# Stochastic resonance in extended bistable systems: The role of potential symmetry

S. Bouzat* and H. S. Wio†
*Comisión Nacional de Energía Atómica, Centro Atómico Bariloche and Instituto Balseiro (CNEA and UNC), 8400-San Carlos de Bariloche, Argentina*
(Received 13 November 1998)

We study the role of potential symmetry in a three-field reaction-diffusion system presenting bistability by means of a two-state theory for stochastic resonance in general asymmetric systems. By analyzing the influence of different parameters in the optimization of the signal-to-noise ratio, we observe that this magnitude always increases with the symmetry of the system's potential, indicating that it is this feature which governs the optimization of the system's response to periodic signals. [S1063-651X(99)12505-4]

PACS number(s): 05.40.−a, 05.10.Gg, 82.40.Ck, 82.20.Fd

## I. INTRODUCTION

Since its original proposal as a mechanism accounting for the periodicity in Earth's ice ages [1], the phenomenon of *stochastic resonance* has been extensively studied, from both the theoretical and experimental points of view [2,3]. Stochastic resonance (SR) is the name coined for the rather counterintuitive fact that the response of a *nonlinear* system to a periodic signal may be *enhanced* through the addition of an optimal amount of noise. One of the key parameters here is the *signal-to-noise ratio* (SNR) at the output.

A vast majority of studies on SR have been done analyzing a paradigmatic system: a bistable one-dimensional double-well system. Among such kinds of models there is one that can be singled out: *the two-state model* [1,4]. Such a model has proven to be extremely useful in the understanding of the SR phenomenon, offering also a simple framework able to provide analytical results. Most of the studies have been carried out in the symmetric potential case. However, even in the earliest account of the two-state model [1] the possibility of potential asymmetry was introduced, with the conclusion that the symmetric case would be the optimal one. Other authors have also analyzed different aspects of this case (see references in [3]), for instance considering equal curvatures of the potential wells [5], or from the point of view of residence times [6]. Also higher order resonant behavior and dc signal detection in nonlinear asymmetrical devices using a perturbative approach have been studied [7]. However, all those studies correspond to the analysis of zero-dimensional or uncoupled systems.

The occurrence of SR in coupled and extended systems has been the focus of several recent studies (see the citations in Ref. [3]). Some of the different aspects that have been analyzed are the effect of global coupling in dynamical and neuron models [8], enhancement of the SR phenomenon due to coupling [9], and spatiotemporal SR-like phenomena [10]. The studies in continuous extended systems are more closely related to the present work [11–15].

In this contribution we analyze the role of the symmetry in the SR phenomenon in extended systems. We start with a general analysis of SR in asymmetric situations, extending the two-state approach [1,4]. In this way we derive general expressions for the power spectral density and for the SNR for a general two-state system. After discussing a simple zero-dimensional example, the results are exploited to analyze the dependence of the system's response on the noise intensity and on the degree of symmetry in a spatially extended reaction-diffusion system. In the last section we conclude with some final remarks on the influence of the different parameters and the central role played by the potential symmetry in the SR of extended systems.

*Electronic address: bouzat@cab.cnea.gov.ar
†Electronic address: wio@cab.cnea.gov.ar

## II. THEORETICAL FRAMEWORK: TWO-STATE MODEL FOR STOCHASTIC RESONANCE

We consider a random system described by a discrete dynamical variable $x$ adopting two possible values: $c_1$ and $c_2$, with probabilities $n_{1,2}(t)$, respectively. Such probabilities satisfy the condition $n_1(t)+n_2(t)=1$. The equation governing the evolution of $n_1(t)$ [with a similar one for $n_2(t)=1-n_1(t)$] is

$$\frac{dn_1}{dt}=-\frac{dn_2}{dt}=W_2(t)n_2(t)-W_1(t)n_1(t)$$
$$=W_2(t)-[W_2(t)+W_1(t)]n_1, \tag{1}$$

where the $W_{1,2}(t)$ are the transition rates *out of* the $x=c_{1,2}$ states.

If the system is subject (through one of its parameters) to a time-dependent signal of the form $A\cos(\omega_s t)$, up to first order in the amplitude (assumed to be small) the transition rates may be expanded as

$$W_1(t)=\mu_1-\alpha_1 A\cos(\omega_s t),$$
$$W_2(t)=\mu_2+\alpha_2 A\cos(\omega_s t), \tag{2}$$

where the constants $\mu_{1,2}$ and $\alpha_{1,2}$ depend on the detailed structure of the system under study. Here we remark that the $\mu_i$'s, which are the (time-independent) values of the $W_i$'s without signal, are in general different from each other as a consequence of the different stability of the two states, and the same happens to the $\alpha_i$'s. These considerations are the main difference between our treatment and the one in Ref. [4] where both $\mu_1=\mu_2$ and $\alpha_1=\alpha_2$ were assumed. Using Eq. (2) we can integrate Eq. (1) with the initial condition

 

$x(t_0)=x_0$ and obtain the conditional probability $n_1(t|x_0,t_0)$. This result allows us to calculate the autocorrelation function, the power spectrum, and finally the SNR. The details of the calculation are shown in the Appendix. When the symmetrical case is considered all the results reduce to those in [4]. For the SNR, up to the relevant (second) order in the signal amplitude $A$, we find the result given by Eq. (A10) in the Appendix.

The independence of the SNR on the signal frequency for small signal amplitude was well known for symmetric systems [4] and here is found to be valid also when the symmetry is broken. Later on, and in order to characterize the SNR independently of both the signal frequency and amplitude we will work with $R=\tilde{R}/A^2$ instead of $\tilde{R}$; however, the results will only be valid for small enough amplitudes. Hence, the form for SNR we will use is

$$R=\frac{\pi(\alpha_2\mu_1+\alpha_1\mu_2)^2}{4\mu_1\mu_2(\mu_1+\mu_2)}. \tag{3}$$

For the sake of completeness and in order to gain insight into the role of symmetry in the SR of a bistable system, we briefly analyze here a simple one-dimensional system using the theory described above. However, similar (and more complete) analyses have been performed in Ref. [7]. In what follows we will work with nondimensional quantities.

