbol{\nu}\cdot\nabla\boldsymbol{\phi}\cdot\boldsymbol{\nu}\rangle\geqslant0,\quad\forall\boldsymbol{\nu}\in\boldsymbol{\Gamma},\quad\forall t\in(0,T) \tag{3.10}$$

which is the condition that the eigenspectrum of the linear self-adjoint operator

$$\mathcal{L}(\boldsymbol{\phi};a)\boldsymbol{\nu}:=a(\nabla\boldsymbol{\phi}+\nabla^T\boldsymbol{\phi})\cdot\boldsymbol{\nu}+\nabla p-2(a-1)\nabla^2\boldsymbol{\nu}$$

is positive semidefinite over $\boldsymbol{\Gamma}$ $\forall t\in(0,T)$. This ensures that the important quadratic terms in $\mathcal{G}_T$ are positive semidefinite. The function set $\boldsymbol{\Omega}_2$,

$$\boldsymbol{\Omega}_2:=\{\boldsymbol{\phi}\in\boldsymbol{\Omega}|\langle(a-1)|\nabla\boldsymbol{\nu}|^2+a\boldsymbol{\nu}\cdot\nabla\boldsymbol{\phi}\cdot\boldsymbol{\nu}\rangle\geqslant0,\ \forall\boldsymbol{\nu}\in\boldsymbol{\Gamma},\quad\forall t\in(0,T)\}, \tag{3.11}$$

collects together all such ''allowable'' background fields. If the selected background field marginally satisfies the spectral constraint, the self-adjoint operator to be inverted in Eq. (3.8) is singular. In this case, there is the solvability condition that the right-hand side in Eq. (3.8) must be orthogonal to the operator's null space, and the solution $\boldsymbol{\nu}$ is only determined up to this null space. This latter feature is actually not important at this stage of the optimization procedure because only the inhomogeneous solution contributes to inf $\mathcal{G}_T$ and therefore affects the dissipation functional. Put another way, it is of no consequence here that this infimum may not be unique although the background trial field must be adjusted to satisfy the solvability condition. Practically, the way to bypass this extra complication is to avoid trial fields which are ''spectrally'' marginal.

The upper bound in Eq. (3.7) can be minimized over all permissible background fields $\boldsymbol{\phi}\in\boldsymbol{\Omega}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Omega}_2$ to give the better bound

$$D_T\leqslant\inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Omega}_2}\left\{\frac{1}{T}\int_0^T\langle|\nabla\boldsymbol{\phi}|^2\rangle+\frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle-\inf_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)}\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a)\right\}. \tag{3.12}$$

This optimization procedure translates into solving the two variational equations

$$\left.\frac{\delta D_T}{\delta \boldsymbol{\nu}}\right|_{\boldsymbol{\phi}} = \left.\frac{\delta D_T}{\delta \boldsymbol{\phi}}\right|_{\boldsymbol{\nu}} = \mathbf{0} \tag{3.13}$$

over the restricted set $\boldsymbol{\phi} \in \boldsymbol{\Omega}_1(\boldsymbol{\phi}_0) \cap \boldsymbol{\Omega}_2$ with $\boldsymbol{\nu} \in \boldsymbol{\Gamma}_1$ given $\boldsymbol{\phi}_0$. It is at this point that any possible degeneracy in solving the first variational equation introduced through a spectrally marginal trial background field (3.8) becomes important. The second variational equation is an equation for $\boldsymbol{\phi}$ then forced by an underdetermined fluctuation field $\boldsymbol{\nu}$. A conceivable resolution of this is that both operators in the two variational equations are simultaneously singular for the realized solutions so that the null space degeneracy in $\boldsymbol{\nu}$, say, would be removed by the solvability condition in the $\boldsymbol{\phi}$ equation and *vice versa*. In this scenario both variational equations would need to be solved simultaneously, a task more difficult than the Navier-Stokes equation itself. The optimal solution of the Doering-Constantin upper bounding problem provides a simplified example of this situation where only one spectral constraint exists and this is marginally satisfied (see [27] for details). Realistically, perhaps only the reduced principle [in which $\boldsymbol{\phi}$ remains a trial field and only Eq. (3.8) is solved] is of practical interest.

Formally, a final minimization over the initial spatial field $\boldsymbol{\phi}_0(\boldsymbol{x})$ produces the lowest upper bound available,

$$D_T \leqslant \inf_{\boldsymbol{\phi}_0 \in \hat{\boldsymbol{\Omega}}} \inf_{\boldsymbol{\phi} \in \boldsymbol{\Omega}_1(\boldsymbol{\phi}_0)\cap \boldsymbol{\Omega}_2} \left\{ \frac{1}{T}\int_0^T \langle |\nabla \boldsymbol{\phi}|^2\rangle + \frac{a}{2T}\langle (\boldsymbol{u}_0 - \boldsymbol{\phi}_0)^2\rangle - \inf_{\boldsymbol{\nu}\in \boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)} \mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) \right\} \tag{3.14}$$

or in a (min) min-max form

$$D_T \leqslant \inf_{\boldsymbol{\phi}_0 \in \hat{\boldsymbol{\Omega}}} \inf_{\boldsymbol{\phi} \in \boldsymbol{\Omega}_1(\boldsymbol{\phi}_0)\cap \boldsymbol{\Omega}_2} \sup_{\boldsymbol{\nu}\in \boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)} \left\{ \frac{1}{T}\int_0^T \langle |\nabla \boldsymbol{\phi}|^2\rangle + \frac{a}{2T}\langle (\boldsymbol{u}_0 - \boldsymbol{\phi}_0)^2\rangle - \mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) \right\}, \tag{3.15}$$

where

$$\hat{\boldsymbol{\Omega}} := \{ \boldsymbol{\phi} \,|\, \phi_i(\boldsymbol{x}) \in C^2(\mathcal{V}), i=1,2,3;\ \nabla\cdot\boldsymbol{\phi}=0,\ \boldsymbol{\phi}=\mathrm{Re}\,\boldsymbol{V}|_{\partial\mathcal{V}}\}. \tag{3.16}$$

B. Maximization problem

The order of optimization can be reversed to produce a max-min procedure in the following way. The functional

$$D_T = -\frac{a}{T}[\langle \tfrac{1}{2}\boldsymbol{\nu}^2\rangle]_0^T + \mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) - \frac{(a-1)}{T}\int_0^T dt\,\langle |\nabla \boldsymbol{\nu}|^2\rangle \tag{3.17}$$

is first optimized over $\boldsymbol{\phi}$ *assuming* that the fluctuation field $\boldsymbol{\nu}$ is known, and then optimized over $\boldsymbol{\nu}$. Since we now have direct control over the trial fluctuation field which must vanish at $t=T$, it is no longer necessary to know $\boldsymbol{u}^{\mathrm{NS}}(\boldsymbol{x},T)$ and we can work with the more general background function set

$$\boldsymbol{\Omega}_3 := \{ \boldsymbol{\phi} \in \boldsymbol{\Omega} \,|\, \boldsymbol{\phi}(\boldsymbol{x},0) = \boldsymbol{\phi}_0(\boldsymbol{x})\}. \tag{3.18}$$

