Energy conservation law for randomly fluctuating electromagnetic fields

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An energy conservation law is derived for electromagnetic fields generated by any random, statistically stationary, source distribution. It is shown to provide insight into the phenomenon of correlation-induced spectral changes. The results are illustrated by an example. [S1063-651X(99)01403-8]

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I. INTRODUCTION

Classical electromagnetic theory deals with deterministic sources and deterministic fields. It follows from Maxwell's equations that such fields obey well-known conservation laws for energy, linear momentum, and angular momentum. The situation regarding conservation laws is rather different when the sources and the fields fluctuate randomly either in space or in time. Such situations are actually very common and are also more realistic, because sources found in nature or produced in laboratories undergo some irregular, unpredictable, fluctuations.

Around 1960, after the rigorous laws of coherence theory of the electromagnetic field had been formulated, various conservation laws for such fields were derived [1]. They turned out to be rather complicated and, probably because of this, little use has been made of them.

About ten years ago the phenomenon of correlation-induced spectral changes was discovered, and it has been extensively studied since then, both theoretically and experimentally [2]. This phenomenon is characterized by changes in the spectrum of the field on propagation, as a consequence of source correlations. In particular the field spectrum may differ from the spectrum of the source, and may be different at different points in space. The source correlations may give rise to shifts of spectral lines, or to broadening or narrowing of the lines, or they may generate much more drastic changes, e.g., producing new lines or suppressing some of the lines present in the source spectrum.

It might appear at first sight that correlation-induced spectral changes violate energy conservation. That this is not so was demonstrated, under somewhat special circumstances, in several papers [3], and this question was examined under more general conditions in Ref. [4], within the framework of scalar theory.

In the present paper we generalize the results of Ref. [4], and we derive an energy conservation law which is valid for all statistically stationary fluctuating electromagnetic fields. We further show that correlation-induced changes of spectra of electromagnetic fields of any state of coherence are consistent with this conservation law, and we illustrate the results by an example.

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II. ENERGY CONSERVATION IN RANDOMLY FLUCTUATING ELECTROMAGNETIC FIELDS

We begin by deriving an energy conservation law for an electromagnetic field generated by a randomly fluctuating statistically stationary source occupying a domain D. Let $\langle \mathbf{F}(\mathbf{r},\omega)\rangle$ represent the expectation value of the flux density vector (the Poynting vector) at frequency ω, at an arbitrary point $\mathbf{r}$ in the field. It is given by the expression (using coherence theory in the space-frequency domain—see Sec. 4.7 of Ref. [5])

$$\langle \mathbf{F}(\mathbf{r},\omega)\rangle = \frac{c}{8\pi}\,\mathrm{Re}\langle \mathbf{E}^*(\mathbf{r},\omega)\times \mathbf{H}(\mathbf{r},\omega)\rangle, \tag{2.1}$$

where Re denotes the real part, and the asterisk denotes the complex conjugate. On taking the divergence of this expression and on using the vector identity

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}), \tag{2.2}$$

it follows that

$$\nabla\cdot\langle \mathbf{F}(\mathbf{r},\omega)\rangle = \frac{c}{8\pi}\,\mathrm{Re}\{\langle \mathbf{H}^*(\mathbf{r},\omega)\cdot[\nabla\times\mathbf{E}(\mathbf{r},\omega)]\rangle - \langle \mathbf{E}^*(\mathbf{r},\omega)\cdot[\nabla\times\mathbf{H}(\mathbf{r},\omega)]\rangle\}. \tag{2.3}$$

The right-hand side of Eq. (2.3) may be simplified by making use of the relations

$$\nabla\times\mathbf{E}(\mathbf{r},\omega) = ik\mathbf{H}(\mathbf{r},\omega), \tag{2.4a}$$

$$\nabla\times\mathbf{H}(\mathbf{r},\omega) = -ik\mathbf{E}(\mathbf{r},\omega) - 4\pi ik\mathbf{P}(\mathbf{r},\omega), \tag{2.4b}$$

which follow from Maxwell's equations. We have assumed that the source is nonmagnetic. Using Eqs. (2.4) in Eq. (2.3), one finds that

