Thermal screening of Darwin interactions in a weakly relativistic plasma

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We study a weakly relativistic and weakly degenerate plasma of electrons at equilibrium, described by thermal quantum electrodynamics. Near the classical limit, at lowest order in $\hbar$, the first contribution (in a perturbative expansion with respect to e^2) to the current correlation is shown to exhibit two different behaviors. At intermediate distances (between the de Broglie thermal wavelength $\sqrt{\beta\hbar^2/m}$ and the thermal photon wavelength $\beta\hbar c$), it decreases slowly, as the Darwin (transverse) potential which accounts for retarded electromagnetic interactions beyond Coulomb in Darwin classical models. At large distances (larger than $\beta\hbar c$), it decays exponentially fast, in agreement with the predictions of the classical field theory. This thermal screening of the Darwin (transverse) interactions allows us to understand the contradictory results on the equilibrium properties of relativistic plasmas found in the literature. Furthermore, we give the physical regime of validity of these models for the description of real plasmas. [S1063-651X(99)04704-2]

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I. INTRODUCTION

The Coulomb interactions are sufficient to describe the properties of systems of charged particles (either classical or quantum), when the mean velocities are small compared to the speed of light c. For weakly relativistic and weakly degenerate plasmas, namely at sufficiently low temperature and low density, the first effects of retarded electromagnetic interactions beyond Coulomb can be studied *a priori* within the so-called Darwin classical (nonquantum) models. In these models [1], the Lorentz equations of motion for the charged particles, truncated at order $1/c^2$, can be recast in Hamiltonian form [2–4]. The Hamiltonian of the system is then a function of the sole canonical coordinates of the particles, the degrees of freedom of the electromagnetic fields being eliminated, as in the Coulomb case. In a simplified version [5], the non-Coulombic potential part of the Hamiltonian is a sum of two-body Darwin (transverse) interactions. (It is simplified in the sense that a Lagrangian truncated at the first order $1/c^2$ leads to a full Hamiltonian with terms of any order in $1/c^2$; Krizan keeps only the term of order $1/c^2$ in his Hamiltonian. We stress that this *simplified* model is consistent at the order $1/c^2$ included. The terms of order higher than $1/c^2$ that appear in the full Darwin Hamiltonian do not make sense, because they do not account for the electromagnetic interactions of the same order that would appear in the equations of motion.) The Darwin potential depends on the momentum of the particles, and it decays like the Coulomb potential, i.e. as $1/r$ at large distances.

In a previous paper [6], we have studied the equilibrium properties of a classical one-component plasma described by the (simplified) Darwin Hamiltonian. At low densities, we have computed the first relativistic corrections to the pure Coulomb contributions for various quantities such as the excess pressure or the current correlations. Use of diagrammatic resummations remove the long-range divergences induced by the Coulomb and Darwin potentials, and they allow us to go beyond previous mean-field calculations [7–9].

As far as real matter is concerned, some predictions of the simplified Darwin model are questionable. For instance, another Darwin model, the Hamiltonian of which differs from ours by terms of order $1/c^4$ and more, predicts a faster (but still algebraic) decay of the current correlations at large distances. This discrepancy has been the source of controversial discussions in the literature [9].

In order to clarify the situation, we must start with a complete description of matter and radiation, where all the electromagnetic interactions between the charges are taken into account. As already explained in [10] and [6], this requires the use of quantum electrodynamics at finite temperature, or *thermal QED*. Our goal is to study various equilibrium properties of the system and to compare the QED results in the weakly relativistic and weakly degenerate limit to the predictions of the different (and mutually disagreeing) classical models. In this paper, we will first consider the current correlation of a dilute and cold electron plasma (with a rigid uniform background) in equilibrium with radiation. In particular, we show that the corresponding lowest-order Feynman graph decays exponentially fast at large distances in the present almost classical and nonrelativistic limit. Such a decay can be interpreted as resulting from the screening of the transverse effective interactions between the charges, generated by the exchange of thermalized photons. We stress that this mechanism is completely different from the classical Debye screening of Coulomb interactions. It is due to the process of thermalization of the mediating photons that becomes crucial at distances larger than the thermal photon wavelength $\lambda_{\text{photon}} \overset{\text{def}}{=} \beta\hbar c$. At intermediate distances, $\lambda_{\text{dB}} \overset{\text{def}}{=} (\beta\hbar^2/m)^{1/2} < r < \lambda_{\text{photon}}$, we recover the classical expression predicted by all the Darwin models. The existence of the thermal photonic screening implies that all these classical models fail in the description of large-distance behaviors for real matter coupled with radiation. According to this mechanism, we shall determine in forthcoming papers the

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physical status of Darwin contributions to the thermodynamic quantities such as the free energy or the excess pressure.

In Sec. II, we recall some general features about thermal quantum electrodynamics. The free thermal propagators for both matter and radiation are presented, as well as the thermodynamical conditions required to describe a weakly relativistic and almost classical system.