We consider the following stochastic system:

$$\dot{u}(t)=-(u^2-1)(u+a)+S(t)+\sqrt{2}\,\xi(t), \tag{4}$$

where $\xi(t)$ is a Gaussian white noise of zero mean and correlation $\langle\xi(t)\xi(t')\rangle=\eta\delta(t-t')$. The corresponding double-well potential,

$$V(u)=\frac{u^4}{4}+\frac{au^3}{3}-\frac{u^2}{2}-[a+S(t)]u, \tag{5}$$

is symmetric for $a=0$ and $S(t)$. Up to first order in $S(t)$, $V(u)$ has minima ($u_1$ and $u_2$) and a maximum ($u_m$) located at

$$u_1=1+\frac{S(t)}{2(1+a)},\quad u_2=-1+\frac{S(t)}{2(1-a)},$$
$$u_m=-a-\frac{S(t)}{1-a^2}. \tag{6}$$

In order to apply the two-state theory described above we set $S(t)=A\cos(\omega_s t)$ and assume that $(\omega_s)^{-1}$ is large compared to the characteristic relaxation times in both wells. The transition rates between the states are given by the Kramers-like formulas

$$W_{u_1\to u_2}\equiv W_1=\frac{\sqrt{|V''(u_m)|V''(u_1)}}{2\pi}\exp\left[-\frac{V(u_m)-V(u_1)}{\eta}\right],$$
$$W_{u_2\to u_1}\equiv W_2=\frac{\sqrt{|V''(u_m)|V''(u_2)}}{2\pi}\exp\left[-\frac{V(u_m)-V(u_2)}{\eta}\right], \tag{7}$$

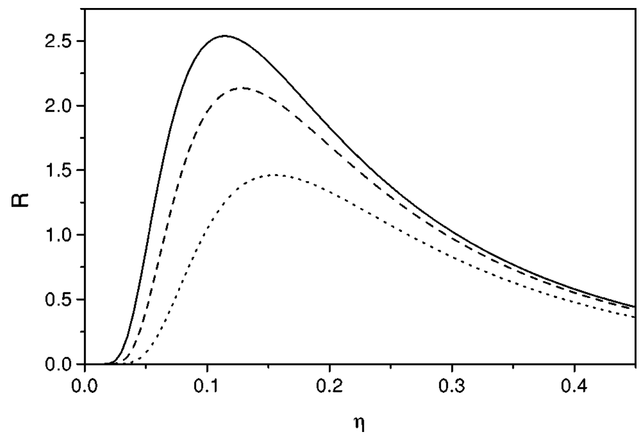


FIG. 1. SNR as a function of the noise intensity for different values of the parameters: the solid line corresponds to modulating around the symmetric situation ($a=0$), the dashed and dotted lines correspond, respectively, to the asymmetric cases with $a=0.1$ and $a=0.2$.

where $V''$ is the second derivative of $V$ with respect to $u$. The parameters $\mu_i$ and $\alpha_i$ result in functions of $a$ and $\eta$ that can be analytically calculated as

$$\mu_1=W_1|_{S(t)=0},\quad \alpha_1=-\left.\frac{dW_1}{dS(t)}\right|_{S(t)=0},$$
$$\mu_2=W_2|_{S(t)=0},\quad \alpha_2=\left.\frac{dW_2}{dS(t)}\right|_{S(t)=0}. \tag{8}$$

From Eqs. (3) and (8) we can compute the SNR ($R$) as a function of $a$ and the noise intensity $\eta$. The parameter $a$ characterizes the symmetry as follows: setting $a=0$ corresponds to modulating around a symmetric situation in which both states are equally stable, while $a\neq 0$ corresponds to asymmetric situations where the most stable state is $u_1$ for $a>0$ and $u_2$ for $a<0$. However, as the system is invariant under the simultaneous transformations $a\to -a, u\to -u$, and $S(t)\to -S(t)$, the results for $R$ evaluated at $a$ are the same as those evaluated at $-a$. Hence, we will only consider the case $a>0$.

In Fig. 1 we depict the results of $R(\eta)$ for different values of $a$. Note that each curve shows an optimum noise intensity where the SNR has a maximum; this is the typical characteristic of the SR phenomenon. Furthermore, it can be appreciated that the value of the maximum of $R$ increases with the symmetry of the system (i.e., with the proximity of $a$ to zero). Actually, from the complex (not shown) analytical expression for $R$ as a function of $a$ and $\eta$, it can be seen that for a fixed value of $\eta$, $R$ is maximized by setting $a=0$. Hence the symmetric situation is the most favorable one for the SR phenomenon. In Fig. 2 we show the value of the maximum of $R$ (regarding $\eta$) plotted as a function of $a$, where the optimization occurring for the symmetric case ($a=0$) is apparent.