As before, there is a spectral constraint (now on the fluctuation field $\boldsymbol{\nu}$) which ensures that the dissipation rate available after this initial optimization over $\boldsymbol{\phi}$ underestimates the true saddle point value. We define $\boldsymbol{\Gamma}_2$,

$$\boldsymbol{\Gamma}_2 := \{ \boldsymbol{\nu} \in \boldsymbol{\Gamma} \,|\, \langle |\nabla(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2)|^2 + a(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2)\cdot \nabla\boldsymbol{\nu}\cdot(\boldsymbol{\phi}_1 - \boldsymbol{\phi}_2)\rangle \geqslant 0,\ \forall\, \boldsymbol{\phi}_1,\boldsymbol{\phi}_2 \in \boldsymbol{\Omega},\ \forall t\in(0,T)\}, \tag{3.19}$$

as the set of fluctuation fields which satisfy this spectral constraint. Membership of this set is determined by examining whether the eigenspectrum of the linear self-adjoint operator

$$\mathcal{L}(\boldsymbol{\nu};a)\boldsymbol{v} := a(\nabla\boldsymbol{\nu} + \nabla^T\boldsymbol{\nu})\cdot\boldsymbol{v} + \nabla p - 2\nabla^2\boldsymbol{v} \tag{3.20}$$

is positive semidefinite over $\boldsymbol{\Gamma}$ $\forall t\in(0,T)$. Then, provided $\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Gamma}_2$, we have the lower bound

$$\frac{a}{2T}\langle (\boldsymbol{u}_0 - \boldsymbol{\phi}_0)^2\rangle + \inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_3} \mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) - \frac{(a-1)}{T}\int_0^T dt\,\langle |\nabla\boldsymbol{\nu}|^2\rangle \leqslant D_T. \tag{3.21}$$

The condition $\boldsymbol{\nu}\in\boldsymbol{\Gamma}_2$ ensures that $\inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_3}\mathcal{F}_T$ exists. This infimum is identified by the solution of $\delta D_T/\delta\boldsymbol{\phi} = \mathbf{0}$,

$$a(\nabla\boldsymbol{\nu} + \nabla^T\boldsymbol{\nu})\cdot\boldsymbol{\phi} + \nabla p_\phi - 2\nabla^2\boldsymbol{\phi} = -a\frac{\partial\boldsymbol{\nu}}{\partial t} - 2a\boldsymbol{\omega}\times\boldsymbol{\nu} - a\boldsymbol{\nu}\cdot\nabla\boldsymbol{\nu} - (a-2)\nabla^2\boldsymbol{\nu}, \tag{3.22}$$

a linear spatial problem for $\boldsymbol{\phi}$. Again, if a trial fluctuation field is chosen which is ''spectrally'' marginal, Eq. (3.22) is subject to a solvability condition. However, as discussed above, this issue can be ignored in the reduced problem by avoiding such marginal trial fields.

A subsequent optimization over $\boldsymbol{\nu}$ leads to a maximization problem,

$$\sup_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Gamma}_2} \inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_3} \left\{ \frac{a}{2T}\langle (\boldsymbol{u}_0 - \boldsymbol{\phi}_0)^2\rangle + \mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) - \frac{(a-1)}{T}\int_0^T dt\,\langle |\nabla\boldsymbol{\nu}|^2\rangle \right\} \leqslant D_T. \tag{3.23}$$

These two optimizations together amount to solving the required two variational equations

$$\left.\frac{\delta D_T}{\delta \boldsymbol{\nu}}\right|_{\boldsymbol{\phi}} = \left.\frac{\delta D_T}{\delta \boldsymbol{\phi}}\right|_{\boldsymbol{\nu}} = \mathbf{0} \tag{3.24}$$

over the restricted set $\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Gamma}_2$ with $\boldsymbol{\phi}\in\boldsymbol{\Omega}_3$. A final maximization over the initial spatial field $\boldsymbol{\phi}_0(\boldsymbol{x})$ produces the greatest lower bound,

$$\sup_{\boldsymbol{\phi}_0\in\hat{\boldsymbol{\Omega}}}\ \sup_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Gamma}_2}\ \inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_3}\left\{\frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle+\mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a)-\frac{(a-1)}{T}\int_0^T dt\,\langle|\boldsymbol{\nabla}\boldsymbol{\nu}|^2\rangle\right\}\leqslant D_T. \tag{3.25}$$

In summary, we have the bracketing of the realized dissipation, thus

$$\sup_{\boldsymbol{\phi}_0\in\hat{\boldsymbol{\Omega}}}\ \sup_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Gamma}_2}\ \inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_3(\boldsymbol{\phi}_0)}\left\{\frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle+\mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a)-\frac{(a-1)}{T}\int_0^T dt\,\langle|\boldsymbol{\nabla}\boldsymbol{\nu}|^2\rangle\right\}$$
$$\leqslant D_T\leqslant \inf_{\boldsymbol{\phi}_0\in\hat{\boldsymbol{\Omega}}}\ \inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Omega}_2}\ \sup_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)}\left\{\frac{1}{T}\int_0^T\langle|\boldsymbol{\nabla}\boldsymbol{\phi}|^2\rangle+\frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle-\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a)\right\}. \tag{3.26}$$

IV. DO THE COMPLEMENTARY VARIATIONAL PROBLEMS INTERSECT?

It is natural to speculate whether the complementary variational principles *attain* the saddle point value; in other words, whether the inequality signs in Eq. (3.26) may be more accurately replaced by equality signs. In terms of the solution $[\boldsymbol{\phi}^{\rm NS}(\boldsymbol{x},t),\boldsymbol{\nu}^{\rm NS}(\boldsymbol{x},t)]$ of the full variational problem [Eqs. (2.22) and (2.24) subject to the initial and final conditions $\boldsymbol{\phi}(\boldsymbol{x},0)+\boldsymbol{\nu}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})$ and $\boldsymbol{\nu}(\boldsymbol{x},T)=\boldsymbol{0}$], this is the condition that

$$\boldsymbol{\phi}^{\rm NS}\in\boldsymbol{\Omega}_2\quad\text{and}\quad\boldsymbol{\nu}^{\rm NS}\in\boldsymbol{\Gamma}_2. \tag{4.1}$$

If only one of these conditions is satisfied, then the saddle point value is only attained from that side although this unsymmetric situation seems unlikely. If both conditions hold, then it can be shown using convexity and concavity arguments that the complementary variational problems intersect at a unique saddle point. Consider the minimization problem first for the trial background field $\tilde{\boldsymbol{\phi}}\in\boldsymbol{\Omega}_1(\boldsymbol{\phi}^{\rm NS}(\boldsymbol{x},0))$ with accompanying fluctuation field $\tilde{\boldsymbol{\nu}}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}^{\rm NS}(\boldsymbol{x},0))$ found by solving $\delta D_T/\delta\boldsymbol{\nu}=\boldsymbol{0}$ and compare its dissipation rate value with that of the Navier-Stokes solution $[\boldsymbol{\phi}^{\rm NS}(\boldsymbol{x},t),\boldsymbol{\nu}^{\rm NS}(\boldsymbol{x},t)]$. Since $D_T(\boldsymbol{\phi},\boldsymbol{\nu})$ is only quadratic in $\boldsymbol{\phi}$ and $\boldsymbol{\nu}$, it is straightforward to show that

$$\begin{aligned}D_T(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}};a)-D_T(\boldsymbol{\phi}^{\rm NS},\boldsymbol{\nu}^{\rm NS};a)&=[D_T(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}};a)-D_T(\tilde{\boldsymbol{\phi}},\boldsymbol{\nu}^{\rm NS};a)]+[D_T(\tilde{\boldsymbol{\phi}},\boldsymbol{\nu}^{\rm NS};a)-D_T(\boldsymbol{\phi}^{\rm NS},\boldsymbol{\nu}^{\rm NS};a)]\\&=\frac{1}{T}\int_0^T dt\left\langle\frac{\delta D_T}{\delta\boldsymbol{\nu}}(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}})\cdot(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})-\frac{1}{2}\frac{\delta^2D_T}{\delta\boldsymbol{\nu}^2}(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}})(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})\right.\\&\quad\left.+\frac{\delta D_T}{\delta\boldsymbol{\phi}}(\boldsymbol{\phi}^{\rm NS},\boldsymbol{\nu}^{\rm NS})\cdot(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})+\frac{1}{2}\frac{\delta^2D_T}{\delta\boldsymbol{\phi}^2}(\boldsymbol{\phi}^{\rm NS},\boldsymbol{\nu}^{\rm NS})(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})\right\rangle.\end{aligned} \tag{4.2}$$