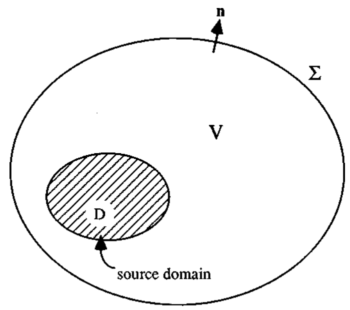

FIG. 1. Illustrating notation relating to the integral form (2.13) of the energy conservation law for fluctuating, statistically stationary, electromagnetic fields.

$$\nabla\cdot\langle\mathbf{F}(\mathbf{r},\omega)\rangle=\frac{kc}{8\pi}\mathrm{Re}\{i\langle\mathbf{H}^*(\mathbf{r},\omega)\cdot\mathbf{H}(\mathbf{r},\omega)\rangle+i\langle\mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{E}(\mathbf{r},\omega)\rangle+4\pi i\langle\mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{P}(\mathbf{r},\omega)\rangle\}. \quad (2.5)$$

The first two terms on the right of Eq. (2.5) are purely imaginary, and hence do not contribute to the left-hand side. Equation (2.5) therefore reduces to

$$\nabla\cdot\langle\mathbf{F}(\mathbf{r},\omega)\rangle=-\frac{kc}{2}\mathrm{Im}\langle\mathbf{E}^*(\mathbf{r},\omega)\cdot\mathbf{P}(\mathbf{r},\omega)\rangle. \quad (2.6)$$

On eliminating the magnetic field from Eqs. (2.4a) and (2.4b), we can solve the resulting equation for the electric field subject to the requirement that it is outgoing at infinity, and we find that

$$\mathbf{E}(\mathbf{r},\omega)=[k^2+\nabla(\nabla\cdot)]\int_D\mathbf{P}(\mathbf{r}',\omega)\frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}'. \quad (2.7)$$

Next we substitute from Eq. (2.7) into Eq. (2.6), and obtain the formula

$$\nabla\cdot\langle\mathbf{F}(\mathbf{r},\omega)\rangle=-\frac{kc}{2}\mathrm{Im}\left\{\left\langle k^2\int_D\mathbf{P}(\mathbf{r},\omega)\cdot\mathbf{P}^*(\mathbf{r}',\omega)\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right\rangle+\left\langle\mathbf{P}(\mathbf{r},\omega)\cdot\nabla\int_D\mathbf{P}^*(\mathbf{r}',\omega)\cdot\nabla\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right\rangle\right\}. \quad (2.8)$$

Let us now introduce the cross-spectral density tensor $W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ of the source polarization, defined by the formula

$$W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle P_i^*(\mathbf{r}_1,\omega)P_j(\mathbf{r}_2,\omega)\rangle, \quad (2.9)$$

where the angular brackets denote averages over the ensemble of the space-frequency realization of the source polarization $\mathbf{P}(\mathbf{r},\omega)$, and the suffixes i and j label Cartesian components. The tensor $W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)$ is a measure of the correlations of the polarization at pairs of points in the source, at frequency ω. On interchanging the order of the various operations on the right-hand side of Eq. (2.8), the formula may be expressed in the more compact form

$$\nabla\cdot\langle\mathbf{F}(\mathbf{r},\omega)\rangle=-\frac{kc}{2}\mathrm{Im}\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)(k^2\delta_{ij}+\partial_i\partial_j)\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r', \quad (2.10)$$

where summation over repeated indices is to be taken.

Equation (2.10) is the *differential form* of an energy conservation law for statistically stationary random electromagnetic fields. We note that when the point $\mathbf{r}$ is outside the source domain D, $W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)=0$, and Eq. (2.10) reduces to the simple form

$$\nabla\cdot\langle\mathbf{F}(\mathbf{r},\omega)\rangle=0. \quad (2.11)$$

The physical significance of formula (2.10) becomes more apparent if one converts it into integral form. Let us, therefore, integrate both sides of Eq. (2.10) over a volume V, bounded by a surface Σ, which completely encloses the source domain D. Making use of the divergence theorem of vector calculus and of the fact that $W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)=0$ for all points $\mathbf{r}$ located outside the domain D, it follows that

$$\int_\Sigma\langle\mathbf{F}(\mathbf{r},\omega)\rangle\cdot\mathbf{n}\,d\Sigma=-\frac{kc}{2}\mathrm{Im}\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)(k^2\delta_{ij}+\partial_i\partial_j)\frac{e^{-ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}d^3r\,d^3r', \quad (2.12)$$