In the next section we will analyze the SR phenomenon in a more complicated situation corresponding to a three-field reaction-diffusion system in one spatial dimension. In that model we will find that symmetry plays the same role of increasing the SNR as in the one-dimensional system. Fur-

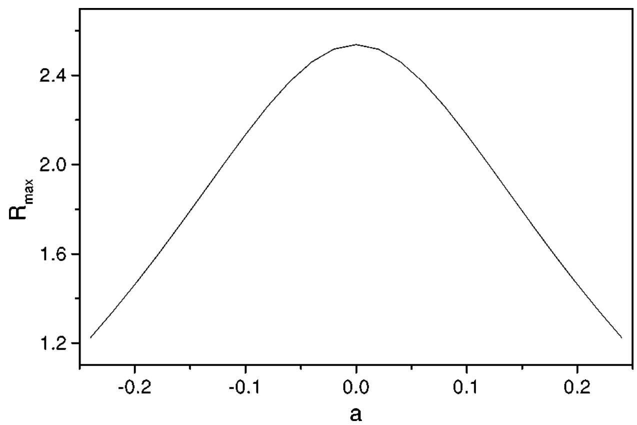


FIG. 2. Maximum of $R$ ($R_{max}$) as a function of $a$. The maximum of $R_{max}$ occurs for $a=0$, which corresponds to modulating around a symmetric situation.

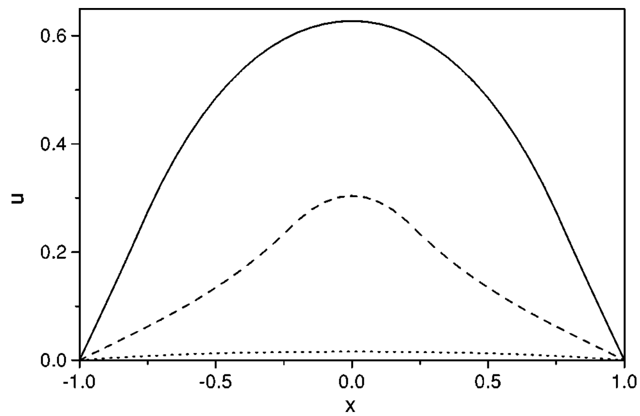


FIG. 3. $u$ fields of the stationary patterns. The solid line corresponds to the stable pattern $u_1$, the dashed line to the unstable pattern $u_m$, and the dotted line to the stable pattern $u_2$. Results for $a=0.25, D=0.3$, and $S(t)=0.025$.

thermore, we will show that symmetry is the key feature in improving the SNR and that its relevance goes, in some sense, beyond that of any other relevant characteristic, such as coupling, for example.

## III. STOCHASTIC RESONANCE IN EXTENDED SYSTEMS

Here, and in order to investigate general trends of the SR phenomenon in extended systems, we will consider a three-field reaction-diffusion system of the activator-inhibitor type in one spatial dimension. The relevance of activator-inhibitor models for the description of pattern formation phenomena in chemical and biological systems is very well known [16]. Recently, and in order to describe the experimental results obtained in chemical systems (Belouzov-Zhabotinsky or CIMA reactions), different forms of three-field models (typically reduced to effective two-field systems) have been studied [17].

We consider a related model given by the equations

$$\frac{\partial u(x,t)}{\partial t}=D\frac{\partial^2 u(x,t)}{\partial x^2}+f(u(x,t))-v(x,t)-w(x,t),$$

$$\frac{\partial v(x,t)}{\partial t}=\beta u(x,t)-\gamma v(x,t), \tag{9}$$

$$\epsilon\frac{\partial w(x,t)}{\partial t}=\nu\frac{\partial^2 w(x,t)}{\partial x^2}+\beta' u(x,t)-\gamma' w(x,t),$$

with $f(u)=-u+\theta(u-a)+S$, where $\theta(u)$ is the unit step function [ $\theta(u)=1$ for $u>0$ and $\theta(u)=0$ for $u<0$] while $a$ and $S$ are two additional parameters.

In analogy with the systems studied in [17], the equation for the first inhibitor ($v$) has no diffusive term. In addition, we will consider that the second inhibitor ($w$) is a fast one fixing $\epsilon=0$. Then for the now *temporally slaved* inhibitor $w$, we have

$$w(x,t)=\beta'\int dx' G(x,x')u(x',t), \tag{10}$$

$G(x,x')$ being the Green function of the third of Eqs. (9) in the indicated limit [18], which depends on the boundary conditions considered. In this limit the system of Eqs. (9) can be reduced to an effective two-component system ($u$ and $v$) with a *nonlocal interaction term*, by inserting Eq. (10) into the first of Eqs. (9). In this way we have obtained a system where the role played by each inhibitor is clearly different: one acts only locally while the other has a nonlocal character.

We will fix Dirichlet boundary conditions on the three fields in $[-L,L]$ $[u(\pm L)=v(\pm L)=w(\pm L)=0]$ for which we have

$$G(x,x')=\begin{cases}\dfrac{\sinh[k(L-x')]\sinh[k(L+x)]}{\nu k\sinh[2kL]}, & x<x'\\[2ex] \dfrac{\sinh[k(L-x)]\sinh[k(L+x')]}{\nu k\sinh[2kL]}, & x>x'\end{cases} \tag{11}$$

where $k=\sqrt{\gamma'/\nu}$.

We will focus our analysis on a region of parameters where the system has two stationary stable patterns [stationary linearly stable solutions of Eqs. (9) for $u,v$, and $w$] and one stationary unstable pattern [stationary linearly unstable solution of Eqs. (9)]. The piecewise linear choice for the nonlinearity $f(u)$ allows us to calculate these patterns as linear combinations of exponentials plus constants [18]. In Fig. 3 we show the $u$ fields for the three patterns for a particular choice of the parameters. We call $U_1$ the large stable pattern which has a central activated region ($u>a$), $U_2$ the small stable pattern which reduces to the homogeneous null solution when $S$ is set equal to zero, and $U_m$ the unstable pattern. A more complete study of the pattern formation of this system, including the analysis of different regions of parameters and discussions on the different role played by each inhibitor, can be found in Ref. [18].

In the region of only two stable patterns we are considering, the deterministic dynamics given by Eqs. (9) drives the system toward one of the patterns (selected depending on the initial condition) which is reached asymptotically. If small fluctuations are present in the system the fields fluctuate around one of the stable patterns and transitions between the two patterns become possible.