Here by initial assumption $\delta D_T/\delta\boldsymbol{\nu}(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}})=\boldsymbol{0}$ and the second functional derivative terms are

$$\left\langle-\frac{1}{2}\frac{\delta^2D_T}{\delta\boldsymbol{\nu}^2}(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}})(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})\right\rangle=\frac{1}{T}\int_0^T\langle(a-1)|\boldsymbol{\nabla}(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})|^2+a(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})\cdot\boldsymbol{\nabla}\tilde{\boldsymbol{\phi}}\cdot(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})\rangle dt, \tag{4.3}$$

$$\left\langle\frac{1}{2}\frac{\delta^2D_T}{\delta\boldsymbol{\phi}^2}(\boldsymbol{\phi}^{\rm NS},\boldsymbol{\nu}^{\rm NS})(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})\right\rangle=\frac{1}{T}\int_0^T\langle|\boldsymbol{\nabla}(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})|^2+a(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})\cdot\boldsymbol{\nabla}\boldsymbol{\nu}^{\rm NS}\cdot(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})\rangle dt. \tag{4.4}$$

With conditions (4.1) satisfied, the last term in Eq. (4.2) is positive semidefinite and then, providing $\tilde{\boldsymbol{\phi}}$ satisfies the spectral constraint that the expression in Eq. (4.3) is positive semidefinite for all $\tilde{\boldsymbol{\nu}}\in\boldsymbol{\Gamma}$ ($\tilde{\boldsymbol{\phi}}\in\boldsymbol{\Omega}_2$), a global minimum is assured at $[\boldsymbol{\phi}^{\rm NS},\boldsymbol{\nu}^{\rm NS}]$. For uniqueness of this minimum, we need to establish the nonexistence of fields $[\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}}]$ distinct from $[\boldsymbol{\phi}^{\rm NS},\boldsymbol{\nu}^{\rm NS}]$ for which Eqs. (4.3) and (4.4) vanish identically. Since the integrands of both these expressions are supposedly positive semidefinite, they must in fact vanish pointwise in time. This implies that the two equations

$$a(\boldsymbol{\nabla}\tilde{\boldsymbol{\phi}}+\boldsymbol{\nabla}^T\tilde{\boldsymbol{\phi}})\cdot(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})+\boldsymbol{\nabla}p_1-2(a-1)\nabla^2(\tilde{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\rm NS})=\boldsymbol{0}, \tag{4.5}$$

$$a(\boldsymbol{\nabla}\boldsymbol{\nu}^{\rm NS}+\boldsymbol{\nabla}^T\boldsymbol{\nu}^{\rm NS})\cdot(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})+\boldsymbol{\nabla}p_2-2\nabla^2(\tilde{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\rm NS})=\boldsymbol{0} \tag{4.6}$$

must be satisfied $\forall\,t\in(0,T)$ in *addition* to the variational equation $\delta D_T/\delta\boldsymbol{\nu}(\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}})=\boldsymbol{0}$. As an overspecified system for $[\tilde{\boldsymbol{\phi}},\tilde{\boldsymbol{\nu}}]$, this indicates (but does not prove) that the global minimum is unique.

In the maximization case, consider a trial fluctuation field $\hat{\boldsymbol{\nu}}\in\boldsymbol{\Gamma}_1(\boldsymbol{\phi}^{\rm NS}(\boldsymbol{x},0))$ and its accompanying background field $\hat{\boldsymbol{\phi}}\in\boldsymbol{\Omega}_1(\boldsymbol{\phi}^{\rm NS}(\boldsymbol{x},0))$ found by solving $\delta D_T/\delta\boldsymbol{\phi}=\boldsymbol{0}$ and compare its dissipation rate value with that of the Navier-Stokes solution $[\boldsymbol{\phi}^{\rm NS}(\boldsymbol{x},t),\boldsymbol{\nu}^{\rm NS}(\boldsymbol{x},t)]$,

$$
\begin{aligned}
&D_T(\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\nu}};a)-D_T(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}};a)\\
&\quad=[D_T(\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\nu}};a)-D_T(\boldsymbol{\phi}^{\mathrm{NS}},\hat{\boldsymbol{\nu}};a)]\\
&\qquad+[D_T(\boldsymbol{\phi}^{\mathrm{NS}},\hat{\boldsymbol{\nu}};a)-D_T(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}};a)]\\
&\quad=\frac{1}{T}\int_0^T\Big\langle\frac{\delta D_T}{\delta\boldsymbol{\nu}}(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}})\cdot(\hat{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\mathrm{NS}})\\
&\qquad+\frac{1}{2}\frac{\delta^2 D_T}{\delta\boldsymbol{\nu}^2}(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}})(\hat{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\mathrm{NS}})\\
&\qquad+\frac{\delta D_T}{\delta\boldsymbol{\phi}}(\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\nu}})\cdot(\hat{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\mathrm{NS}})\\
&\qquad-\frac{1}{2}\frac{\delta^2 D_T}{\delta\boldsymbol{\phi}^2}(\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\nu}})(\hat{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\mathrm{NS}})\Big\rangle dt. \qquad (4.7)
\end{aligned}
$$

Here $\delta D_T/\delta\boldsymbol{\phi}(\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\nu}})=\mathbf{0}$ by initial assumption and

$$
\begin{aligned}
&\Big\langle-\frac{1}{2}\frac{\delta^2 D_T}{\delta\boldsymbol{\nu}^2}(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}})(\hat{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\mathrm{NS}})\Big\rangle\\
&\quad=\frac{1}{T}\int_0^T\langle(a-1)|\boldsymbol{\nabla}(\hat{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\mathrm{NS}})|^2\\
&\qquad+a(\hat{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\mathrm{NS}})\cdot\boldsymbol{\nabla}\boldsymbol{\phi}^{\mathrm{NS}}\cdot(\hat{\boldsymbol{\nu}}-\boldsymbol{\nu}^{\mathrm{NS}})\rangle dt, \qquad (4.8)
\end{aligned}
$$

$$
\begin{aligned}
&\Big\langle\frac{1}{2}\frac{\delta^2 D_T}{\delta\boldsymbol{\phi}^2}(\hat{\boldsymbol{\phi}},\hat{\boldsymbol{\nu}})(\hat{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\mathrm{NS}})\Big\rangle\\
&\quad=\frac{1}{T}\int_0^T\langle|\boldsymbol{\nabla}(\hat{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\mathrm{NS}})|^2\\
&\qquad+a(\hat{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\mathrm{NS}})\cdot\boldsymbol{\nabla}\hat{\boldsymbol{\nu}}\cdot(\hat{\boldsymbol{\phi}}-\boldsymbol{\phi}^{\mathrm{NS}})\rangle dt. \qquad (4.9)
\end{aligned}
$$

With conditions (4.1) satisfied, the second term of Eq. (4.7) is positive semidefinite and then, providing $\hat{\boldsymbol{\nu}}$ satisfies the dual spectral constraint that the expression in Eq. (4.8) is positive semidefinite for all $\hat{\boldsymbol{\phi}}\in\boldsymbol{\Omega}$ ($\hat{\boldsymbol{\nu}}\in\boldsymbol{\Gamma}_2$), a global maximum is assured at $[\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}}]$. Uniqueness depends on the same arguments as before.