In Sec. III, we present the Feynman graphs which describe the current correlation function at the lowest order in the e^2 expansion. The behavior of this function with respect to the distance will be shown to be completely different below and above the photon thermal wavelength λ_{photon}. These two behaviors will be compared to the predictions of the Darwin models and classical field theory. The previous results are then generalized to all the graphs. Furthermore, a dynamical interpretation of thermal screening is given, and we briefly sketch the discussion about the regime of validity of the Darwin approach which will be detailed in a next paper.

Eventually, we conclude in Sec. IV and we clarify the paradox arising from the mutually disagreeing Darwin models [8,9,5,11].

II. GENERAL FRAMEWORK

In the following, we shall call ''real world'' a system of electrons and positrons at equilibrium, with electromagnetic interactions, regardless to any other kind of particles and interactions, in a world fully described by special relativity. A fully relativistic and quantum description is necessary to determine the regimes of validity of the Darwin model.

Matter, made of electrons and positrons, is described by a Dirac bispinor field $\psi(\boldsymbol{x})$, with fermionic nature, a function of the space variable $\boldsymbol{x}$ (time does not appear in this equilibrium theory). It should be noticed that, although the classical limit of our system will be a *one component plasma*, this bispinor field describes both electrons and positrons, indissociable in a relativistic and quantum theory. (The disappearance of the positrons will follow from the specific choice of the thermodynamic parameters in the regime of interest.) Radiation is described by a quadrivector field $A_\mu(\boldsymbol{x})$. In the grand canonical ensemble, the chemical potential of the photons is identically zero, whereas the chemical potentials of the electrons and the positrons are opposite : $\mu_{\text{pos}}=-\mu_{\text{el}}=-\mu$, as a consequence of the equilibrium between the annihilation of electron-positron pairs into photons, and the converse process of creation. Notice that in the relativistic case, we have $\mu=mc^2+\mu^*$, where μ^* is the usual chemical potential.

Our system, made of electrons and positrons coupled to photons with the coupling constant $\alpha \overset{\text{def}}{=} e^2/\hbar c$, is immersed in a rigid, homogeneous background, with charge density $-e\rho_{\text{B}}$, which creates an electrostatic potential. As already discussed in [4] and [6], this bath has a great physical importance, for it ensures the stability of the system; however, it will not enter explicitly in most perturbative calculations except by removing some divergent Feynman graphs in the Dyson expansion with respect to e^2 (as in the Mayer expansions for classical models [6]).

In the following, the specific effects arising from a quantum field description, such as pair creation (annihilation) and renormalization, will not be considered, for we are only interested here in contributions having a classical equivalent. Therefore, the physical constants $(m,e,\ldots)$ involved in the expression of interest are supposed to be the renormalized observable quantities.

A. Hamiltonian description and free thermal propagators

The Hamiltonian of our system can be written as the sum of three terms:

$$\mathcal{H} \overset{\text{def}}{=} \mathcal{H}^0_{\text{mat}}+\mathcal{H}^0_{\text{rad}}+\mathcal{H}_{\text{int}}, \tag{2.1}$$

where the free matter Hamiltonian is

$$\mathcal{H}^0_{\text{mat}} \overset{\text{def}}{=} \int \overline{\psi}(\boldsymbol{x})(-i\hbar c\,\boldsymbol{\gamma}\cdot\boldsymbol{\nabla}+mc^2)\psi(\boldsymbol{x})\,d\boldsymbol{x}, \tag{2.2}$$

[$\boldsymbol{\gamma}$ being the 3-vector of Dirac matrices $\gamma^1,\gamma^2,\gamma^3$, and $\overline{\psi}(\boldsymbol{x}) \overset{\text{def}}{=} \psi^\dagger(\boldsymbol{x})\gamma^0$], the free electromagnetic field Hamiltonian $\mathcal{H}^0_{\text{rad}}$ is derived from the Lagrangian

$$\mathcal{L}^0_{\text{em}} \overset{\text{def}}{=} -(1/16\pi)\int F_{\mu\nu}F^{\mu\nu}\,d\boldsymbol{x}, \tag{2.3}$$

[there are several ways to derive a Hamiltonian from the electromagnetic Lagrangian, due to the gauge symmetry; however, the explicit form of the chosen Hamiltonian is not important here, for we will express the correlation functions in terms of the sole free propagators (which of course still depend on the chosen gauge)], and the interaction Hamiltonian is

$$\mathcal{H}_{\text{int}} \overset{\text{def}}{=} e\int \overline{\psi}(\boldsymbol{x})\gamma^\mu\psi(\boldsymbol{x})A_\mu(\boldsymbol{x})\,d\boldsymbol{x}. \tag{2.4}$$