where $\mathbf{n}$ denotes the unit outward normal to Σ at the point $\mathbf{r}$ (see Fig. 1). Noting that $W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)$, summed over the subscripts i and j, is Hermitian, and that the expression $e^{-ik|\mathbf{r}-\mathbf{r}'|}/|\mathbf{r}-\mathbf{r}'|$ is symmetric with respect to $\mathbf{r}$ and $\mathbf{r}'$, Eq. (2.12) may be rewritten in the form

$$\int_\Sigma\langle\mathbf{F}(\mathbf{r},\omega)\rangle\cdot\mathbf{n}d\Sigma=\frac{k^2c}{2}\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)(k^2\delta_{ij}+\partial_i\partial_j)\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}d^3r\,d^3r'. \quad (2.13)$$

This formula is the *integral form* of the conservation law. It shows that the rate at which the source radiates energy across any surface Σ which completely encloses the source domain D depends on the second-order correlation properties of the source polarization, represented by the cross-spectral density tensor $W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)$. The conservation laws (2.10) and (2.13) are generalizations to electromagnetic fields of energy conservation laws derived not long ago for fluctuating scalar fields [Ref. [4], Eqs. (3.4) and (3.6)].

III. SOURCE SPECTRUM AND THE SPECTRUM OF THE RADIATED FIELD

We now apply the energy conservation law to elucidate the phenomenon of correlation-induced spectral changes [2]. Let us consider the field in the far zone of the source, at a point specified by the position vector $R\mathbf{u}$, $(\mathbf{u}^2=1)$. The electric and the magnetic fields are given by the expressions [6]

$$\mathbf{E}(R\mathbf{u},\omega)\sim(2\pi)^3k^2\frac{e^{ikR}}{R}\{[\mathbf{u}\times\tilde{\mathbf{P}}(k\mathbf{u},\omega)]\times\mathbf{u}\} \quad (3.1a)$$

and

$$\mathbf{H}(R\mathbf{u},\omega)\sim(2\pi)^3k^2\frac{e^{ikR}}{R}[\mathbf{u}\times\tilde{\mathbf{P}}(k\mathbf{u},\omega)], \quad (3.1b)$$

where

$$\tilde{\mathbf{P}}(\mathbf{k},\omega)=\frac{1}{(2\pi)^3}\int_D\mathbf{P}(\mathbf{r},\omega)e^{-i\mathbf{k}\cdot\mathbf{r}}d^3r \quad (3.2)$$

is the spatial Fourier transform of the source polarization [7]. In tensor notation, Eqs. (3.1a) and (3.1b) take the forms

$$E_i(R\mathbf{u},\omega)\sim(2\pi)^3k^2\frac{e^{ikR}}{R}(\delta_{ij}-u_iu_j)\tilde{\mathbf{P}}_j(k\mathbf{u},\omega), \quad (3.3a)$$

$$H_i(R\mathbf{u},\omega)\sim(2\pi)^3k^2\frac{e^{ikR}}{R}\varepsilon_{ijk}u_j\tilde{\mathbf{P}}_k(k\mathbf{u},\omega), \quad (3.3b)$$

where δ_{ij} is the Kroenecker delta symbol, and ε_{ijk} is the completely antisymmetric unit tensor of Levi-Civita.

Let us now define the cross-spectral density tensors $W_{ij}^{(E)}$ and $W_{ij}^{(H)}$ of the field by formulas analogous to that by which the polarization tensor was introduced [Eq. (2.9)], viz.

$$W_{ij}^{(E)}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle E_i^*(\mathbf{r}_1,\omega)E_j(\mathbf{r}_2,\omega)\rangle, \quad (3.4a)$$

$$W_{ij}^{(H)}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\langle H_i^*(\mathbf{r}_1,\omega)H_j(\mathbf{r}_2,\omega)\rangle. \quad (3.4b)$$

