Ising spin glass with arbitrary spin beyond the mean field theory

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We consider the Ising spin glass for the arbitrary spin S with the short-ranged interaction using the Bethe-Peierls approximation previously formulated by Serva and Paladin [Phys. Rev. E. **54**, 4637 (1996)] for the same system but limited to $S=1/2$. Results obtained by us for arbitrary S are not a simple generalization of those for $S=1/2$. In this paper we mainly concentrate our studies on the calculation of the critical temperature and the linear susceptibility in the paramagnetic phase as functions of the dimension of the system and spin number S. These dependences are illustrated by corresponding plots. [S1063-651X(99)16605-4]

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I. INTRODUCTION

The study of glasses is today one of the most relevant and actual problems in condensed matter physics. In particular, research around spin glasses in finite dimensions is very active, since it is still unclear if they share some the qualitative features of the mean-field theory of the Sherrington-Kirkpatrick (SK) model [1–3] However, there are recent studies [4–6] that indicate difficulties in extending the molecular field approximation (MFA) scenario to realistic spin glasses with short-range interaction and deciding *a priori* which properties survive and which must be appropriately modified. Recently, in an interesting paper [7], an approach beyond the MFA has been achieved for a d-dimensional Ising spin glass (SG) model ($S=1/2$) with short-range interactions on a real lattice using an extension of the Bethe-Peierls approximation (BPA) [8] to the spin glass problem via the replica trick. This approach seems to be very promising for establishing a direct contact with the results obtained by different authors for the infinite-ranged version and for controlling possible deviations for short-ranged glasses from the well acquired MFA scenario. Quite recently [9] the Parisi [3] ansatz has been investigated for the Ising spin glass with $S=1/2$, using the generalized form of the Bethe-Peierls method called by the authors ''a variational approach,'' where finite clusters of spins interact and the sample averaging is properly taken into account. The result is qualitatively similar to that obtained in the frame of the MFA with some quantitative-modifications due to short-range order interactions.

All studies mentioned above have been performed for the standard Ising model with $S=1/2$. Therefore it seems to be quite interesting to extend the methods applied to the Ising spin glass, where the number of spin is arbitrary. In spite of numerous works on the Ising model $S=1/2$, little attention has been devoted to the same system with arbitrary S. In the spin glasses solutions for the arbitrary S there is not a simple generalization such as that for the Ising model with $S=1/2$. The reason is that for higher spins, $(S_i)^n \neq S_i$ or const, which leads to parameters that are diagonal in replica indices. This considerably affects the results for the $S=1/2$ Ising SG.

We will use the Bethe-Peierls approximation (BPA) [7] with some necessary modifications. As we will see the critical temperature depends as well on the dimension of the system as it does on the spin number. In addition, linear susceptibility, even in the paramagnetic phase, is a nontrivial function of the temperature.

The paper is organized as follows. In Sec. II we briefly present the idea of the Bethe-Peierls approach for the the Ising spin glass [7] and give equations for the critical temperature and linear susceptibility in the paramagnetic phase when the spin number S is arbitrary. Section III contains plots showing the dependence of the critical temperature on the dimension and the value of the spin number. Finally in Sec. IV some conclusions are drawn.

II. BETHE-PEIERLS METHOD FOR SPIN GLASSES

Our starting point is the Hamiltonian of the Ising model with arbitrary spin:

$$H=-\frac{1}{2}\sum_{i,j} J_{i,j}S_iS_j, \tag{1}$$

where $S_i=S_i^z$ is the z component of the spin. As usual, eigenvalues of S_i run from $-S$ to S, where S is arbitrary. In Eq. (1) the summation over i,j comprises only nearest neighbors. Interaction parameters $J_{i,j}$ are variables obeying random distribution; that is, dichotomic, Gaussian, etc. For simplicity, in order not to complicate the main idea, we will assume that $J_{i,j}$ is a random variable with $J_{i,j}=\pm J$, with equal probability for a $+$ or $-$ sign.

Using the replica trick the free energy can be written as follows:

$$-\beta F=\lim_{n\to 0}\frac{1}{n}\mathrm{Tr}\,\exp\left[-\beta\sum_{\alpha} H_{\alpha}\right]_{\mathrm{av}}, \tag{2}$$

where $[\quad]_{av}$ denotes a sample averaging and H_α is the replicated Hamiltonian (1).