In order to investigate the equilibrium statistical mechanics of the system at a given temperature $k_{\text{B}}T=1/\beta$ and chemical potential μ, through perturbative expansions with respect to $\mathcal{H}_{\text{int}}$, it is convenient to introduce the imaginary-time free evolved operators

$$\psi(\tau,\boldsymbol{x}) \overset{\text{def}}{=} \exp\left[\tau\left(\mathcal{H}^0_{\text{mat}}-\mu\int \overline{\psi}(\boldsymbol{x})\gamma^0\psi(\boldsymbol{x})\,d\boldsymbol{x}\right)\right] \times\psi(\boldsymbol{x})\exp\left[-\tau\left(\mathcal{H}^0_{\text{mat}}-\mu\int \overline{\psi}(\boldsymbol{x})\gamma^0\psi(\boldsymbol{x})\,d\boldsymbol{x}\right)\right] \tag{2.5}$$

and

$$A_\mu(\tau,\boldsymbol{x}) \overset{\text{def}}{=} \exp[\tau\mathcal{H}^0_{\text{rad}}]A_\mu(\boldsymbol{x})\exp[-\tau\mathcal{H}^0_{\text{rad}}]. \tag{2.6}$$

Following [12–14], we now define the matter free propagator as

$$\mathcal{G}(\tau,\boldsymbol{x};\tau',\boldsymbol{x}') \overset{\text{def}}{=} \langle \mathrm{T}\overline{\psi}(\tau,\boldsymbol{x})\psi(\tau',\boldsymbol{x}')\rangle_0, \tag{2.7}$$

where $\langle\ \rangle_0$ is an equilibrium average, evaluated with the free Gibbs distribution

$$\exp\left\{-\beta\left(\mathcal{H}^0_{\rm mat}-\mu\int\overline{\psi}\gamma^0\psi\,d\boldsymbol{x}\right)\right\}.$$

The time-ordered product T has been used. It is well known that this propagator depends only on the difference of its arguments, and satisfies antiperiodicity relations that allow us to decompose it in Fourier series on the τ variable:

$$\mathcal{G}(\tau,\boldsymbol{x};\tau',\boldsymbol{x}')=\frac{1}{\beta}\sum_{l\in\mathbb{Z}}\int e^{i\omega_l(\tau'-\tau)}e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}'-\boldsymbol{x})/\hbar}\times\mathcal{G}(\omega_l,\boldsymbol{p})\frac{d\boldsymbol{p}}{(2\pi\hbar)^3},\tag{2.8}$$

where the Matsubara frequencies ω_l's are odd multiples of π/β:

$$\omega_l\overset{\rm def}{=}\frac{(2l+1)\pi}{\beta}\quad\text{for all}\quad l\in\mathbb{Z}.\tag{2.9}$$

As usual, we will write, in a condensed and convenient way,

$$\mathbf{p}\overset{\rm def}{=}(p^0,c\boldsymbol{p})=(i\omega_l+\mu,c\boldsymbol{p}).\tag{2.10}$$

As in (dynamical) QED, the propagator (2.7) is the Green function for the Dirac operator, and reads, in terms of momentum and Matsubara frequencies,

$$\mathcal{G}(\omega_l,\boldsymbol{p})=\frac{1}{\not{\mathbf{p}}-mc^2}=\frac{\not{\mathbf{p}}+mc^2}{\mathbf{p}^2-m^2c^4}=\frac{\gamma^0p^0-c\boldsymbol{\gamma}\cdot\boldsymbol{p}+mc^2}{(p^0)^2-\boldsymbol{p}^2c^2-m^2c^4}.\tag{2.11}$$

Similarly, the free photonic propagator defined by

$$\mathcal{D}_{\alpha\delta}(\tau,\boldsymbol{x},;\tau',\boldsymbol{x}')\overset{\rm def}{=}\langle \mathrm{T}A_\alpha(\tau,\boldsymbol{x})A_\delta(\tau',\boldsymbol{x}')\rangle_0\tag{2.12}$$

satisfies periodicity relations over τ, and its Fourier components with *even* multiples of π/β as Matsubara frequencies read

$$\mathcal{D}_{\alpha\delta}(\tau,\boldsymbol{x};\tau',\boldsymbol{x}')=\frac{1}{\beta}\sum_{n\in\mathbb{Z}}\int e^{-i\omega_n(\tau-\tau')}e^{-i\boldsymbol{k}\cdot(\boldsymbol{x}'-\boldsymbol{x})}\times\mathcal{D}_{\alpha\delta}(\omega_n,\hbar\boldsymbol{k})\frac{d\boldsymbol{k}}{(2\pi)^3},\tag{2.13}$$

$$\omega_n=2n\pi/\beta.\tag{2.14}$$

Notice that we have chosen here to write the momentum of the photon as $\hbar\boldsymbol{k}$, while the momentum of the electrons and positrons was $\boldsymbol{p}$. Now, the expression of the photon propagator depends on the choice of gauge. We can work within the gauge that is most likely to be convenient for our purpose. Unusually, it will not be the Lorentz gauge, although the expression of the propagator is simpler therein, for the Lorentz invariance is lost anyhow at finite temperature. On the other hand, the Coulomb gauge has the most likeable property to separate clearly the longitudinal Coulomb potential, which corresponds to the nonrelativistic limit of the electromagnetic field, and the transverse field that corresponds to relativistic corrections (retardation and magnetic terms, see [4]). In this gauge, the photon propagator reads