It is difficult to establish whether $\boldsymbol{\phi}^{\mathrm{NS}}\in\boldsymbol{\Omega}_2$ or $\boldsymbol{\nu}^{\mathrm{NS}}\in\boldsymbol{\Gamma}_2$ since these are global properties over the whole spaces $\boldsymbol{\Gamma}$ and $\boldsymbol{\Omega}$, respectively. This very fact, however, naturally suggests that these conditions be interpreted as some sort of ''stability'' criteria. Doering and Constantin [26] have already noticed some similarity between the spectral constraint and energy stability arguments in their upper bound formulation. However, this connection is not borne out in the broader variational context discussed here. Instead, their interpretation seems to lie simply with the attribute of making the dissipation extremal. If $\boldsymbol{\nu}^{\mathrm{NS}}\in\boldsymbol{\Gamma}_2$, then from Eq. (4.2)

$$
D_T(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}})\leqslant D_T(\boldsymbol{\phi},\boldsymbol{\nu}^{\mathrm{NS}}),\quad\forall\boldsymbol{\phi}\in\boldsymbol{\Omega} \qquad (4.10)
$$

so that dissipation (for $\boldsymbol{\nu}=\boldsymbol{\nu}^{\mathrm{NS}}$) is minimized by the realized background field, whereas if $\boldsymbol{\phi}^{\mathrm{NS}}\in\boldsymbol{\Omega}_2$, then

$$
D_T(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu})\leqslant D_T(\boldsymbol{\phi}^{\mathrm{NS}},\boldsymbol{\nu}^{\mathrm{NS}}),\quad\forall\boldsymbol{\nu}\in\boldsymbol{\Gamma}. \qquad (4.11)
$$

Unfortunately, this interpretation is merely that and does not offer any clues as to how the conditions (4.1) may be established. An observation already made in Secs. III A and III B regarding ''spectrally'' marginal trial fields is worth revisiting here. If one of the realized fields, say $\boldsymbol{\phi}^{\mathrm{NS}}$, *is* spectrally marginal, then the operator to be inverted in Eq. (3.8) (to identify $\boldsymbol{\nu}^{\mathrm{NS}}$) is singular. The solvability condition(s) then indirectly imposed on $\boldsymbol{\phi}^{\mathrm{NS}}$ together with the null space degeneracy in $\boldsymbol{\nu}^{\mathrm{NS}}$ seem to necessitate that the operator in the other variational equation also be singular with the same dimensional null space. Talking broadly, this null space degeneracy in $\boldsymbol{\phi}^{\mathrm{NS}}$ is then of the right dimension to accommodate the solvability condition(s) in the $\boldsymbol{\nu}^{\mathrm{NS}}$ equation and *vice versa*. In this scenario, it does not necessarily follow that $\boldsymbol{\nu}^{\mathrm{NS}}$ is also spectrally marginal but merely that the eigenspectrum of the operator in Eq. (3.20) has a zero. Nevertheless, this argument at least indicates that Eq. (4.1) is not obviously flawed and perhaps hints at a way forward. This issue is, of course, of fundamental interest since proving Eq. (4.1) would appear to guarantee that D_T can only have one stationary point and therefore that the Navier-Stokes equations have a unique solution for given initial conditions.

V. IMPLEMENTATION: THE MINIMIZATION PROBLEM

So far the focus has been to formally establish the existence of complementary variational principles based upon certain spectral constraints and to discuss whether they meet in a unique saddle point. Here we concentrate upon their implementation by discussing reduced versions of these variational principles with a view to realizing both upper and lower bounds on the dissipation in plane Couette flow.

The minimization problem formulated in Sec. III A required the knowledge of the final realized velocity $\boldsymbol{u}^{\mathrm{NS}}(\boldsymbol{x},T)$ which will usually be unavailable. Consequently, we present here a more useful and only very slightly ''less-sharp'' variational problem which upper bounds the functional

$$
\begin{aligned}
\mathcal{D}_T&:=D_T+\frac{a}{2T}\langle\boldsymbol{\nu}^2(\boldsymbol{x},T)\rangle\\
&=\frac{1}{T}\int_0^T dt\,\langle|\boldsymbol{\nabla}\boldsymbol{\phi}|^2\rangle+\frac{a}{2T}\langle\boldsymbol{\nu}^2(\boldsymbol{x},0)\rangle-\mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) \qquad (5.1)
\end{aligned}
$$

and therefore D_T. Provided the trial background field $\boldsymbol{\phi}$ is chosen in $\boldsymbol{\Omega}_3(\boldsymbol{\phi}_0)\cap\boldsymbol{\Omega}_2$ and fluctuation fields which solve

$$
\left.\frac{\delta D_T}{\delta\boldsymbol{\nu}}\right|_{\boldsymbol{\phi}}=\mathbf{0} \qquad (5.2)
$$

are in

$$
\boldsymbol{\Gamma}_3(\boldsymbol{\phi}_0)=\{\boldsymbol{\nu}\in\boldsymbol{\Gamma}\,|\,\boldsymbol{\nu}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})-\boldsymbol{\phi}_0(\boldsymbol{x})\}, \qquad (5.3)
$$

then

$$D_T \leqslant \frac{1}{T}\int_0^T dt\,\langle|\nabla\boldsymbol{\phi}|^2\rangle + \frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle - \inf_{\boldsymbol{\nu}\in\boldsymbol{\Gamma}_3(\boldsymbol{\phi}_0)} \mathcal{G}_T(\boldsymbol{\phi},\boldsymbol{\nu};a). \tag{5.4}$$

The key step is being able to remove the boundary term $\langle\frac{1}{2}\boldsymbol{\nu}^2(\boldsymbol{x},T)\rangle$ from variational consideration: this lifts the troublesome requirement that $\boldsymbol{\nu}(\boldsymbol{x},T)=\boldsymbol{0}$. Certainly in the case of turbulent flows we can anticipate that the final kinetic energy term which separates $\mathcal{D}_T$ and D_T will become less significant relative to the dissipative terms as T increases so that ultimately $\mathcal{D}_T\rightarrow D_T$ as $T\rightarrow\infty$.

The exact form of the trial background field chosen has profound consequences on the constraints actually being applied to the upper bound minimization problem. We illustrate this in the case of plane Couette flow where an incompressible fluid in the region $(x,y,z)\in\mathbb{R}^2\times[-\frac{1}{2},\frac{1}{2}]$ is subjected to shearing boundary conditions $\boldsymbol{u}(x,y,\pm\frac{1}{2})=\mp\frac{1}{2}\mathrm{Re}\,\hat{\boldsymbol{x}}$.