In order to analyze the phenomenon of stochastic resonance between the stable patterns $U_1$ and $U_2$ we will now consider the presence of fluctuations in the system and also we will introduce a periodic signal. Fluctuations will be introduced in the effective two-equation system equivalent to Eqs. (9) as additive Gaussian white noise sources of zero mean by writing

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t}&=D\frac{\partial^2 u(x,t)}{\partial x^2}+f(u(x,t))-v(x,t)\\&\quad-\beta'\int dx' G(x,x')u(x',t)+g_1^u\xi_1(x,t)\\&\quad+g_2^u\xi_2(x,t),\\\frac{\partial v(x,t)}{\partial t}&=\beta u(x,t)-\gamma v(x,t)+g_1^v\xi_1(x,t)+g_2^v\xi_2(x,t),\end{aligned}\tag{12}$$

with the $\xi_i$'s satisfying

$$\langle \xi_i(x,t)\xi_j(x',t')\rangle=\eta\,\delta_{ij}\delta(t-t')\,\delta(x-x').\tag{13}$$

Note that the $g_i^\mu$ are constants that couple the noises to the system while the intensity of the fluctuations is determined by the parameter $\eta$.

It has been shown in Ref. [18] that, in a certain region of parameters (that includes the bistable one on which our analysis is focused), the nonequilibrium potential [19] for the system of Eqs. (12) for Neumann or Dirichlet boundary conditions on the three fields in $(-L<x<L)$ is

$$\Phi[u,v]=\int dx\left[\frac{D}{Q_u}(\nabla u)^2+V(u,v)+\frac{\beta'}{Q_u}\int dx' G(x,x')u(x)u(x')\right],\tag{14}$$

where

$$V(u,v)=-\frac{2}{Q_u}\int^u f(u')du'+\frac{2Q_{uv}\beta}{Q_uQ_v}u^2+\frac{\gamma}{Q_v}v^2-2\frac{\beta}{Q_v}uv,\tag{15}$$

$Q_u=(g_1^u)^2+(g_2^u)^2$, $Q_v=(g_1^v)^2+(g_2^v)^2$, and $Q_{uv}=g_1^ug_1^v+g_2^ug_2^v$. The nonequilibrium potential has stationary points (vanishing functional derivatives) at the stationary patterns, minima at the stable patterns, and maxima or generalized saddle points at the unstable patterns. It also determines the stationary solution $\mathcal{P}$ of the Fokker-Planck equation associated to Eq. (12) (in the sense of Ito), in the limit of small $\eta$, that is given by

$$\mathcal{P}=Z\exp\left(-\frac{\Phi}{\eta}\right),\tag{16}$$

where $Z$ is a normalization constant [18]. Hence, the nonequilibrium potential characterizes the most important stationary properties of the system.

It is worth mentioning that the nonequilibrium potential given in Eq. (14) is valid for the system in Eq. (12) in arbitrary spatial dimension, for an arbitrary nonlinear function $f(u)$, and with the parameter region of validity being independent of the choice of $f(u)$ [18]. The consideration of only one spatial dimension and the particular election of $f(u)$ are in order to simplify the calculations, particularly regarding pattern formation.

The signal will be introduced as a (slow) modulation through the parameter $S$ by setting $S=S(t)=A\cos(\omega_s t)$. With this modulation the system becomes nonstationary but we make an adiabatic assumption similar to the one we adopted in the preceding section (considering small signal frequencies) that makes the nonequilibrium potential valid at each time for the corresponding value of the signal.

We now analyze the SR phenomenon in our spatially extended system using the theory presented in Sec. II. To proceed with such an analysis we identify the two stable patterns ($U_1$ and $U_2$) with the states of the two-state theory. Hence, the discrete variable $x$ will adopt values $c_1$ and $c_2$ according to the system being in the states $U_1$ and $U_2$, respectively, yielding the result for the SNR in Eq. (3). The same result can be obtained considering the space-time correlation function of the field $u(x,t)$ that, similarly to what was discussed in [13,15], shall give a factorized expression with a temporal factor that is coincident with the result for the autocorrelation of $x(t)$ obtained in the Appendix [Eq. (A3)], leading to Eq. (3) for the SNR. The other factor, which includes the space dependence of the $U_1$ pattern, is not relevant for the present study. However, the changes induced in the patterns by the variation of some model parameter will be reflected in changes in the values of $\mu_i$ and $\alpha_i$ and, accordingly, will affect the results for the SNR.

In what follows we fix $L=1,\beta=\beta'=1,\gamma=10.026,\gamma'=\nu=10,g_1^u=1,g_2^u=0,g_1^v=0.05$, and $g_2^v=0.01$, and leave $D$, $a$, and $\eta$ (the noise intensity) as free parameters. Note that with the chosen values for the $g_i^\mu$'s, the only relevant noise term in the system [Eq. (12)] is $g_1^u\xi_1(t)$ in the equation for $u$ that appears added to the signal (hence it can be considered as coming together with the signal). The parameters $g_1^v$ and $g_2^v$ are set different from zero to keep the system inside the parameter region where $\Phi[u,v]$ as defined in Eq. (14) is valid as a nonequilibrium potential [18].