$$\mathcal{D}_{ij}(\mathbf{k})=\frac{4\pi\hbar^2c^2}{\mathbf{k}^2}\left(\delta_{ij}-\frac{k_ik_j}{\boldsymbol{k}^2}\right),\quad i,j=1,2,3,\tag{2.15}$$

$$\mathcal{D}_{00}(\mathbf{k})=-\frac{4\pi}{\boldsymbol{k}^2}\tag{2.16}$$

and the other components vanish. In this expression, we have set $\mathbf{k}=_{\rm def}(k^0,\hbar c\boldsymbol{k})=(i\omega_n,\hbar c\boldsymbol{k})$. The time component $\mathcal{D}_{00}$ is obviously related to the (instantaneous) Coulomb potential. The total expression is of course not relativistically explicitly covariant.

B. Perturbative expansions

The fine-structure constant $\alpha=e^2/\hbar c$ being small, one may perform expansions of various equilibrium quantities in powers of α. The equilibrium averages resulting from the Dyson expansion of the Gibbs factor in powers of $\mathcal{H}_{\rm int}$ are treated using Wick's theorem, and can be represented by series of Feynman graphs.

The free fermionic propagators will be represented as usual by straight lines, whereas the free photon propagators will appear as wiggly lines. Each graph will contain fermionic loops, connected by photonic lines.

C. The classical and nonrelativistic limit

In order to study the weakly relativistic and almost classical limit of the system, we take μ^* negative and such that

$$k_{\rm B}T\ll|\mu^*|\ll mc^2.\tag{2.17}$$

In order to check that this regime really defines the above limit, we can, for instance, compute the free charge density of electrons and positrons of the noninteracting system, i.e.,

$$q_{\rm free}\overset{\rm def}{=}e\langle\overline{\psi}(0,\boldsymbol{x})\gamma^0\psi(0,\boldsymbol{x})\rangle_0$$

(the total charge density including the contribution $-e\rho_{\rm B}$ of the background vanishes). A straightforward calculation shows that $q_{\rm free}$ can be written as $e\int N(\boldsymbol{p})\,d\boldsymbol{p}/(2\pi\hbar)^3$, with

$$N(\boldsymbol{p})=\frac{1}{\beta}\sum_l \mathrm{tr}\,\gamma^0\mathcal{G}(\omega_l,\boldsymbol{p})=\sum_l\frac{4(i\beta\omega_l+\beta\mu)}{(i\beta\omega_l+\beta\mu)^2-\beta^2E^2(\boldsymbol{p})},$$

where we have set $E(\boldsymbol{p})=_{\rm def}\sqrt{m^2c^4+\boldsymbol{p}^2c^2}$. If we combine in this sum the opposite terms in l and $-l$, we see that each term decreases as $1/l^2$ and the sum is therefore convergent. Using integration in the complex plane and the method of residues, one finds that

$$N(\boldsymbol{p})=2N_{\rm F}^{\rm el}(\boldsymbol{p})-2N_{\rm F}^{\rm pos}(\boldsymbol{p}),\tag{2.18}$$

where $N_{\rm F}$ refers to the electron or positron Fermi distribution

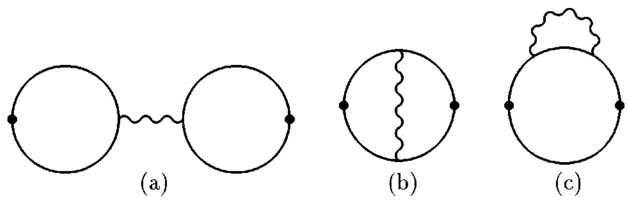

FIG. 1. The first three Feynman graphs in the Dyson expansion of the current correlations.

$$N_{\rm F}^{\rm el}(\boldsymbol{p}) \stackrel{\rm def}{=} \frac{1}{\exp\{\beta[E(\boldsymbol{p})-\mu]\}+1},$$

$$N_{\rm F}^{\rm pos}(\boldsymbol{p}) \stackrel{\rm def}{=} \frac{1}{\exp\{\beta[E(\boldsymbol{p})+\mu]\}+1}, \tag{2.19}$$

and the factor 2 comes from the two different spin states. The condition (2.17) implies $E(\boldsymbol{p})+\mu \simeq 2mc^2 \gg k_{\rm B}T$, and thus the positronic contribution is completely negligible. For the electronic contribution, $E(\boldsymbol{p})-\mu \simeq -\mu^* + \boldsymbol{p}^2/2m$, and $e^{-\beta\mu^*} \ll 1$, so at low densities we have

$$N_{\rm F}^{\rm el}(\boldsymbol{p}) \simeq N_{\rm MB}(\boldsymbol{p}) = e^{\beta\mu} e^{-\beta E(\boldsymbol{p})}, \tag{2.20}$$

where $N_{\rm MB}$ is the Maxwell-Boltzmann (relativistic) distribution, which reduces, in the weakly relativistic limit, to the familiar Gaussian, i.e.,

$$N_{\rm MB}(\boldsymbol{p}) \simeq e^{\beta\mu^*} e^{-\beta \boldsymbol{p}^2/2m}. \tag{2.21}$$

Therefore, and as expected, the system under the condition (2.17) is mainly made of classical nonrelativistic electrons.