Using Eqs. (3.3) in Eqs. (3.4), we find that at points in the far zone of the source the field correlation tensors are given by the expressions

$$W_{ij}^{(E)}(R\mathbf{u}_1,R\mathbf{u}_2,\omega)=\frac{(2\pi)^6k^4}{R^2}(\delta_{im}-u_{1i}u_{1m})\times(\delta_{jn}-u_{2j}u_{2n})\tilde{W}_{mn}^{(P)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega), \quad (3.5a)$$

$$W_{ij}^{(H)}(R\mathbf{u}_1,R\mathbf{u}_2,\omega)=\frac{(2\pi)^6k^4}{R^2}\varepsilon_{imn}\varepsilon_{jpq}u_{1m}u_{2p}\times\tilde{W}_{nq}^{(P)}(-k\mathbf{u}_1,k\mathbf{u}_2,\omega), \quad (3.5b)$$

where $u_{\alpha i}$, $(i=1,2,3)$, is the ith component of the unit vector $\mathbf{u}_\alpha$, and

$$\tilde{W}_{ij}^{(P)}(\mathbf{k}_1,\mathbf{k}_2,\omega)=\frac{1}{(2\pi)^6}\int_D\int_D W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)e^{-i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)}d^3r_1\,d^3r_2 \quad (3.6)$$

is the six-dimensional Fourier transform of the cross-spectral density of the source polarization.

Let us now determine the field spectrum in the far zone. The power spectrum $S^{(\infty)}(R\mathbf{u},\omega)$ of the field in the far zone at distance R from the source, in a direction specified by a unit vector $\mathbf{u}$, may be identified with the ensemble average of the energy density multiplied by the speed of light, [see Ref. [5], Eqs. (5.7-31)] viz.

$$S^{(\infty)}(R\mathbf{u},\omega)\equiv c\langle U^{(\infty)}(R\mathbf{u},\omega)\rangle=\frac{c}{16\pi}\langle E_i^*(R\mathbf{u},\omega)E_i(R\mathbf{u},\omega)\rangle+\frac{c}{16\pi}\langle H_i^*(R\mathbf{u},\omega)H_i(R\mathbf{u},\omega)\rangle$$
$$=\frac{c}{16\pi}[W_{ii}^{(E)}(R\mathbf{u},R\mathbf{u},\omega)+W_{ii}^{(H)}(R\mathbf{u},R\mathbf{u},\omega)]. \quad (3.7)$$

On making use of Eqs. (3.5) we obtain for the spectrum of the field in the far zone expression [8]

$$S^{(\infty)}(R\mathbf{u},\omega)=\frac{8\pi^5k^4c}{R^2}[(\delta_{ij}-u_iu_j)\tilde{W}_{ij}^{(P)}(-k\mathbf{u},k\mathbf{u},\omega)]. \quad (3.8)$$

The spectrum of each Cartesian component of the source polarization may be defined by the expression

$$S_i^{(P)}(\mathbf{r},\omega)\equiv W_{ii}^{(P)}(\mathbf{r},\mathbf{r},\omega)\quad(\text{no summation}). \quad (3.9)$$

Let us define the spectral degree of coherence of the source polarization by the formula

$$\mu_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)=\frac{W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)}{\sqrt{S_i^{(P)}(\mathbf{r}_1,\omega)}\sqrt{S_j^{(P)}(\mathbf{r}_2,\omega)}}. \qquad (3.10)$$

Using elementary properties of the source polarization tensor and the Schwarz inequality, it is not difficult to show that

$$0\leqslant|\mu_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega)|\leqslant 1. \qquad (3.11)$$

Evidently $\mu_{ij}^{(P)}$ represents the correlation between Cartesian components of the polarization.

If we substitute for $\widetilde{W}_{ij}^{(P)}$ in Eq. (3.8) from Eq. (3.6), we find that

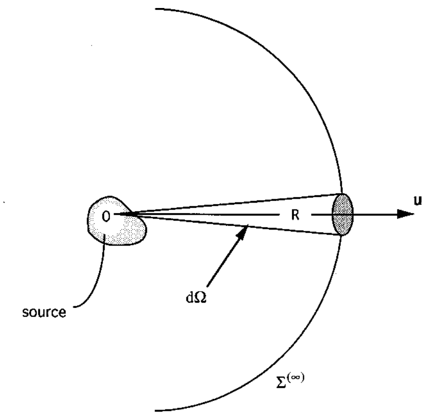

FIG. 2. Illustrating notation relating to the spectrum of the radiated field in the far zone of a fluctuating source polarization.

$$S^{(\infty)}(R\mathbf{u},\omega)=\frac{1}{8\pi}\frac{k^4c}{R^2}\left[(\delta_{ij}-u_iu_j)\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\,d^3r'\right]. \qquad (3.12)$$