A. One-dimensional steady background fields: $\boldsymbol{\phi}=\boldsymbol{\phi}(z)\hat{\boldsymbol{x}}$

The choice of a steady, scalar background field $\boldsymbol{\phi}=\phi(z)\hat{\boldsymbol{x}}$ is the simplest possible one consistent with the boundary conditions in plane Couette flow. The variational equation $\delta D_T/\delta\boldsymbol{\nu}|_{\phi}=\boldsymbol{0}$ is now simply

$$a\phi'\begin{bmatrix}\nu_3\\0\\\nu_1\end{bmatrix}+\nabla p_\nu-2(a-1)\nabla^2\boldsymbol{\nu}=(a-2)\phi''\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{5.5}$$

[[28], Eq. (2.39)]. However, there is no guarantee that the steady solution to this equation will have the correct initial conditions $\boldsymbol{\nu}_0=\boldsymbol{u}_0-\phi_0(z)\hat{\boldsymbol{x}}$. In this case it is clear that only the limiting case $T\rightarrow\infty$ in Eq. (5.4) makes sense since then the initial conditions drop entirely from the problem. The subsequent upper bound on D_∞ is then a supremum over *all* possible initial conditions. This is the upper bound problem initially proposed by Doering and Constantin [23,24] and improved by Nicodemus *et al.* [30]. Simple trial fields have been used to deduce rigorous bounds on the dissipation [23,30,34] as well as more sophisticated examples in a detailed numerical study by Nicodemus, Grossman, and Holthaus [32,33]. This latter work has proved especially important not only in establishing the practicality of this upper bound problem but also in indirectly confirming previous asymptotic work by Busse [5] in the dual problem (see [28]).

Given the larger variational context within which this reduced problem lies, it is now clear what constraints are actually being imposed. In the limit $T\rightarrow\infty$, the functional given in Eq. (2.5) is

$$D_\infty := \lim_{T\rightarrow\infty}\frac{1}{T}\int_0^T\langle|\nabla\boldsymbol{u}|^2\rangle\,dt - \lim_{T\rightarrow\infty}\frac{a}{T}\int_0^T\langle\boldsymbol{u}\cdot\boldsymbol{\mathcal{N}}\rangle\,dt + \int_{-1/2}^{1/2}\phi\lim_{T\rightarrow\infty}\frac{a}{T}\int_0^T\overline{\mathcal{N}_1 dt}\;dz, \tag{5.6}$$

where

$$\overline{A}(z,t) := \lim_{L_x,L_y\rightarrow\infty}\frac{1}{4L_xL_y}\int_{-L_x}^{+L_x}dx\int_{-L_y}^{+L_y}dy\,A(\boldsymbol{x},t)$$

is the horizontal mean and $\boldsymbol{\mathcal{N}}=\boldsymbol{0}$ indicates the Navier-Stokes equations (here with $\boldsymbol{\omega}=\boldsymbol{0}$),

$$\boldsymbol{\mathcal{N}} := \frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}+\nabla p-\nabla^2\boldsymbol{u},$$

so that $\mathcal{N}_1=0$ is the $\hat{\boldsymbol{x}}$ component. Both ϕ and a signify Lagrange multipliers in this expression linked with specific constraints. Taking variations in ϕ requires that the long-time average (or equivalently statistical average) of the horizontally averaged Navier-Stokes equation in the $\hat{\boldsymbol{x}}$ direction be satisfied (i.e., vanish). Subsequent optimization over the balance parameter a imposes the total power integral as noticed before [[28], Eq. 2.67)].

The result of this minimization procedure is therefore the maximum dissipation possible subject to this subset of constraints derived from the Navier-Stokes equations, and as such constitutes an upper bound on the true dissipation. The upper bound may also be estimated from *below* by reversing the order of optimization. Eliminating $\phi(z)$ by solving $\delta D_\infty/\delta\phi|_{\boldsymbol{\nu}}=0$ leaves a maximization problem over $\boldsymbol{\nu}$. This is essentially how Busse [5] obtained his first estimate of this upper bound from below. Rather than dealing with the full functional D_∞ and using the degenerate representation $\boldsymbol{u}=\phi(z)\hat{\boldsymbol{x}}+\boldsymbol{\nu}(\boldsymbol{x},t)$, Busse appealed to a number of plausible physical arguments to lead directly to a maximization problem over $\boldsymbol{v}=\boldsymbol{\nu}-\overline{\boldsymbol{\nu}}$, thereby shortcutting the formal procedure outlined above (see [28] for details). It is clear here that the optimization procedure over ϕ and a incorporates precisely the physical information that Busse built directly into his problem.

The fact that this maximization problem can surpass the true dissipation is of course tied in with the restricted form of the background field. Choosing a trial fluctuation field $\boldsymbol{\nu}(\boldsymbol{x},t)$ and minimizing over $\boldsymbol{\phi}=\phi(z)\hat{\boldsymbol{x}}$ is no longer guaranteed to lower bound the true dissipation since $\boldsymbol{\nu}(\boldsymbol{x},t)+\phi(z)\hat{\boldsymbol{x}}$ is now not general enough to encompass all possible velocity fields for a *given* $\boldsymbol{\nu}(\boldsymbol{x},t)$.

B. Three-dimensional background fields: $\boldsymbol{\phi}=\boldsymbol{\phi}(\boldsymbol{x})$

Allowing the trial background field to be three-dimensional permits more information to be incorporated from the Navier-Stokes equations and hence should lead to a lower (better) upper bound. In particular, Eq. (5.6) becomes

$$D_\infty := \lim_{T\rightarrow\infty}\frac{1}{T}\int_0^T\langle|\nabla\boldsymbol{u}|^2\rangle\,dt - \lim_{T\rightarrow\infty}\frac{a}{T}\int_0^T\langle\boldsymbol{u}\cdot\boldsymbol{\mathcal{N}}\rangle\,dt + \left\langle\boldsymbol{\phi}\cdot\lim_{T\rightarrow\infty}\frac{a}{T}\int_0^T\boldsymbol{\mathcal{N}}dt\right\rangle. \tag{5.7}$$

Optimization with respect to the Lagrange multiplier $\boldsymbol{\phi}$ then incorporates the requirement that the steady Navier-Stokes equations be satisfied. Improvement of the bound, however, is not automatically guaranteed unless the new constraints imposed disallow previous optimizing solutions. Section 3.3 of [28] discusses the use of the two-dimensional z-dependent background field choice $\boldsymbol{\phi}=\phi_1(z)\hat{\boldsymbol{x}}+\phi_2(z)\hat{\boldsymbol{y}}$ in plane Couette flow. The extra constraint applied compared to using the background field $\boldsymbol{\phi}=\phi(z)\hat{\boldsymbol{x}}$ is that the steady, horizontally averaged y component of the Navier-Stokes equation must vanish. Since Busse's optimizing solution already satisfies this new constraint, the optimal choice of $\phi_2(z)$ is trivially 0 and no improvement results.

Again this upper bound may be estimated from below by reversing the order of optimizations to produce a maximization problem over $\boldsymbol{\nu}$.