III. CORRELATIONS AND EFFECTIVE INTERACTIONS

Within the above QED framework, we now calculate the first relativistic corrections to the current correlation function and the charge correlation functions, in the regime (2.17), and we compare them to the results found within classical descriptions such as the simplified Darwin model or the classical field theory.

A. General expression of the current correlations

We study the lowest order term in e^2 of the 4-current correlations

$$\mathcal{J}^{\mu\nu}(\boldsymbol{r}) \stackrel{\rm def}{=} \langle J^\mu(\boldsymbol{0}) J^\nu(\boldsymbol{r})\rangle \quad \text{with} \quad J^\mu(\boldsymbol{r}) \stackrel{\rm def}{=} e\,\overline{\psi}(\boldsymbol{r})\gamma^\mu \psi(\boldsymbol{r}), \tag{3.1}$$

where $\langle\ \rangle$ is the equilibrium value for the interacting system. With these notations, the charge correlations are $\langle \rho(\boldsymbol{0})\rho(\boldsymbol{r})\rangle = \mathcal{J}^{00}(\boldsymbol{r})$ and the current correlations are

$$\langle \boldsymbol{j}(\boldsymbol{0})\cdot \boldsymbol{j}(\boldsymbol{r})\rangle = c^2 \sum_{i=1}^{3} \mathcal{J}^{ii}(\boldsymbol{r}). \tag{3.2}$$

In the perturbative Dyson expansion, we now apply Wick's theorem to contract field operators by pairs and make the propagators appear; the first terms are then represented by the three topologically distinct Feynman graphs of Fig. 1.

The second diagram 1(b) corresponds to a purely quantum exchange term; it decreases exponentially with the distance, on a characteristic scale depending on Planck's constant. (In the purely Coulomb nonrelativistic case [the electromagnetic interaction ''transported'' by the photon in Fig. 1(b) is then replaced by the Coulomb potential], the exchange term is Gaussian with a covariance length equal to the de Broglie thermal wavelength $\lambda_{\rm dB} = \hbar\sqrt{\beta/m}$ at low density.) The last diagram 1(c) is related to the first contribution to the mass renormalization of the fermions. Since the latest two have no classical equivalent, their contribution will be ignored in the following, and we shall restrict ourselves to the study of the first graph 1(a). As a matter of fact, all the renormalization processes are omitted here, assuming that our propagators and vertices are expressed, in the final expressions and at the order considered, in terms of the dressed masses and charges.

For technical reasons, we shall consider the Fourier transform of $\mathcal{J}^{\mu\nu}(\boldsymbol{r})$. The corresponding contribution of graph 1(a) is

$$\begin{aligned}\tilde{\mathcal{J}}^{\mu\nu}(\boldsymbol{k}) = \frac{e^4}{\beta^3} \sum_{l,l' \in \mathbb{Z}} \sum_{n\in\mathbb{Z}} \int\int \mathrm{tr}[\gamma^\alpha \mathcal{G}(\omega_l, \boldsymbol{p})\gamma^\mu \mathcal{G} \\ \times(\omega_l+\omega_n, \boldsymbol{p}+\hbar\boldsymbol{k})]\mathcal{D}_{\alpha\delta}(\omega_n, \hbar\boldsymbol{k}) \\ \times \mathrm{tr}[\gamma^\delta \mathcal{G}(\omega_{l'}, \boldsymbol{q})\gamma^\nu \\ \times \mathcal{G}(\omega_{l'}-\omega_n, \boldsymbol{q}-\hbar\boldsymbol{k})]\frac{d\boldsymbol{p}}{(2\pi\hbar)^3}\frac{d\boldsymbol{q}}{(2\pi\hbar)^3}.\end{aligned} \tag{3.3}$$

The calculation of this quantity is straightforward but tedious. After evaluating the traces, we have to sum over the fermionic indices l and l'. For this purpose, we can use complex integration and the method of residues. The positronic contributions can be ignored at this stage, by omitting some exponentially small terms proportional to $e^{-2\beta mc^2}$. The photonic index n can then be summed over, still using the method of residues. Some details of the calculation are given in Appendix A. The final formula for $\mathcal{J}^{\mu\nu}(\boldsymbol{r})$ can be evaluated in two different regimes, according to the fact that the distance r is larger or smaller than the thermal photon wavelength $\beta\hbar c$.