In order to evaluate the transition rates between both states we discretize the space and the fields as

$$x\to x_i,\quad (u(x),v(x))\to(\tilde{u}_1,\tilde{u}_2,\ldots,\tilde{u}_N,\tilde{v}_1,\ldots,\tilde{v}_N)\tag{17}$$

and use the Kramers-like formula [20]

$$W_{U_i\to U_j}\equiv W_i=\frac{\lambda_+}{2\pi}\sqrt{\frac{\Phi_i''}{|\Phi_m''|}}\exp\left[-\frac{(\Phi_m-\Phi_i)}{\eta}\right],\tag{18}$$

where $\lambda_+$ is the unstable eigenvalue of the deterministic flux at the unstable state $U_m$, $\Phi_i''$ and $\Phi_m''$ indicate the determinants of the matrix of second order derivatives of the nonequilibrium potential with respect to the discretized fields in the states $U_i$ and $U_m$, respectively, and $\Phi_i$ and $\Phi_m$ are the values of the nonequilibrium potential evaluated at the sta-

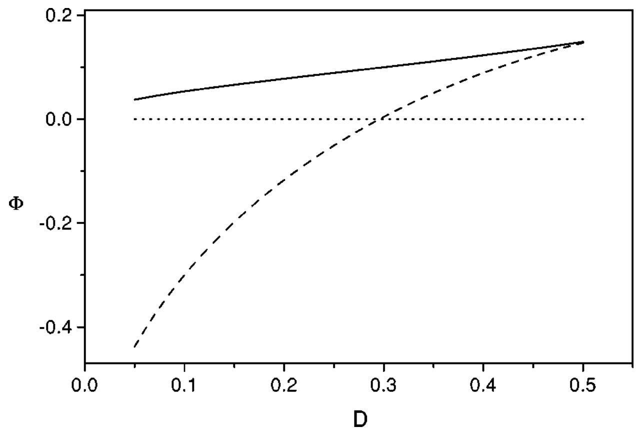


FIG. 4. Nonequilibrium potential evaluated at the stationary patterns as a function of the activator diffusion $D$ for $a=0.25$ and $S(t)=0$. The solid line corresponds to the unstable pattern $u_m$, the dashed line to the stable pattern $u_1$, and the dotted line to the stable pattern $u_2$.

tionary states $U_i$ and $U_m$, $i=1,2$. Finally, in order to compute the SNR as indicated in Sec. II, we calculate the parameters $\mu_i$ and $\alpha_i$ numerically as

$$\mu_1 = W_1\big|_{S(t)=0}, \quad \alpha_1 = -\left.\frac{dW_1}{dS(t)}\right|_{S(t)=0},$$
$$\mu_2 = W_2\big|_{S(t)=0}, \quad \alpha_2 = \left.\frac{dW_2}{dS(t)}\right|_{S(t)=0}. \tag{19}$$

It is worth noting that the dependence of the Kramers rates in Eq. (18) on the signal $S(t)$ comes in two ways: first, through the explicit dependence of the nonequilibrium potential on $S(t)$ that only affects the exponential factor, since the second derivatives of the potential do not depend explicitly on $S(t)$. Second, there is an implicit dependence that affects not only $\Phi_i$ and $\Phi_m$ but also $\Phi_i''$, $\Phi_m''$, and $\lambda_+$, and comes through the dependence of the stationary patterns on $S(t)$. In obtaining $\alpha_1$ and $\alpha_2$ we neglected the implicit dependence of $\Phi_i''$, $\Phi_m''$, and $\lambda_+$ on $S(t)$, but kept exactly the dependence of the exponential factor.

In Fig. 4 we show the nonequilibrium potential evaluated on the different patterns as a function of $D$, the diffusion constant of the activator, for $S(t)=0$ and $a=0.25$. It can be appreciated that the symmetric situation (where the two stable states have equal values of the potential) occurs near $D=0.3$ (actually at $D=D_s=0.2956$). For $D<D_s$ the pattern $U_1$ is more stable than $U_2$ while for $D>D_s$ we have the opposite situation. Near $D=0.5$ the stable pattern $U_1$ and the unstable pattern $U_m$ coalesce and they disappear for larger values of $D$ [18].

In Fig. 5 we show the results for the SNR ($R$) as a function of the noise intensity for different values of $D$ and $a$. We see that while keeping constant $a=0.25$ [Fig. 5(a)], the largest values of $R$ are those for $D=D_s$, which is the symmetric situation. Also, if we fix $D=D_s$ [Fig. 5(b)], any departure of $a$ from the value 0.25 (that is, any departure from the symmetric situation) reduces the values of $R$. Hence, the symmetric situation is found to be the most favorable one concerning the improvement of SNR. Note that the maximum of

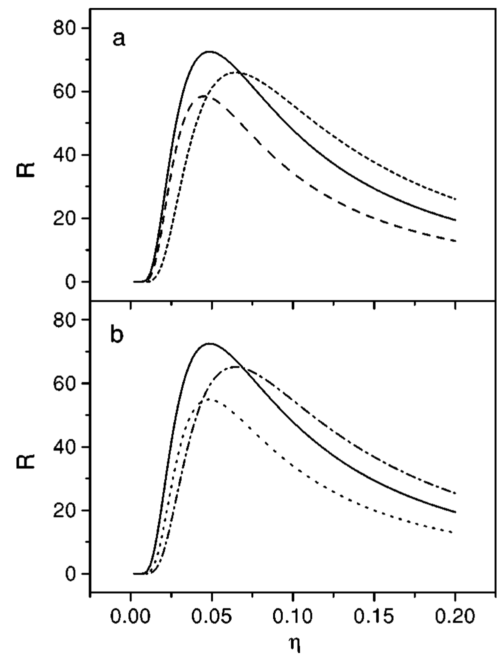


FIG. 5. (a) SNR as a function of the noise intensity for $a=0.25$ and different values of $D$. The solid line corresponds to the symmetric situation $D=D_s$, the long-dashed line to $D=0.35$, and the short-dashed line to $D=0.25$. (b) SNR as a function of the noise intensity for $D=D_s$ and different values of $a$. The solid line corresponds to the symmetric situation $a=0.25$, the dotted line corresponds to $a=0.27$, and the dot-dashed line corresponds to $a=0.23$.

the $R$ vs $\eta$ curve (for fixed values of $a$ and $D$), which we will call $R_{\max}$, increases with symmetry and reaches its largest value for the symmetric situation. In Fig. 6 we show $R_{\max}$ plotted as a function of $D$ for $a=0.25$, where it is apparent that the optimum value of diffusion is $D=D_s$, corresponding, as indicated, to the symmetric case.