C. One-dimensional unsteady background fields: $\boldsymbol{\phi}=\boldsymbol{\phi}(z,t)\hat{\boldsymbol{x}}$

A time-dependent, one-dimensional background field depending only on z is the simplest choice which allows upper bounds dependent on initial conditions to be explored. The functional is

$$D_T:=\frac{1}{T}\int_0^T\langle|\nabla\boldsymbol{u}|^2\rangle\,dt-\frac{a}{T}\int_0^T\langle\boldsymbol{u}\cdot\boldsymbol{\mathcal{N}}\rangle\,dt+\frac{a}{T}\int_0^T\int_{-1/2}^{1/2}\phi\,\overline{\mathcal{N}_1\,dz}\,dt,\qquad(5.8)$$

which means that the horizontally averaged first component of the Navier-Stokes equation is the applied constraint. As mentioned before, since $\delta D_T/\delta\boldsymbol{\nu}|_\phi=\mathbf{0}$ is a purely spatial problem for $\boldsymbol{\nu}$ given ϕ, the initial condition $\boldsymbol{\nu}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x})-\phi_0(z,0)\hat{\boldsymbol{x}}$ is in fact a condition on $\partial\phi/\partial t(z,0)$. Given the restricted form of the background field, only a subset of initial conditions will be accessible. The challenge then becomes one of finding a time-dependent background field which gives the best (lowest) upper bound. The spectral constraint is a pointwise condition in time and so only the spatial structure of the background field is important in this respect. Constructing a valid background field amounts to specifying a smooth time path across sets of allowable spatial fields parametrized by time. Since the forcing boundary conditions are steady for plane Couette flow, one set of allowable background spatial fields at a given Reynolds number,

$$\tilde{\Omega}_2:=\{\,\phi\,|\,\phi(z)\in C^2[-\tfrac{1}{2},\tfrac{1}{2}],\ \phi(\pm\tfrac{1}{2})=\mp\tfrac{1}{2}\mathrm{Re},\ \langle\,(a-1)|\nabla\boldsymbol{\nu}|^2+a\boldsymbol{\nu}\cdot\nabla\boldsymbol{\phi}\cdot\boldsymbol{\nu}\,\rangle\geqslant 0,\quad\forall\ \boldsymbol{\nu}\in\boldsymbol{\Gamma}\},\qquad(5.9)$$

can be used to select background profiles throughout the time interval. It is straightforward to show that the set $\tilde{\Omega}_2$ is convex; if $\phi_1(z)$ and $\phi_2(z)$ satisfy the spectral constraint, then so does $\lambda\phi_1+(1-\lambda)\phi_2$ for $\lambda\in[0,1]$. Additionally, the set is nonempty for any Reynolds number Re. For a background spatial field $\phi(z)$ to be in $\tilde{\Omega}_2$, the sign-indeterminate term $a/(a-1)\langle\phi'\nu_1\nu_3\rangle$ must be dominated by the positive-definite term $\langle|\nabla\boldsymbol{\nu}|^2\rangle$ for all permissable fluctuation fields $\boldsymbol{\nu}$. Since trial background fields will scale with Re through the boundary condition, the former term can be made as small as required relative to the latter positive definite term by reducing Re. As a result, every practical background field has a critical Reynolds number below which it satisfies the spectral constraint. In particular, the steady laminar solution $\boldsymbol{u}=-\mathrm{Re}\,z\,\hat{\boldsymbol{x}}$ is in $\tilde{\Omega}_2$ for $\mathrm{Re}\leqslant\mathrm{Re}_{\mathrm{abs}}$, where $\mathrm{Re}_{\mathrm{abs}}$ is the energy stability limit [30,35]. For these Reynolds numbers, the background field should be designed to evolve from its initial state to the steady laminar state since this *is* the correct (long-time) Navier-Stokes solution (with $\boldsymbol{\nu}\rightarrow\mathbf{0}$). At higher Reynolds numbers above the energy stability threshold, the fluid may or may not be attracted to the laminar state depending on the initial conditions. Since the laminar state is no longer a valid background field for these Reynolds numbers, other nearby choices must be made. These other choices, of course, still permit the laminar state to be approached ultimately but now necessarily through an accompanying nontrivial fluctuation field.

Constructing more general background fields clearly allows more information from the Navier-Stokes equations to be built into the variational problem as constraints; see expressions (5.6), (5.7), and (5.8). Also clear from these is the nontrivial role played by the parameter a. Although upper bounds emerge for any $a>1$, these bounds can be further optimized over a. This is in contrast to the maximization problem, where a merely acts to renormalize $\boldsymbol{\nu}$.

VI. IMPLEMENTATION: THE MAXIMIZATION PROBLEM

The interestingly novel feature in the maximization problem is the spectral constraint on the trial fluctuation field. Unlike that on the background field, this does not depend on the Reynolds number since the fluctuation field has homogeneous boundary conditions. The set of permissible fluctuation spatial fields is then unique for a given system (modulo a renormalization in a). The Reynolds-number dependence is associated with the background field and enters through the solution of $\delta D_T/\delta\boldsymbol{\phi}|_{\boldsymbol{\nu}}=\mathbf{0}$ into the dissipation functional.

The emphasis in producing best lower bounds is also distinctly different from that in the upper bounding case. There the underlying philosophy is generally to select the background profile of least dissipation which is still consistent with the spectral contraint. In the lower bound problem, fluctuation fields which will *maximize* the dissipation are sought subject to the spectral constraint, which suggests selecting the most complicated field available. It is therefore, for example, of little consequence that $\boldsymbol{\nu}=\mathbf{0}$ is always allowed as a trial function since beyond $\mathrm{Re}_{\mathrm{abs}}$ this is not assured to give the best (largest) lower bound on the dissipation.

A. Steady fluctuation fields: $\boldsymbol{\nu}=\boldsymbol{\nu}(x)$

As before in the background field case, choosing a steady fluctuation field forces the long-time limit to be taken and means that only a dissipation infimum can be produced over all initial conditions. Since initial conditions appropriate to the laminar flow solution, $\boldsymbol{u}=-\mathrm{Re}\,z\,\hat{\boldsymbol{x}}$, are possible for all Re, this infimum will always be the laminar dissipation and hence trivial. A time-dependent trial fluctuation field is required to avoid this scenario.

B. Unsteady fluctuation fields: $\boldsymbol{\nu}=\boldsymbol{\nu}(\boldsymbol{x},t)$

A fully three-dimensional, unsteady fluctuation field dependent on all three spatial variables potentially imposes the complete Navier-Stokes equations as constraints on the lower bound problem [see Eq. (2.5)]. Provided $\boldsymbol{\nu} \in \boldsymbol{\Gamma}_1(\boldsymbol{\phi}_0)\cap\boldsymbol{\Gamma}_2$, we have the bound

$$\frac{a}{2T}\langle(\boldsymbol{u}_0-\boldsymbol{\phi}_0)^2\rangle+\inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_3}\mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a)-\frac{(a-1)}{T}\int_0^T dt\,\langle|\boldsymbol{\nabla}\boldsymbol{\nu}|^2\rangle \leqslant D_T, \tag{6.1}$$