B. Behavior at short distances

In order to avoid some purely quantum effects due to the strong overlapping of the electron wave functions at short distances, we restrict the analysis to distances larger than the de Broglie thermal wavelength,

$$\lambda_{\rm dB} \stackrel{\rm def}{=} \sqrt{\frac{\beta\hbar^2}{m}}, \tag{3.4}$$

where a classical behavior of matter is expected. The condition $r \gg \lambda_{\rm dB}$ can be written in Fourier components, thanks to the duality between positions and momenta, as

$$\frac{\beta\hbar^2 \boldsymbol{k}^2}{m} \ll 1. \tag{3.5}$$

This condition will be used in the expansion of the expressions derived in Appendix A.

Now, another length appears in these expressions, which is the mean thermal wavelength of the photons $\lambda_{\text{photon}} \overset{\text{def}}{=} \beta\hbar c$. This length separates two crucially different behaviors of the correlation function. In this section, we derive expansions under the condition

$$r \ll \beta\hbar c \quad \text{or} \quad \beta\hbar c|\boldsymbol{k}| \gg 1. \tag{3.6}$$

We point out that the double condition $\lambda_{\text{dB}} \ll r \ll \lambda_{\text{photon}}$ can be fulfilled only for a weakly relativistic plasma, for $\lambda_{\text{dB}} \ll \lambda_{\text{photon}}$ is equivalent to $\beta mc^2 \gg 1$. We also use a third inequality which is

$$\hbar|\boldsymbol{k}| \ll |\boldsymbol{p}|. \tag{3.7}$$

This inequality is at first sight a bit surprising, for we have to integrate $\boldsymbol{p}$ over $\mathbb{R}^3$ in the expression of $\langle \boldsymbol{j}(\boldsymbol{0}) \cdot \boldsymbol{j}(\boldsymbol{r})\rangle$. However, the presence of the Maxwell-Boltzmann distribution in the integral and the volume factor $d\boldsymbol{p}$ imply that the leading contributions arise from momenta of order $\sqrt{m/\beta}$, which do satisfy Eq. (3.7) by virtue of Eq. (3.5).

We are therefore led to a double expansion of Eq. (3.3) in powers of the small parameters $\beta\hbar^2\boldsymbol{k}^2/m$ and $1/\beta\hbar c|\boldsymbol{k}|$. We stress that $\hbar$ appears at both the denominator and the numerator of these parameters, so *negative* powers of $\hbar$ will appear in the previous expansion. The presence of these inverse powers of $\hbar$ exemplifies that some retarded effective interactions do not have a well-behaved classical limit [$\hbar \to 0$] (see below.) The classical contribution can be found, *in fine*, by retaining the terms of order $\hbar^0$. Once this expansion is done, we find, at the orders $1/c^2$ and $\hbar^0$, and at the lowest order in the density,

$$\begin{aligned}\widetilde{\langle \boldsymbol{j}\cdot\boldsymbol{j}\rangle} &= c^2 \sum_{i=1}^{3} \mathcal{J}^i(\boldsymbol{k}) \\ &= 4\pi \int \frac{d\boldsymbol{p}}{(2\pi\hbar)^3} \int \frac{d\boldsymbol{q}}{(2\pi\hbar)^3} N_{\text{MB}}(\boldsymbol{p}) N_{\text{MB}}(\boldsymbol{q}) \\ &\quad \times \Bigg[\frac{4\beta e^4}{m^2\boldsymbol{k}^2}\boldsymbol{p}\cdot\boldsymbol{q} - \frac{4e^4}{m^3c^2\boldsymbol{k}^2}[\boldsymbol{p}^2+\boldsymbol{q}^2-(\boldsymbol{p}\cdot\hat{\boldsymbol{k}})^2-(\boldsymbol{q}\cdot\hat{\boldsymbol{k}})^2] \\ &\quad - \frac{4\beta e^4}{m^2\boldsymbol{k}^2}\Bigg\{ \frac{\boldsymbol{p}^2+\boldsymbol{q}^2}{2m^2c^2}\boldsymbol{p}\cdot\boldsymbol{q} + \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m^2c^2} \\ &\quad \times [\boldsymbol{p}\cdot\boldsymbol{q} - (\boldsymbol{p}\cdot\hat{\boldsymbol{k}})(\boldsymbol{q}\cdot\hat{\boldsymbol{k}})]\Bigg\} + \cdots \Bigg].\end{aligned} \tag{3.8}$$