If we then express $W_{ij}^{(P)}$ in Eq. (3.12) in terms of the spatial degree of coherence and the spectral densities by the use of Eq. (3.10), we finally obtain for the spectrum of the field in the far zone the expression

$$S^{(\infty)}(R\mathbf{u},\omega)=\frac{1}{8\pi}\frac{k^4c}{R^2}(\delta_{ij}-u_iu_j)\int_D\int_D\sqrt{S_i^{(P)}(\mathbf{r}',\omega)}\sqrt{S_j^{(P)}(\mathbf{r},\omega)}\,\mu_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\,d^3r'. \qquad (3.13)$$

It is evident from this equation that the spectrum of the far field depends not only on the source spectrum, but also on the correlations between Cartesian components of the polarization. Hence, except perhaps in some special cases, the spectrum of the far field will differ from the source spectrum, and will also depend upon the direction of observation **u**.

We will now show that in spite of the fact that source correlations induce spectral changes in the far field, formula (3.12) is consistent with our new energy conservation law (2.13). For this purpose we integrate both sides of Eq. (3.12) over all directions **u**, and multiply them by R^2. We then obtain the formula

$$\int_{\Sigma^{(\infty)}}S^{(\infty)}(R\mathbf{u},\omega)d\Sigma^{(\infty)}=\frac{1}{8\pi}k^4c\int_{(4\pi)}d\Omega(\delta_{ij}-u_iu_j)\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\,d^3r', \qquad (3.14)$$

where we used the fact that $R^2d\Omega=d\Sigma^{(\infty)}$ is the differential surface element of a large sphere $\Sigma^{(\infty)}$ centered in the source region (see Fig 2). The product u_iu_j on the right side of Eq. (3.14) may be expressed as a differential operator acting on the exponent, and Eq. (3.14) then becomes

$$\int_{\Sigma^{(\infty)}}S^{(\infty)}(R\mathbf{u},\omega)d\Sigma^{(\infty)}=\frac{1}{8\pi}k^2c\int_{(4\pi)}d\Omega\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)(k^2\delta_{ij}+\partial_i\partial_j)e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d^3r\,d^3r', \qquad (3.15)$$

where the integral with respect to Ω is taken over the whole 4π solid angle generated by the real unit vector **u**. On making use of the identity (see the footnote on p. 123 of Ref. [5])

$$\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}=\frac{1}{4\pi}\int_{(4\pi)}e^{-ik\mathbf{u}\cdot(\mathbf{r}-\mathbf{r}')}d\Omega, \qquad (3.16)$$

formula (3.15) may be rewritten as

$$\int_{\Sigma^{(\infty)}}S^{(\infty)}(R\mathbf{u},\omega)d\Sigma^{(\infty)}=\frac{k^2c}{2}\int_D\int_D W_{ij}^{(P)}(\mathbf{r}',\mathbf{r},\omega)[k^2\delta_{ij}+\partial_i\partial_j]\frac{\sin k|\mathbf{r}-\mathbf{r}'|}{k|\mathbf{r}-\mathbf{r}'|}d^3r\,d^3r'. \qquad (3.17)$$

The right-hand side of this equation is identical to the right-hand side of the integral form of the energy conservation law (2.13). The left-hand sides are also equal to each other because of the well-known relations between the average flux vector $\langle \mathbf{F}^{(\infty)} \rangle$ and the spectral density $\langle S^{(\infty)} \rangle$ in the far field viz. $\langle \mathbf{F}^{(\infty)}(R\mathbf{u},\omega) \rangle = S^{(\infty)}(R\mathbf{u},\omega)\mathbf{u}$ (see, for instance, Eqs. (5.7-32) of Ref. [5]). Hence the two equations (2.13) and (3.17) are equivalent and consequently correlation-induced spectral changes are consistent with energy conservation.