A fact that arises from these results is that, while keeping all the other parameters of the system fixed, there exists an optimal value of diffusion (coupling of the distributed system) that maximizes SNR. The interesting aspect is that such an optimal value is the one that makes the potential symmetric.

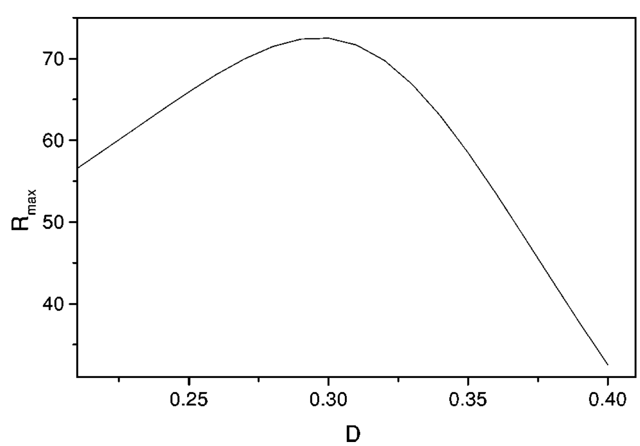


FIG. 6. Maximum of SNR ($R_{\max}$) as a function of the activator diffusion $D$ for $a=0.25$. The maximum of $R_{\max}$ occurs for the symmetric situation $D=D_s$.

It is worth mentioning that these results do not contradict but complete those in [14] where enhancement due to coupling was found, since in that work only symmetric situations were analyzed. Roughly speaking, the main result in [14] can be summarized saying that, given two different symmetric situations (each one necessarily having different values of $D$ and $a$), the one with the higher value of $D$ produces higher values of SNR. However, we must keep in mind that for a too large value of $D$, some of the approximations involved in the calculations may break down [14].

Here we have studied the dependence of the SNR on the parameters $D$ and $a$, however, similar results, leading to identical conclusions, are obtained when the dependence of SNR on other parameters of the system, such as $\beta, \gamma$, or $\nu$, is considered. $R_{\mathrm{max}}$ is always enhanced when the parameters are varied in the direction of increasing the potential symmetry and diminish when the asymmetry grows.

It is worth pointing out here that for too large asymmetries, some of our approximations will break down. For example, consider an extremely asymmetric situation where the barrier for, say, the transition from state $U_1$ to $U_2$, is much larger than the barrier for the opposite transition. In such a case, the values of the noise intensity leading to reasonable jumping rates from $U_1$ to $U_2$ will be far beyond the validity of the Kramers-like approximation for the inverse transition.

## IV. FINAL REMARKS

In this paper we have analyzed the role of potential symmetry in the SR for a bistable system with spatial extension, for the case of small signal amplitudes. We started by presenting an extension of the two-state theory of stochastic resonance [1,4] in order to include situations with different stabilities.

As a first step in the analysis of the role of the potential symmetry, we have used the extended theory to analyze SR in a simple example: a (space-independent) double-well system. For this case we have found that the symmetric situation is the optimal one in order to improve the SNR. It is worth mentioning that we have obtained essentially the same results in other different bistable systems: one corresponding to a cusp-shaped bistable potential [in which the Kramers approach for the transition rates is different from the one in Eq. (7) [20]], and also for an activator-inhibitor bistable (two-variable) system. Furthermore, this behavior (improvement of SNR with symmetry) seems to be independent of the way in which the signal is introduced in the system since similar results have been found when the modulation was introduced in other system parameters [for example, the threshold parameter $a$ of Eq. (4)].

Besides the analysis of the influence of symmetry on stochastic resonance, it is important to remark that the mere consideration of asymmetric situations has its own relevance. This is because such bistable asymmetric models provide, for example, the appropriate framework for describing SR in voltage-dependent ion channels, as proposed in [22]. In those systems, the conducting state is associated to a higher-energy well than the nonconducting one.

We also remark here that our analysis and results have some differences with those found in [23]. First, our result for the SNR shows that the well known independence of the SNR on the signal frequency for a small signal amplitude for symmetric systems [4] is also found to be valid when the symmetry is broken, at variance with those of [23]. Secondly, if in the system described by Eq. (12) we adopt $\beta' = 0$ and consider spatial independence ($D = \nu = 0$) it reduces to the same FitzHugh-Nagumo model discussed in Ref. [23]. For the bistable region of this (nongradient) resulting system, the nonequilibrium potential [19] is known for a general way of introducing fluctuations, and is given by $V(u,v)$, as defined in Eq. (15) [18]. Hence, the claim of *SR in a nonpotential system* made for the system analyzed in Ref. [23] is incorrect. However, it is worth remarking here that there are systems without a potential that show SR [21].

The main goal of our work was the analysis of SR in an extended three-field reaction-diffusion system. We have focused in the parameter region where the system is bistable, that is, where there are only two stationary stable patterns. In order to use the two-state theory of SR, we have evaluated the transition rates between the two stable patterns using a Kramers-like approach exploiting the nonequilibrium potential presented in [18]. The analysis of the results for the SNR in this extended system shows the central role played by the symmetry in improving the SNR. We studied the behavior of $R_{\mathrm{max}}$, that is, the maximum of the SNR vs $\eta$ curve, as the different model parameters are (not simultaneously) varied, finding that $R_{\mathrm{max}}$ always increases with the symmetry of the potential. This fact leads us to our main result: the optimal values of the different model parameters (for instance, diffusivity or threshold), regarding the maximization of $R_{\mathrm{max}}$, correspond to those making the potential more symmetric in each situation.