where

$$\inf_{\boldsymbol{\phi}\in\boldsymbol{\Omega}_3}\mathcal{F}_T(\boldsymbol{\phi},\boldsymbol{\nu};a) = \frac{1}{2T}\int_0^T\Big\langle -a\,\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\phi}^*}{\partial t}+a\,\boldsymbol{\nu}\cdot\boldsymbol{\nabla}\boldsymbol{\nu}\cdot\boldsymbol{\phi}^* +(a-2)\boldsymbol{\nu}\cdot\nabla^2\boldsymbol{\phi}^*\Big\rangle +[\overline{\boldsymbol{\phi}^*\cdot\partial_z(2\boldsymbol{\phi}^*-(a-2)\boldsymbol{\nu})}]_{z=-1/2}^{z=1/2}\,dt \tag{6.2}$$

and $\boldsymbol{\phi}^*$ solves $\delta D_T/\delta\boldsymbol{\phi}=\boldsymbol{0}$. Formally, for given initial conditions on $\boldsymbol{u}$, the trial field $\boldsymbol{\nu}$ must be designed so that the solution of $\delta D_T/\delta\boldsymbol{\phi}|_{\boldsymbol{\nu}}=\boldsymbol{0}$ is $\boldsymbol{u}_0(\boldsymbol{x})-\boldsymbol{\nu}(\boldsymbol{x},0)$ at $t=0$. This requires that

$$\frac{\partial\boldsymbol{\nu}}{\partial t}(\boldsymbol{x},0)=\frac{1}{a}\{-a(\boldsymbol{\nabla}\boldsymbol{\nu}+\boldsymbol{\nabla}^T\boldsymbol{\nu})\cdot\boldsymbol{\phi}-\boldsymbol{\nabla}p_{\phi} +2\nabla^2\boldsymbol{\phi}-a\,\boldsymbol{\nu}\cdot\boldsymbol{\nabla}\boldsymbol{\nu}-(a-2)\nabla^2\boldsymbol{\nu}\}_{\boldsymbol{\phi}=\boldsymbol{u}_0-\boldsymbol{\nu}(\boldsymbol{x},0)}, \tag{6.3}$$

where p_{ϕ} merely ensures that $\boldsymbol{\nu}$ remains incompressible. Since the set of spatial fluctuation fields (call it $\tilde{\boldsymbol{\Gamma}}_2$) which satisfy the spectral constraint is convex as before, both $\boldsymbol{\nu}(\boldsymbol{x},0)$ and $\partial\boldsymbol{\nu}/\partial t(\boldsymbol{x},0)$ should be members of $\tilde{\boldsymbol{\Gamma}}_2$ to ensure that $\boldsymbol{\nu}(\boldsymbol{x},t>0)$ is too. Convexity of $\tilde{\boldsymbol{\Gamma}}_2$ means that the trial background field can smoothly evolve to visit various known spatial fluctuation fields in $\tilde{\boldsymbol{\Gamma}}_2$. Finally, since $\boldsymbol{\nu}=\boldsymbol{0}$ is in $\tilde{\boldsymbol{\Gamma}}_2$ reaching the required end point $\boldsymbol{\nu}(\boldsymbol{x},T)=\boldsymbol{0}$ does not present a problem. Practically, $\boldsymbol{u}_0$ should not be specified at the onset but determined *a posteriori* given an allowable background field at $t=0$. Once the initial conditions have been set, then the evolution of $\boldsymbol{\nu}$ for $t>0$ is open to adjustment in the search for the largest dissipation lower bound.

This procedure has obvious possibilities for exploring the instability of laminar flows from the prospective of an initial value problem. In the particular example of plane Couette flow, presumed linearly stable at all Reynolds numbers, information concerning threshold amplitudes for instability may be available by demonstrating that the dissipation must exceed the laminar dissipation for given initial conditions as $T\to\infty$. This application makes it clear how retaining time dependence enriches these bounding problems since only then is the fundamental dynamical concept of stability included. The simplest such problem available has a time-dependent trial fluctuation field of the form $\boldsymbol{\nu}=\nu(z,t)\hat{\boldsymbol{x}}$ which leads to the dissipation functional

$$D_T:=\frac{1}{T}\int_0^T\langle|\boldsymbol{\nabla}\boldsymbol{u}|^2\rangle\,dt-\frac{a}{T}\int_0^T\int_{-1/2}^{1/2}\nu\,\overline{\mathcal{N}_1\,dz}\,dt. \tag{6.4}$$

The horizontally averaged x component of the Navier-Stokes equations is imposed as a constraint and therefore only stability characteristics associated with this one time-dependent equation can be incorporated. Given the restricted form of the trial fields, only a special subset of initial conditions can be possible. Nevertheless, providing these conditions are different from the laminar profile, there is still the possibility of establishing bounds on the threshold amplitude for instability of the laminar state. As before in the trial background field case, these lower bounds can all be estimated from above by reversing the order of optimization.

VII. DISCUSSION

In this paper we have revealed how to embed the dual upper bound formulations of Doering-Constantin and Busse within a grander variational framework found by exploiting the full degeneracy of the background-fluctuation velocity decomposition: see Fig. 1. In this framework, it is clear how to manufacture successively more constrained upper and now lower bounding problems on the fluid's dissipation until ultimately the Navier-Stokes solution itself is the optimal solution. At this point the upper and lower bounding problems represent complementary variational principles for estimating the true dissipation *and* the Navier-Stokes solution (in the dissipation norm). Presently it is unclear whether these dual principles actually touch at the Navier-Stokes solution. Establishing this would prove the uniqueness of the Navier-Stokes solution for given initial and boundary conditions.

The novel introduction or rather retention of time dependence in the variational problems discussed here certainly offers new challenges for their implementation. Time dependence undoubtedly adds another layer of complexity to the numerical study recently completed by Nicodemus *et al.* [32,33] where the spectral constraint has already been successfully implemented. However, the corresponding rewards seem encouragingly high. Aspects of hydrodynamic stability can now be included along with the sensitivity to initial conditions now known to be so important for most hydrodynamic stability problems. In particular, only a lower bound problem incorporating time dependence can ''rise above'' the trivial laminar dissipation infimum.

From the prospective of ''upper bound theory'' [1,2] or the ''optimal theory of turbulence'' [3–6], it is natural to ask whether other functionals apart from the dissipation may also share the variational structure revealed here. In other words, is the dissipation functional special in some sense? Certainly there is nothing profound in the initial construction of the functional D_T given in Eq. (2.5). However, written in terms of $\boldsymbol{\phi}$ and $\boldsymbol{\nu}$, it is clear that to obtain the crucial saddle point structure, the constraint expression must contribute exactly the right term to make the overall functional negative-definite in the highest derivative term involving $\boldsymbol{\nu}$, i.e., $(1/T)\int_0^T\langle|\boldsymbol{\nabla}\boldsymbol{\nu}|^2\rangle dt$. If the base functional (which of course