IV. EXAMPLE

We will illustrate our main results by considering a quasi-homogeneous, isotropic source with a source spectrum which is taken to be scalar. For such a source the cross-spectral density tensor can be well approximated by (cf. Ref. [5], Sec. 5.2.2)

$$W_{ij}^{(P)}(\mathbf{r}_1,\mathbf{r}_2,\omega) \approx S((\mathbf{r}_1+\mathbf{r}_2)/2,\omega)\mu_{ij}(\mathbf{r}_2-\mathbf{r}_1,\omega), \tag{4.1}$$

where $S(\mathbf{r},\omega)$ is assumed to vary much more slowly with $\mathbf{r}$ than $\mu_{ij}(\mathbf{r}',\omega)$ varies with $\mathbf{r}'$. Because the source is assumed to be isotropic, it must have the form (cf. Ref. [9])

$$\mu_{ij}(\mathbf{r},\omega) = \delta_{ij}A(r,\omega) + B(r,\omega)r_i r_j, \tag{4.2}$$

where r_i is the ith component of the vector $\mathbf{r}$. The normalization $\mu_{ii}(0,\omega)=1$ (no summation) implies that

$$A(0,\omega)=1, \tag{4.3a}$$

$$r^2 B(r,\omega) \to 0 \quad \text{as} \quad r \to 0. \tag{4.3b}$$

In this case the six-dimensional Fourier transform of the source polarization tensor (4.1) is given by the expression

$$\tilde{W}_{ij}^{(P)}(-k\mathbf{u},k\mathbf{u},\omega) = \frac{1}{(2\pi)^3}\int d^3r[\delta_{ij}A(r,\omega) + r_i r_j B(r,\omega)]e^{-ik\mathbf{u}\cdot\mathbf{r}} \times \frac{1}{(2\pi)^3}\int S(\mathbf{R},\omega)d^3R. \tag{4.4}$$

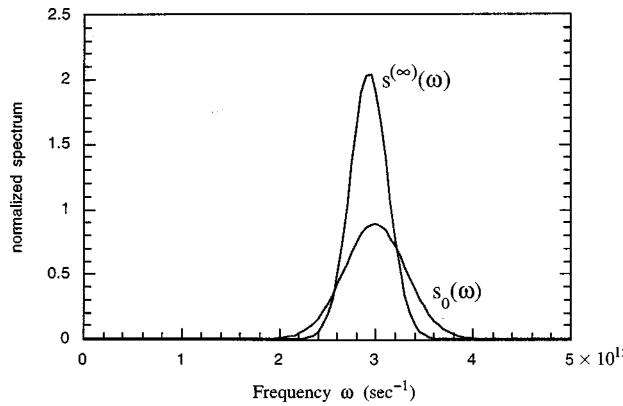

FIG. 3. Normalized spectrum $s_0(\omega) \equiv S_0(\omega)/\int_0^\infty S_0(\omega')d\omega'$ of a homogeneous, isotropic source [represented by Eqs. (4.1), (4.2), and (4.9)] and the normalized spectrum $s^{(\infty)}(\omega) \equiv S^{(\infty)}(\omega)/\int_0^\infty S^{(\infty)} \times (\omega')d\omega'$ of the far field generated by the source, when

$$S(\mathbf{R},\omega) \equiv S_0(\omega) = \frac{I_0}{\sqrt{2\pi}\delta}\exp[-(\omega-\omega_0)^2/2\delta^2]$$

with $\sigma/c = 10^{-15}$ sec, $\omega_0 = 3\times 10^{15}$ sec^{-1}, and $\delta = 2\times 10^{14}$ sec^{-1}.

If $\tilde{A}(q,\omega)$ and $\tilde{B}(q,\omega)$ denote the Fourier transforms of $A(r,\omega)$ and $B(r,\omega)$, respectively, i.e.,

$$\tilde{A}(q,\omega) = \frac{1}{(2\pi)^3}\int A(r,\omega)e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r,$$

$$\tilde{B}(q,\omega) = \frac{1}{(2\pi)^3}\int B(r,\omega)e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r, \tag{4.5}$$

and we make use of the identity

$$\frac{1}{(2\pi)^3}\int r_i r_j B(r,\omega)e^{-i\mathbf{q}\cdot\mathbf{r}}d^3r = -\frac{\partial^2}{\partial q_i \partial q_j}\tilde{B}(q,\omega) = -\left(\delta_{ij} - \frac{q_i q_j}{q^2}\right)\frac{1}{q}\frac{d}{dq}\tilde{B}(q,\omega) - \frac{q_i q_j}{q^2}\frac{d^2}{dq^2}\tilde{B}(q,\omega), \tag{4.6}$$

formula (4.4) becomes

$$\tilde{W}_{ij}^{(P)}(-k\mathbf{u},k\mathbf{u},\omega) = \tilde{S}(0,\omega)\left\{\delta_{ij}\left[\tilde{A}(k,\omega) - \frac{1}{k}\frac{d}{dk}\tilde{B}(k,\omega)\right] + u_i u_j\left[\frac{1}{k}\frac{d}{dk}\tilde{B}(k,\omega) - \frac{d^2}{dk^2}\tilde{B}(k,\omega)\right]\right\}. \tag{4.7}$$