Working directly on the real lattice, the basic idea of the BPA for spin glasses [7] is to take into account the correct interactions inside replicated clusters (cl), consisting of a central spin S_0 and its $2d$ nearest neighbors $S_i,\{i=1,\cdots,2d\}$, and to describe the interactions of the cluster borders with the remnant of the system by means of effective couplings among replicas to be determined self-consistently. Therefore the free energy in this approximation can be written as follows:

$$-\beta F=\lim_{n\to 0}\frac{1}{n}\ln Z_n, \tag{3}$$

where

$$Z_n=K(\beta)\mathrm{Tr}\Bigg\{\Bigg[\exp\Bigg(\beta\sum_{i=1,\alpha}^{2d} J_{0,i}S_{0,\alpha}S_{i,\alpha}\Bigg)\Bigg]_{\mathrm{av}} \times\exp\Bigg(\frac{\beta^2J^2}{2}\mu\sum_{i=1}^{2d}\sum_{\alpha} S_{i,\alpha}^2\Bigg) \times\exp\Bigg(\frac{\beta^2J^2}{2}\sum_{i=1}^{2d}\sum_{\alpha\neq\alpha'}\mu_{\alpha,\alpha'}S_{i,\alpha}S_{i,\alpha'}\Bigg)\Bigg\}. \tag{4}$$

In Eq. (4) $\alpha=1,\cdots,n$ denotes the replica indices; $K(\beta)$ is a multiplicative constant that depends on the temperature but not on the lateral spins of the cluster; and $\mu_{\alpha,\alpha'}$ and $\mu\equiv\mu_{\alpha,\alpha}$, according to the Bethe-Peierls ansatz, describe the interaction between the ''external world'' and the lateral spins of the replicated cluster. The difference between Eq. (4) and the corresponding formula of Ref. [7] is that in Eq. (4) we have the additional parameter μ. This is a consequence of the fact that for an arbitrary spin $S_{k,\alpha}^2\neq 1/4$.

Effective couplings $\mu_{\alpha,\alpha'}$ and μ are calculated from the following equations:

$$\langle S_{i,\alpha}S_{i,\alpha'}\rangle=\langle S_{0,\alpha}S_{0,\alpha'}\rangle, \tag{5}$$

with $i=1,\cdots,2d$, where

$$\langle\cdots\rangle=\frac{\mathrm{Tr}\exp(-\beta H_{\mathrm{eff}})\cdots}{\mathrm{Tr}\exp(-\beta H_{\mathrm{eff}})}, \tag{6}$$

with

$$H_{\mathrm{eff}}=\frac{-1}{\beta}\ln\Bigg[\exp\Bigg(\beta\sum_{i,\alpha} J_{0,i}S_{0,\alpha}S_{i,\alpha}\Bigg)\Bigg]_{\mathrm{av}}-\frac{\beta J^2}{2}\mu\sum_{i,\alpha} S_{i,\alpha}^2 -\frac{\beta J^2}{2}\sum_{\alpha\neq\alpha'}\sum_{i=1}^{2d}\mu_{\alpha,\alpha'}S_{i,\alpha}S_{i,\alpha'}. \tag{7}$$

Above and at the critical point the spin glass parameters $\langle S_{k,\alpha}S_{k,\alpha'}\rangle=q_{\alpha,\alpha'}=0$ for $\alpha\neq\alpha'$ (but not for $\alpha=\alpha'$). It is easy to calculate that the effective couplings $\mu_{\alpha\neq\alpha'}$ obey the same conditions. If we are interested in the calculation of the critical temperature it is sufficient to formulate Eq. (5) to the lowest order in $\mu_{\alpha\neq\alpha'}$. After some straightforward algebra we get for $\alpha\neq\alpha'$ the following result:

$$\langle S_{i,\alpha}S_{i,\alpha'}\rangle\approx\beta^2J^2\mu_{\alpha,\alpha'}\sum_{j=1}^{2d}[\langle S_iS_j\rangle_0]_{\mathrm{av}}, \tag{8}$$

with $i,j=1,\cdots,2d$ and

$$\langle S_{0,\alpha}S_{0,\alpha'}\rangle\approx\beta^2J^2\mu_{\alpha,\alpha'}\sum_{i=1}^{2d}[\langle S_0S_j\rangle_0]_{\mathrm{av}}, \tag{9}$$