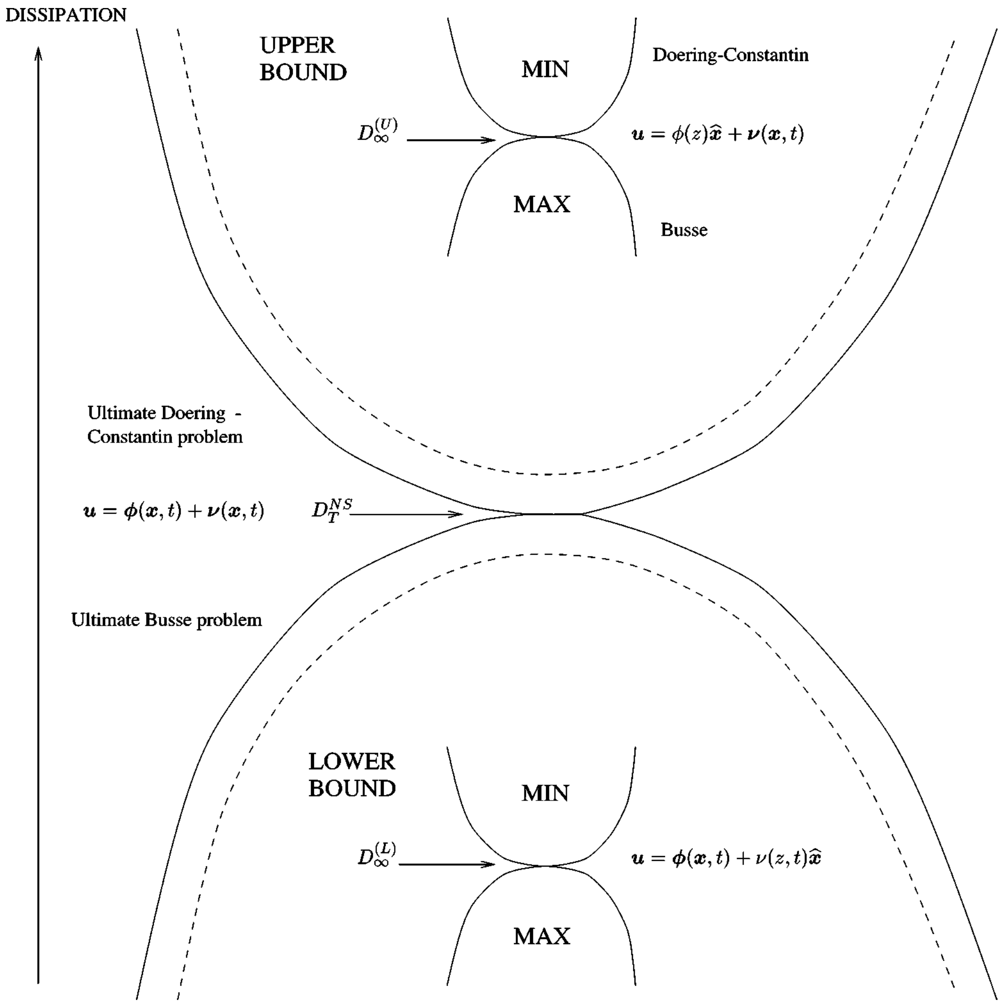

FIG. 1. This schematic summarizes the variational framework. The upper bound problem for $D_\infty^{(U)}$ is based upon the representation $\boldsymbol{u}=\phi(z)\hat{\boldsymbol{x}}+\boldsymbol{\nu}(\boldsymbol{x},t)$. The minimization/maximization procedures are the original Doering-Constantin/Busse problems. The first feasible lower problem for $D_\infty^{(L)}$ is based upon $\boldsymbol{u}=\boldsymbol{\phi}(\boldsymbol{x},t)+\nu(z,t)\hat{\boldsymbol{x}}$. The ultimate Doering-Constantin and Busse problems revolve around the fully degenerate form $\boldsymbol{u}=\boldsymbol{\phi}(\boldsymbol{x},t)+\boldsymbol{\nu}(\boldsymbol{x},t)$. The dashed lines acknowledge the possibility that these problems may not actually intersect at the Navier-Stokes solution $D_T^{\rm NS}$.

has to be expressible in terms of the physical velocity $\boldsymbol{u}=\boldsymbol{\phi}+\boldsymbol{\nu})$ is of higher order, for example, $(1/T)\int_0^T\langle|\nabla^2\boldsymbol{u}|^2\rangle dt$, the constraint (in its present form at least) is too weak to transform this into a saddle point over $(\boldsymbol{\phi},\boldsymbol{\nu})$ (leaving aside any boundary condition issue). Base functionals of lower order can lead to saddle point structure but problems arise in trying to solve the variational equation associated with $\boldsymbol{\phi}$ since it will now have too many boundary conditions. The kinetic energy functional provides an example of this. Defining

$$K_T:=\frac{1}{T}\int_0^T\langle\,\tfrac{1}{2}|\boldsymbol{u}|^2\,\rangle\,dt-\frac{a}{T}\int_0^T\left\langle\;\boldsymbol{\nu}\cdot\left\{\frac{\partial\boldsymbol{u}}{\partial t}+2\,\boldsymbol{\omega}\times\boldsymbol{u}\right.\right.$$
$$\left.\left.+\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}+\boldsymbol{\nabla}p-\nabla^2\boldsymbol{u}\right\}\;\right\rangle\,dt \tag{7.1}$$

leads to the expression

$$\frac{a}{T}[\langle\tfrac{1}{2}\boldsymbol{\nu}^2\rangle]_0^T+K_T$$
$$=\frac{1}{T}\int_0^T dt\bigg\langle(\tfrac{1}{2}|\boldsymbol{\phi}|^2+a\,\boldsymbol{\phi}\cdot\boldsymbol{\nabla}\boldsymbol{\nu}\cdot\boldsymbol{\phi})$$
$$-(a|\boldsymbol{\nabla}\boldsymbol{\nu}|^2-\tfrac{1}{2}|\boldsymbol{\nu}|^2+a\,\boldsymbol{\nu}\cdot\boldsymbol{\nabla}\boldsymbol{\phi}\cdot\boldsymbol{\nu})$$
$$+\boldsymbol{\phi}\cdot\boldsymbol{\nu}-a\,\boldsymbol{\nu}\cdot\frac{\partial\boldsymbol{\phi}}{\partial t}-2a\,\boldsymbol{\nu}\cdot\boldsymbol{\omega}\times\boldsymbol{\phi}+a\,\boldsymbol{\nu}\cdot\nabla^2\boldsymbol{\phi}\bigg\rangle. \tag{7.2}$$

Here, the spectral constraints on $\boldsymbol{\phi}$ and $\boldsymbol{\nu}$ are, respectively, that

$$\langle a|\boldsymbol{\nabla}\boldsymbol{\nu}|^2-\tfrac{1}{2}|\boldsymbol{\nu}|^2+a\,\boldsymbol{\nu}\cdot\boldsymbol{\nabla}\boldsymbol{\phi}\cdot\boldsymbol{\nu}\rangle\geqslant 0,\quad\forall\boldsymbol{\nu}\in\boldsymbol{\Gamma}, \tag{7.3}$$

$$\langle\tfrac{1}{2}|\boldsymbol{v}|^2+a\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{\nu}\cdot\boldsymbol{v}\rangle\geqslant 0,\quad\forall\boldsymbol{v}\in\boldsymbol{\Gamma}.$$

The variational equation $\delta K_T/\delta\boldsymbol{\phi}|_{\boldsymbol{\nu}}=\boldsymbol{0}$ given $\boldsymbol{\nu}$ is purely algebraic in $\boldsymbol{\phi}$, so that the solution is forced to be discontinuous in general at the boundaries. This, however, suggests that the kinetic energy plus a nonvanishing dissipation part should work as would presumably other lower order functionals added to the dissipation.

Finally, it is worth remarking that the ideas discussed here generalize in a perfectly straightforward way when further governing equations are present; for example, the induction equation in magnetohydrodynamics and the heat equation in convection. This merits separate discussion elsewhere.

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