On substituting from Eq. (4.7) into Eq. (3.8), and carrying out the summations, we find that

$$S^{(\infty)}(R\mathbf{u},\omega) = \frac{8\pi^5 k^4 c}{R^2}2\left[\tilde{A}(k,\omega) - \frac{1}{k}\frac{d}{dk}\tilde{B}(k,\omega)\right]\tilde{S}(0,\omega). \tag{4.8}$$

Formula (4.8) shows that the spectrum of the field produced by a source of the kind we are considering is independent of the direction of observation $\mathbf{u}$.

As a specific example, let us choose

$$A(r,\omega) = e^{-r^2/2\sigma^2}, \tag{4.9a}$$

$$B(r,\omega)=\frac{1}{\sigma^2}e^{-r^2/2\sigma^2}, \tag{4.9b}$$

where σ is a positive constant, assumed to be independent of ω. In this case,

$$\tilde{A}(k,\omega)=\frac{\sigma^3}{(2\pi)^{3/2}}e^{-\sigma^2k^2/2}, \tag{4.10a}$$

$$\tilde{B}(k,\omega)=\frac{1}{\sigma^2}\tilde{A}(k,\omega). \tag{4.10b}$$

If we assume that the source spectrum is the same at each source point, i.e., that

$$\begin{aligned} S(\mathbf{r},\omega)&\equiv S_0(\omega), \quad \mathbf{r}\in D \\ &=0, \quad \mathbf{r}\notin D, \end{aligned} \tag{4.11}$$

the formula (4.8) becomes

$$S^{(\infty)}(R\mathbf{u},\omega)=\frac{\sqrt{2\pi}k^4c}{R^2}\sigma^3e^{-\sigma^2k^2/2}V_0S_0(\omega), \tag{4.12}$$

where V_0 is the volume of the domain D occupied by the source.

We see that the normalized spectrum $S^{(\infty)}(R\mathbf{u},\omega)$ of the field in the far zone differs from the source spectrum $S^{(0)}(\omega)$. This is illustrated for a specific case in Fig. 3. In spite of the difference between the two spectra, the result is consistent with the law of conservation of energy, as we showed earlier on general grounds.

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[6] See W. H. Carter and E. Wolf, Phys. Rev. A **36**, 1258 (1987) where the current density **J** rather than the polarization density **P** was used. These two quantities are related by the continuity equation which, in the space-frequency domain, takes the form $\mathbf{J}(\mathbf{r},\omega)=i\omega\mathbf{P}(\mathbf{r},\omega)$.
[7] Formula (3.1a) is sometimes expressed in the more compact form

$$E_i(R\mathbf{u},\omega)\sim(2\pi)^3k^2\frac{e^{ikR}}{R}\tilde{P}_i^{(t)}(k\mathbf{u},\omega),$$

where

$$\tilde{P}_i^{(t)}(k\mathbf{u},\omega)\equiv([\mathbf{u}\times\tilde{\mathbf{P}}(k\mathbf{u},\omega)]\times\mathbf{u})_i=(\delta_{ij}-u_iu_j)\tilde{P}_j(k\mathbf{u},\omega)$$

are components of the transverse polarization. (cf. Ref. [6]).
[8] The left-hand side of Eq. (3.8) is invariant with respect to a rotation of axes, and therefore so must be the right-hand side. That this is so follows at once from the following relation involving the cross-spectral density tensor of the polarization W_{ij}^P and the cross-spectral density tensor of the *transverse polarization* $W_{ij}^{P^{(t)}}$

$$(\delta_{ij}-u_iu_j)\tilde{W}_{ij}^P(-k\mathbf{u},k\mathbf{u},\omega)=\mathrm{Tr}\{\tilde{W}_{ij}^{P^{(t)}}(-k\mathbf{u},k\mathbf{u},\omega)\},$$

where Tr denotes the trace. [cf. Ref. [6], Eq. (D7)].
[9] G. K. Batchelor, *The Theory of Homogeneous Turbulence* (Cambridge University Press, Cambridge, 1986), Secs. 3.3 and 3.4.