where $\alpha\neq\alpha'$ and

$$\langle\cdots\rangle_0=\frac{\mathrm{Tr}\exp\Big(\beta\sum_{i=1}^{2d}J_{0,i}S_0S_i+\frac{\beta^2J^2}{2}\mu\sum_{i=1}^{2d}S_i^2\Big)\cdots}{\mathrm{Tr}\exp\Big(\beta\sum_{i=1}^{2d}J_{0,i}S_0S_i+\frac{\beta^2J^2}{2}\mu\sum_{i=1}^{2d}S_i^2\Big)} =\frac{\int_{-\infty}^{\infty}\prod_{i=1}^{2d}\mathcal{D}x_i\mathrm{Tr}\exp\Big[\beta\sum_{i=1}^{2d}(J_{0,i}S_0+J\mu^{1/2}x_i)S_i\Big]\cdots}{\int_{-\infty}^{\infty}\prod_{i=1}^{2d}\mathcal{D}x_i\mathrm{Tr}\exp\Big[\beta\sum_{i=1}^{2d}(J_{0,i}S_0+J\mu^{1/2}x_i)S_i\Big]}, \tag{10}$$

with

$$\mathcal{D}x_i=\frac{1}{\sqrt{2\pi}}\exp(-x_i^2/2)dx_i. \tag{11}$$

Thus due to the translational symmetry for averaged over disorder correlation functions the equation for the critical temperature takes the form

$$[\langle S_i^2\rangle_0^2]_{\mathrm{av}}+(2d-1)[\langle S_iS_j\rangle_0^2]_{\mathrm{av}}=2d[\langle S_0S_i\rangle_0^2]_{\mathrm{av}}, \tag{12}$$

where $i\neq j$ numbers of arbitrary lateral spins of the cluster. Additionally we must to take into account the equation

$$[\langle S_i^2\rangle_0^2]_{\mathrm{av}}=[\langle S_0^2\rangle_0^2]_{\mathrm{av}}. \tag{13}$$

With Eqs. (9) and (10), after detailed calculations we can formulate the equation for the critical temperature in terms of functions F_0 to F_4, which depend on the temperature, dimension of the system, and the spin number. The sample averaged correlation function with $i\neq j$ has the form

$$[\langle S_iS_j\rangle_0^2]_{\mathrm{av}}=\frac{F_1^2}{F_0^2}, \tag{14}$$

whereas

$$[\langle S_0S_i\rangle_0^2]_{\mathrm{av}}=\frac{F_3^2}{F_0^2}, \tag{15}$$

$$[\langle S_i^2\rangle_0^2]_{\mathrm{av}}=\frac{F_2^2}{F_0^2}, \tag{16}$$

and

$$[\langle S_0^2\rangle_0^2]_{\mathrm{av}}=\frac{F_4^2}{F_0^2}. \tag{17}$$

The form of functions $F_l (l=0,\cdots,4)$ is the following:

$$F_0=\sum_{M=-S}^{S}\left\{\int_{-\infty}^{\infty}\mathcal{D}xQ_S[\beta JS(M+\mu^{1/2}x)]\right\}^{2d}, \tag{18}$$

$$F_1 = S^2 \sum_{M=-S}^{S} \left\{ \int_{-\infty}^{\infty} \mathcal{D}x B_S[\beta JS(M+\mu^{1/2}x)] Q_S \times [\beta JS(M+\mu^{1/2}x)] \right\}^2 \times \left\{ \int_{-\infty}^{\infty} \mathcal{D}x Q_S[\beta JS(M+\mu^{1/2}x)] \right\}^{2d-2}, \quad (19)$$

$$F_2 = S \sum_{M=-S}^{S} \left(\int_{-\infty}^{\infty} \mathcal{D}x \{ B_S'[\beta JS(M+\mu^{1/2}x)] + B_S^2[\beta JS(M+\mu^{1/2}x)] \} Q_S[\beta JS(M+\mu^{1/2}x)] \right) \times \left\{ \int_{-\infty}^{\infty} \mathcal{D}x Q_S[\beta JS(M+\mu^{1/2}x)] \right\}^{2d-1}, \quad (20)$$