As indicated above, the present analysis complements the results in Ref. [14] where only symmetric situations were considered (although in a one-field system). The study of the influence of the potential symmetry in other forms of characterizing SR (for instance, those based on information theoretical approaches), as well as the analysis of SR in extended systems with aperiodic signals, are under way.

## ACKNOWLEDGMENTS

The authors thank R. Graham, M. Kuperman, R. Toral, and G. Nicolis for useful discussions, and V. Grunfeld for a revision of the manuscript. Partial support from CONICET, through Grant No. PIP Nro.4593/96, ANPCyT, through Grant No. 03-00000-00988, and CEB, Bariloche, is also acknowledged.

## APPENDIX: CALCULATION OF THE SIGNAL-TO-NOISE RATIO

Here we follow the procedure of Ref. [4] in order to compute the SNR, generalizing that treatment to include the asymmetric case ($\mu_1 \neq \mu_2$ and $\alpha_1 \neq \alpha_2$). Once Eq. (1) is integrated we can calculate the correlation function $\langle x(t+\tau)x(t)|x_0,t_0\rangle$ as

$$\langle x(t+\tau)x(t)|x_0,t_0\rangle$$
$$=c_1^2 n_1(t+\tau|c_1,t)n_1(t|x_0,t_0)+c_1c_2 n_1(t+\tau|c_2,t)n_2(t|x_0,t_0)+c_1c_2 n_2(t+\tau|c_1,t)n_1(t|x_0,t_0)+c_2^2 n_2(t+\tau|c_2,t)n_2(t|x_0,t_0). \quad (A1)$$

For the $t$-averaged correlation function

$$C(\tau)=\Big\langle \lim_{t_0\to-\infty}\langle x(t+\tau)x(t)|x_0,t_0\rangle\Big\rangle_t, \quad (A2)$$

we obtain

$$C(\tau)=R_0+R_1\exp(-\mu|\tau|)+R_2\exp(-\mu|\tau|)\cos(\omega_s\tau)+R_3\exp(-\mu|\tau|)\sin(\omega_s\tau)+R_4\cos(\omega_s\tau). \quad (A3)$$

Here $\mu=\mu_1+\mu_2$ and the constants $R_i$ are given by

$$R_0=\left(\frac{c_2\mu_1+c_1\mu_2}{\mu_1+\mu_2}\right)^2,$$

$$R_1=\frac{(c_2-c_1)^2\mu_1\mu_2}{\mu^2}+\frac{A^2(c_1-c_2)[c_2(\alpha_2^2\mu_1+\alpha_1\alpha_2\mu_2)-c_1(\alpha_1^2\mu_2+\alpha_1\alpha_2\mu_1)]}{2\mu(\mu^2+\omega^2)},$$

$$R_2=\frac{A^2(c_2-c_1)(\alpha_2-\alpha_1)\sqrt{R_0}(\alpha_2\mu_1+\alpha_1\mu_2)}{2\mu(\mu^2+\omega^2)}, \quad (A4)$$

$$R_3=\frac{A^2(c_2-c_1)(\alpha_2-\alpha_1)\sqrt{R_0}(\alpha_2\mu_1+\alpha_1\mu_2)}{2\omega(\mu^2+\omega^2)},$$

$$R_4=\frac{A^2(c_1-c_2)^2(\alpha_2\mu_1+\alpha_1\mu_2)^2}{2\mu^2(\mu^2+\omega^2)}.$$

Then, noting that $R_0$ is just the square of the mean value of $x$ in the absence of signal ($R_0=\langle x\rangle^2|_{A=0}$), we compute the *t-averaged* power spectral density (PSD) [$\langle\tilde{S}(\omega)\rangle_t$] as the Fourier transform of $C(\tau)-R_0$. After that, we compute the *one-sided t*-averaged PSD [$S(\omega)$], defined for $\omega>0$, as

$$S(\omega)=\langle\tilde{S}(\omega)\rangle_t+\langle\tilde{S}(-\omega)\rangle_t. \quad (A5)$$

We get

$$S(\omega)=4R_1\frac{\mu}{(\mu^2+\omega^2)}+2R_2\phi_2(\omega)+2R_3\phi_3(\omega)+2\pi R_4\delta(\omega-\omega_s), \quad (A6)$$

where

$$\phi_2(\omega)=\frac{2\mu(\mu^2+\omega^2+\omega_s^2)}{\mu^4+2\mu^2\omega^2+\omega^4+2\mu^2\omega_s^2-2\omega^2\omega_s^2+\omega_s^4},$$

$$\phi_3(\omega)=\frac{2\omega_s(\mu^2-\omega^2+\omega_s^2)}{\mu^4+2\mu^2\omega^2+\omega^4+2\mu^2\omega_s^2-2\omega^2\omega_s^2+\omega_s^4}. \quad (A7)$$

In the one-sided $t$-averaged PSD [Eq. (A6)], two contributions can be distinguished: the signal output which is given by the $\delta$ function centered at the signal frequency and the broadband noise output, given by the Lorentzian term (which is the dominant [$o(A^0)$] part) plus the two additional terms containing the $\phi_i$ functions.

If, when calculating the PSD, instead of $C(\tau)-R_0$ only $C(\tau)$ is considered, an extra term [$4\pi R_0\delta(\omega)$] appears in Eq. (A6). Note that a nonvanishing value of $R_0$ can be originated either by an asymmetric choice of the values of $c_1$ and $c_2$ ($c_1\neq-c_2$) or by a difference in the stabilities of both states ($\mu_1\neq\mu_2$) even when $c_1=-c_2$ is considered.