In order to interpret this expression, we can first notice that

$$2\int N_{\text{MB}}(\boldsymbol{p}) \frac{d\boldsymbol{p}}{(2\pi\hbar)^3} = \rho_{\text{ideal}}^{\text{class}}, \tag{3.9}$$

where $\rho_{\text{ideal}}^{\text{class}}$ is the density of an ideal classical relativistic gas with fugacity μ. In the low density limit, we can identify ρ and $\rho_{\text{ideal}}^{\text{class}}$. After integration over the momenta $\boldsymbol{p}$ and $\boldsymbol{q}$, the Coulomb contribution of order $1/c^0$ vanishes. This was expected, for there are no current correlations in a classical Coulomb plasma. However, it should be noticed that we have neglected some quantum, nonrelativistic terms, of order $1/c^0$ and $\hbar^{2n}$. For the other classical terms, of order $1/c^2$, we can use the nonrelativistic expression (2.21) of $N_{\text{MB}}(\boldsymbol{p})$, which leads to

$$\widetilde{\langle \boldsymbol{j}\cdot\boldsymbol{j}\rangle} \simeq -8\pi \frac{\rho^2 e^4}{\beta m^2 c^2 \boldsymbol{k}^2}. \tag{3.10}$$

The inverse Fourier transform of Eq. (3.10) then gives

$$\langle \boldsymbol{j}(\boldsymbol{0}) \cdot \boldsymbol{j}(\boldsymbol{r})\rangle = -2\frac{e^4\rho^2}{\beta mc^2}\frac{1}{r} \quad \text{for} \quad \lambda_{\text{dB}} \ll r \ll \lambda_{\text{photon}}. \tag{3.11}$$

C. Behavior at large distances

At distances $r \gg \lambda_{\text{photon}}$, the contribution of graph 1(a) to the correlation function at order $\hbar^0$ and $1/c^2$ should decay exponentially fast as shown in Appendix A. Under the condition $\beta\hbar c\boldsymbol{k} \ll 1$, the expansion of $\widetilde{\mathcal{J}}(\boldsymbol{k})$ leads to an expression *regular* at $\boldsymbol{k}=\boldsymbol{0}$, at the orders $\hbar^0$ and $1/c^2$. For a given Matsubara frequency ω_n $(n \neq 0)$, the photon propagator is proportional to $1/(\omega_n^2 + \hbar^2 c^2 \boldsymbol{k}^2)$. It should lead, in the real world, to an exponentially decreasing contribution to $\mathcal{J}(\boldsymbol{r})$, over a typical length $\lambda_{\text{photon}}/n$. The sum of all these terms is therefore exponentially decreasing at the order considered. As for the static $(\omega_0 = 0)$ terms, their contributions to the correlations vanish in the classical limit, at large distances.

D. Interpretation in terms of classical descriptions

Now, we interpret the behaviors at short and large distances of the correlations in terms of classical models. As noticed in [6], whereas in the classical one-component plasma with Darwin interactions, algebraic tails appear in the current correlations, within the fully relativistic theory of classical fields such correlations vanish identically (see also [15].) The intermediate distance behavior may therefore be obtained within the simplified Darwin model, while the large distance behavior coincides with the predictions of the classical field theory.

1. TQFT at intermediate distances and the simplified Darwin model

The behavior (3.11) can be interpreted in terms of the simplified Darwin model. Let us recall what this model is.

A simple approach of weakly relativistic classical plasmas consists in considering classical point-particles, with charges e_i and masses m_i, interacting *via* a classical electromagnetic field. Any given charge is assumed to move in the total electromagnetic field created by all the other ones. This field involves retardation effects, as well as magnetic contributions, which can be expanded in powers of $1/c$, under weakly relativistic conditions. The corresponding Lorentz equations of motion for the charges, once truncated at order $1/c^2$, can be integrated into a Hamiltonian form, which mixes the positions $\boldsymbol{r}_i$ and momenta $\boldsymbol{\pi}_i$ of the particles. A simplified version of this Hamiltonian reads

$$\mathcal{H}_{\rm D}\overset{\rm def}{=}\sum_{i=1}^{N}\sqrt{m_i^2c^4+\boldsymbol{\pi}_i^2c^2}+\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\frac{e_ie_j}{|\boldsymbol{r}_i-\boldsymbol{r}_j|}-\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\frac{e_ie_j}{2m_im_jc^2|\boldsymbol{r}_i-\boldsymbol{r}_j|}[\boldsymbol{\pi}_i\cdot\boldsymbol{\pi}_j+(\boldsymbol{\pi}_i\cdot\boldsymbol{n}_{ij})(\boldsymbol{\pi}_j\cdot\boldsymbol{n}_{ij})]. \tag{3.12}$$

In this equation, $\boldsymbol{n}_{ij}$ is the unit vector between $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$. This Hamiltonian is the sum of a kinetic part, the classical Coulomb interaction energy, and an interaction energy term associated to the two-body relativistic Darwin potential of order $1/c^2$. The velocities are related to the momenta by the relation

$$\boldsymbol{v}_i=\frac{\boldsymbol{\pi}_i}{m_i}-\frac{\boldsymbol{\pi}_i^2}{2m_i^3c^2}\boldsymbol{\pi}_i-\sum_{\substack{i,j\\ i\neq j}}\frac{e_ie_j}{2m_im_jc^2|\boldsymbol{r}_i-\boldsymbol{r}_j|}[\boldsymbol{\pi}_j+\boldsymbol{n}_{ij}(\boldsymbol{\pi}_i\cdot\boldsymbol{n}_{ij})]+o(1/c^2). \tag{3.13}$$

A model of one-component plasma described by the simplified Darwin Hamiltonian (3.12) has been studied in previous papers [4,6]. The main result is that collective effects are responsible for a weak (and oscillating) screening of the Darwin interactions on a scale $\xi=\sqrt{mc^2/4\pi\rho e^2}$ at low densities [6]. This length is equal to the Debye screening length multiplied by $\sqrt{\beta mc^2}$.