$$F_3 = S \sum_{M=-S}^{S} \left(M \left\{ \int_{-\infty}^{\infty} \mathcal{D}x B_S[\beta JS(M+\mu^{1/2}x)] Q_S \times [\beta JS(M+\mu^{1/2}x)] \right\} \times \left\{ \int_{-\infty}^{\infty} \mathcal{D}x Q_S[\beta JS(M+\mu^{1/2}x)] \right\}^{2d-1} \right), \quad (21)$$

and

$$F_4 = \sum_{M=-S}^{S} M^2 \left\{ \int_{-\infty}^{\infty} \mathcal{D}x Q_S[\beta JS(M+\mu^{1/2}x)] \right\}^{2d}. \quad (22)$$

In Eqs. (16)–(20)

$$Q_S(y) = \frac{\sinh\left[y\left(1+\frac{1}{2S}\right)\right]}{\sinh\left(\frac{y}{2S}\right)} \quad (23)$$

where

$$B_S(y) = \left(1+\frac{1}{2S}\right)\coth\left[\left(1+\frac{1}{2S}\right)y\right] - \frac{1}{2S}\coth\left(\frac{y}{2S}\right), \quad (24)$$

is the Brillouin function, and

$$B_S'(y) = \frac{dB_S'(y)}{dy}. \quad (25)$$

Taking into account Eqs. (12) and (13) together with Eqs. (14)–(17) we can write equations for the critical temperature as follows:

$$(2d-1)F_1^2 + F_2^2 = 2dF_3^2 \quad (26)$$

and

$$F_2 = F_4. \quad (27)$$

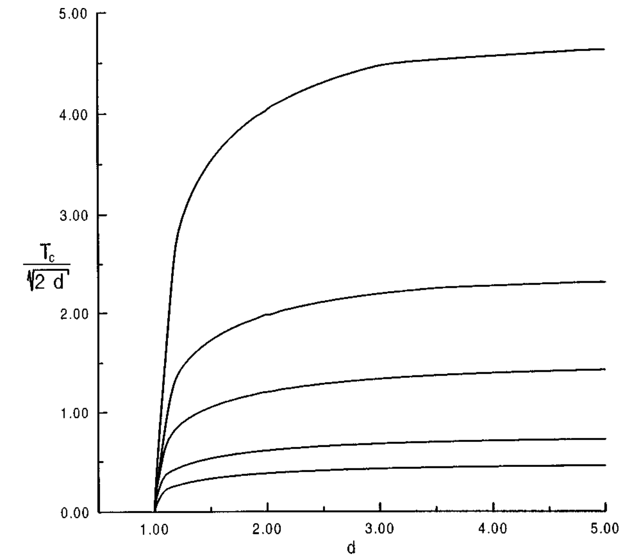

FIG. 1. Variation of the critical temperature T rescaled by the factor $(2d)^{-1/2}$ with the dimension d of the system for spin numbers $S=1/2, 1, 3/2, 2,$ and 3 . The larger the spin, the higher the corresponding line. Here $J\equiv1$.

Obviously the solution of Eqs. (26) and (27) needs numerical calculations.

As concerns the linear susceptibility in zero magnetic field, we define it as follows:

$$\chi = \frac{1}{N}\sum_{i}^{N}\left[\left(\frac{d\langle S_i\rangle_{T,h}}{dh}\right)_{|h=0}\right]_{\rm av}, \quad (28)$$

where $\langle\ \rangle_{T,h}$ denotes the thermal averaging with the Hamiltonian (1), when the term $-h\Sigma_i^N S_i$ is added. A first step is to calculate the susceptibility in the paramagnetic phase with local magnetizations $\langle S_i\rangle_T=0$, where $\langle S_i\rangle_T=\langle S_i\rangle_{T,h=0}$. In that case

$$\chi = \frac{\beta}{N}\sum_{i,j}^{N}[\langle S_iS_j\rangle_T]_{\rm av}. \quad (29)$$

It is easy to show that for $h=0$ and the symmetric probability distribution for $J_{i,j}$,

$$[\langle S_iS_j\rangle_T]_{\rm av} = \delta_{i,j}[\langle S_i^2\rangle_T]_{\rm av}. \quad (30)$$

After some calculation (see, for example, Ref. [10]) we get that

$$\chi = \beta[\langle S_k^2\rangle_0]_{\rm av}, \quad (31)$$

with $k=0,\cdots,2d$.