If we consider the symmetric case ($\mu_1=\mu_2\equiv\tilde{\alpha}_0/2$ and $\alpha_1=\alpha_2\equiv\tilde{\alpha}_1/2$) and also fix $c_2=-c_1\equiv c$, the constants $R_0,R_2$, and $R_3$ vanish and we recover exactly the result of [4] which is a Lorentzian plus a $\delta$ function centered at the signal frequency

$$S(\omega)|_{Sym}=\left(1-\frac{A^2\tilde{\alpha}_1{}^2}{2(\tilde{\alpha}_0{}^2+\omega_s^2)}\right)\frac{4\tilde{\alpha}_0c^2}{\tilde{\alpha}_0{}^2+\omega^2}+\frac{\pi A^2\tilde{\alpha}_1{}^2c^2}{\tilde{\alpha}_0{}^2+\omega_s^2}\times\delta(\omega-\omega_s). \quad (A8)$$

For the general asymmetric case we define $\tilde{R}$, the SNR, as the ratio between the strength of the output signal and the broadband noise output evaluated at the signal frequency, obtaining

$$\tilde{R}=\frac{\pi R_4}{R_1 2\mu/(\mu^2+\omega_s^2)+R_2\phi_2(\omega_s)+R_3\phi_3(\omega_s)}. \quad \text{(A9)}$$

For sufficiently low signal amplitudes (i.e., small $A$) we can neglect the terms of second order in $A$ in the denominator in Eq. (A9) and obtain the following approximation for $\tilde{R}$, which is independent of the signal frequency:

$$\tilde{R}=\frac{A^2\pi(\alpha_2\mu_1+\alpha_1\mu_2)^2}{4\mu_1\mu_2(\mu_1+\mu_2)}. \quad \text{(A10)}$$


[1] C. Nicolis, Tellus **34**, 1 (1982).

[2] Proceedings of the NATO Advanced Research Workshop on Stochastic Resonance in Physics and Biology, edited by F. Moss *et al.* [J. Stat. Phys. **70**, 1/2 (1993)]; Proceedings of the 2nd International Workshop on Fluctuations in Physics and Biology, edited by A. Bulsara *et al.* [Nuovo Cimento D **17**, (1995)].

[3] L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1988).

[4] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).

[5] P. Jung and R. Bartussek, in *Fluctuations and Order: The New Synthesis*, edited by M. Millones (Springer, New York, 1988), p. 35.

[6] R. Löfstedt and S.N. Coppersmith, Phys. Rev. E **49**, 4821 (1994).

[7] A.R. Bulsara, M.E. Inchiosa, and L. Gammaitoni, Phys. Rev. Lett. **77**, 2162 (1996); M.E. Inchiosa, A.R. Bulsara, and L. Gammaitoni Phys. Rev. E **55**, 4049 (1997); M.E. Inchiosa and A.R. Bulsara, *ibid.* **58**, 115 (1998).

[8] A. Bulsara and G. Schmera, Phys. Rev. E **47**, 3734 (1993); P. Jung, U. Behn, E. Pantazelou, and F. Moss, Phys. Rev. A **46**, R1709 (1992).

[9] J.F. Lindner, B.K. Meadows, W.L. Ditto, M.E. Inchiosa, and A. Bulsara, Phys. Rev. Lett. **75**, 3 (1995); Phys. Rev. E **53**, 2081 (1996).

[10] P. Jung and G. Mayer-Kress, Phys. Rev. Lett. **74**, 2130 (1995).

[11] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **18**, 2239 (1985).

[12] F. Marchesoni, L. Gammaitoni, and A. Bulsara, Phys. Rev. Lett. **76**, 2609 (1996).

[13] H.S. Wio, Phys. Rev. E **54**, R3075 (1996).

[14] F. Castelpoggi and H.S. Wio, Europhys. Lett. **38**, 91 (1997).

[15] F. Castelpoggi and H.S. Wio, Phys. Rev. E **57**, 5112 (1998).

[16] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993); E. Meron, Phys. Rep. **218**, 1 (1992); A. S. Mikhailov, *Foundations of Synergetics I* (Springer-Verlag, Berlin, 1990); J. D. Murray, *Mathematical Biology* (Springer-Verlag, Berlin, 1985).

[17] I. Lengyel and I.R. Epstein, Proc. Natl. Acad. Sci. USA **89**, 3977 (1992); John E. Pearson, Physica A **188**, 178 (1992); S. Kádár, J. Wang, and K. Showalter, Nature (London) **391**, 770 (1998).

[18] S. Bouzat and H.S. Wio, Phys. Lett. A **247**, 297 (1998).

[19] R. Graham, in *Instabilities and Nonequilibrium Structures*, edited by E. Tirapegui and D. Villaroel (Reidel, Dordrecht, 1987); R. Graham and T. Tel, in *Instabilities and Nonequilibrium Structures III*, edited by E. Tirapegui and W. Zeller (Kluwer, Dordrecht, 1991); M. San Miguel and R. Toral, in *Instabilities and Nonequilibrium Structures VI*, edited by E. Tirapegui and W. Zeller (Kluwer, Dordrecht, 1998); H. S. Wio, in *Fourth Granada Seminar in Computational Physics*, edited by P. Garrido and J. Marro (Springer-Verlag, Berlin, 1997), p. 135.

[20] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[21] See, for instance, Sec. IV C of Ref. [3], and the references included therein.

[22] S.M. Bezrukov and I. Vodyanoy, Nature (London) **378**, 362 (1995).

[23] T. Alarcón, A. Pérez-Madrid, and J.M. Rubí, Phys. Rev. E **57**, 4979 (1998).