III. RESULTS

Our results are illustrated by plots in Figs. 1–4. In Fig. 1 the dependence of the critical temperature T_c (in units of the constant J) scaled by $\sqrt{2d}$ of the dimension of the system d for a few values of spin $S=1/2, 1, 3/2, 2,$ and 3 is given. The larger the spin, the higher the corresponding line. In Fig. 2 variations of the $T_c/\sqrt{2d}$ with spin number for $d=2$ (the lower line) and $d=3$ (the upper line) are plotted. It is seen

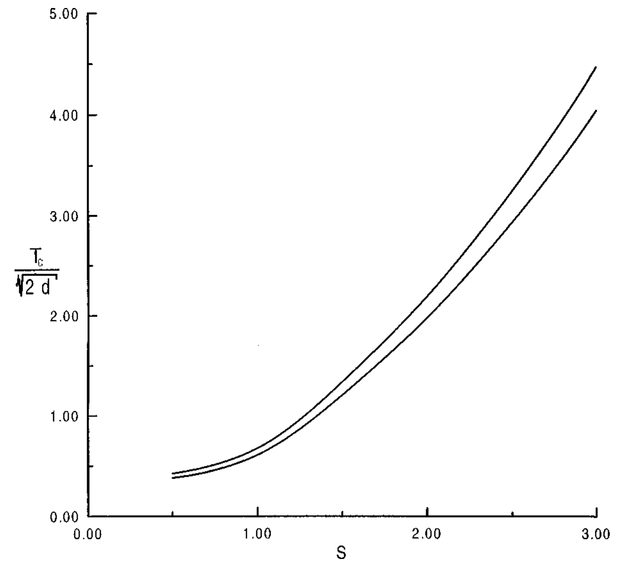

FIG. 2. Rescaling critical temperature $T_c/(2d)^{-1/2}$ versus the spin S for $d=2$ and $d=3$ marked by lower and upper lines, respectively. Here $J\equiv1$.

that the critical temperature, in general, increases with the increasing of the spin number, but this dependence cannot be represented in an explicit form and is more nontrivial compared to simple magnetic systems, such as, for example, a ferromagnet where $T_c\sim S$. The reason for the scaling of the crtitical temperature by $\sqrt{2d}$ will be explained in Sec. V. In Figs. 3 and 4 we show the dependence of the linear susceptibility in the paramagnetic phase of the temperature ($J\equiv1$) for $d=2$ and $d=3$, respectively. The values of S are $1, 3/2$, and 2. The larger the spin, the higher the corresponding line. Obviously the lines in Figs. 3 and 4 terminate at the critical temperature, since to calculate χ below T_c we must enter into theory the spin glass order parameters. At present our purpose is only to show that the linear susceptibility,

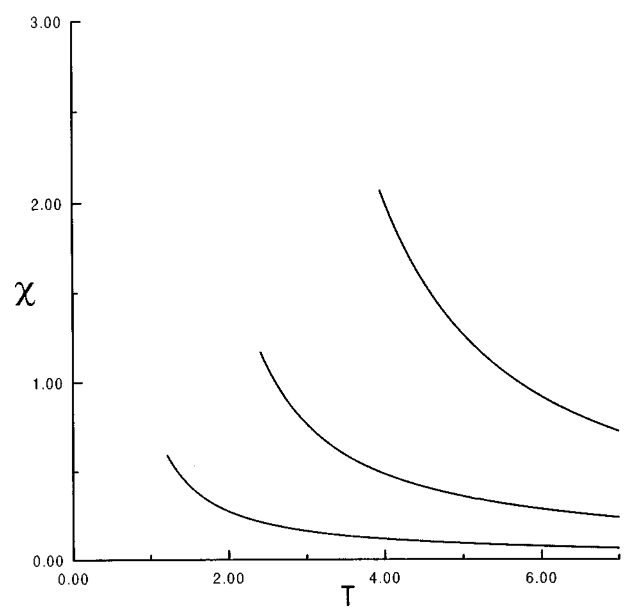

FIG. 3. Linear susceptibility χ in the paramagnetic state as a function of the temperature in units of J for $d=2$ and $S=1$, 3/2, and 2. The larger the spin, the higher the corresponding line.

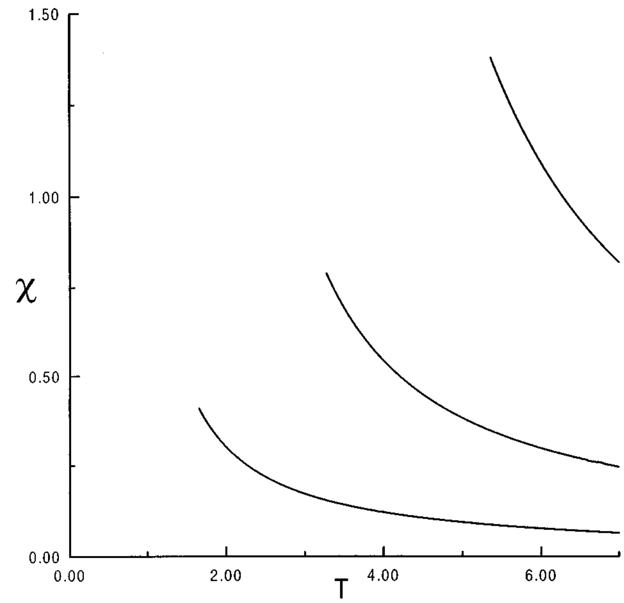

FIG. 4. Linear susceptibility χ in the paramagnetic state as a function of the temperature in units of J for $d=3$ and $S=1$, 3/2, and 2. The larger the spin, the higher the corresponding line.

even in the paramagnetic phase, is a nontrivial function of the temperature for spins higher than $S=1/2$. From plots in Figs. 3 and 4 a tendency for χ to increase with increasing of S is seen.

IV. FINAL REMARKS

Studies of the spin glasses with short-ranged interaction are undoubtedly a difficult problem among the theories of amorphous systems, since the complicated nature of the randomness interplays spatial correlations of spins. At present there is practically no developed systematic method to investigate such systems, as in the case of the long-ranged (more strictly, infinite-ranged) Sherrington-Kirkpatrick type models where MFA is valid [1,2]. It is expected that the BPA will be able to give a more accurate estimation of the critical temperature for the spin glass systems with short-range interaction than the MFA. As concerns our problem, a natural question arises about what will result when the dimension of the system is infinite. It can be easily shown that if $d\to\infty$, one obtains the Sherrington-Kirkpatrick theory for the Ising spin glass with an arbitrary spin. After rescaling $J_{i,j}\to J_{i,j}/\sqrt{2d}$, $J\to J/\sqrt{2d}$ and changing $\mu_{\alpha,\alpha'}=2dq_{\alpha\neq\alpha'}$, $\mu=2dp$ with $p=q_{\alpha,\alpha}$, proceeding in a line similar to that in Ref. [7], one obtains

$$q_{\alpha\neq\alpha'}=\langle S_\alpha S_{\alpha'}\rangle \tag{32}$$

and

$$p=\langle S_\alpha^2\rangle, \tag{33}$$

where

$$\langle \cdots \rangle = \frac{\mathrm{Tr}\exp\left(\frac{\beta^2 J^2}{2}\sum_{\alpha\neq\alpha'} q_{\alpha,\alpha'}S_\alpha S_{\alpha'} + p\sum_\alpha \S^{2}_{\alpha}\right)\cdots}{\mathrm{Tr}\exp\left(\frac{\beta^2 J^2}{2}\sum_{\alpha\neq\alpha'} q_{\alpha,\alpha'}S_\alpha S_{\alpha'} + p\sum_\alpha S^2_\alpha\right)} + \mathcal{O}(d^{-1/2}), \tag{34}$$

where S_α is the z component of the spin operator referred to an arbitrary site. Hence in Figs. 1 and 2, in order to obtain reasonable results for higher spins, we scaled the critical temperature, expressed in units J, by $\sqrt{2d}$. We are aware that our consideration is a first step toward recognizing some properties of the Ising SG with an arbitrary spin. It would be interesting to obtain the properties of the system in the SG phase, at least in a replica symmetric theory. This is a complicated task even for $S=1/2$; therefore, further work will be necessary to elucidate this problem.

ACKNOWLEDGMENT

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