In the considered regime, and at distances $r\ll\lambda_{\rm photon}$, no collective screening effects occur, since $\lambda_{\rm photon}\ll\xi$. Therefore, the first term of order e^4 and $1/c^2$ of the Darwin current correlation can be obtained from a straightforward expansion with respect to the interactions. In the grand canonical ensemble, the Darwin current correlation reads

$$\langle\boldsymbol{j}(\boldsymbol{0})\cdot\boldsymbol{j}(\boldsymbol{r})\rangle_{\rm Darwin}=\frac{e^2}{\Xi}\sum_{N=0}^{\infty}\frac{z^N}{(N-2)!}\int\delta(\boldsymbol{r}_1)\,\delta(\boldsymbol{r}_2-\boldsymbol{r})\times[\boldsymbol{v}_1\cdot\boldsymbol{v}_2]e^{-\beta(\mathcal{H}_0+\mathcal{A})}\prod_i\frac{d\boldsymbol{\pi}_i\,d\boldsymbol{r}_i}{(2\pi\hbar)^3}$$

for $r\neq 0$, where the $\boldsymbol{\pi}_i$'s are the canonical momenta of the particles, related to the velocities by Eq. (3.13), $\mathcal{H}_0$ is the free relativistic Hamiltonian

$$\mathcal{H}_0\overset{\rm def}{=}\sum_i\sqrt{m^2c^4+\boldsymbol{\pi}_i^2c^2},$$

$\mathcal{A}$ is the sum of the Coulomb and Darwin interaction energies, and Ξ is the grand-canonical partition function. We then use the perturbative expansion

$$\langle\boldsymbol{j}(\boldsymbol{0})\cdot\boldsymbol{j}(\boldsymbol{r})\rangle_{\rm Darwin}=\langle\boldsymbol{j}(\boldsymbol{0})\cdot\boldsymbol{j}(\boldsymbol{r})\rangle_0-\beta\langle\mathcal{A}\boldsymbol{j}(\boldsymbol{0})\cdot\boldsymbol{j}(\boldsymbol{r})\rangle_0+\cdots,$$

where $\langle\ \rangle_0$ is an equilibrium average with the free Hamiltonian $\mathcal{H}_0$. [The truncated term $\beta\langle\mathcal{A}\rangle_0\langle\boldsymbol{j}(\boldsymbol{0})\cdot\boldsymbol{j}(\boldsymbol{r})\rangle_0$ vanishes.] In the Fourier world, after straightforward calculations, this leads to

$$\widetilde{\langle\boldsymbol{j}\cdot\boldsymbol{j}\rangle}_{\rm Darwin}=4\pi\int\frac{d\boldsymbol{\pi}_1}{(2\pi\hbar)^3}\int\frac{d\boldsymbol{\pi}_2}{(2\pi\hbar)^3}N_{\rm MB}(\boldsymbol{\pi}_1)N_{\rm MB}(\boldsymbol{\pi}_2)\times\Bigg[\frac{4\beta e^4}{m^2\boldsymbol{k}^2}\boldsymbol{\pi}_1\cdot\boldsymbol{\pi}_2-\frac{4e^4}{m^3c^2\boldsymbol{k}^2}[\boldsymbol{\pi}_1^2+\boldsymbol{\pi}_2^2-(\boldsymbol{\pi}_1\cdot\hat{\boldsymbol{k}})^2-(\boldsymbol{\pi}_2\cdot\hat{\boldsymbol{k}})^2]-\frac{4\beta e^4}{m^2\boldsymbol{k}^2}\Bigg\{\frac{\boldsymbol{\pi}_1^2+\boldsymbol{\pi}_2^2}{2m^2c^2}\boldsymbol{\pi}_1\cdot\boldsymbol{\pi}_2+\frac{\boldsymbol{\pi}_1\cdot\boldsymbol{\pi}_2}{m^2c^2}[\boldsymbol{\pi}_1\cdot\boldsymbol{\pi}_2-(\boldsymbol{\pi}_1\cdot\hat{\boldsymbol{k}})(\boldsymbol{\pi}_2\cdot\hat{\boldsymbol{k}})]\Bigg\}+\cdots\Bigg], \tag{3.14}$$

which is exactly the same expression as the right-hand side of Eq. (3.8), if we identify $\boldsymbol{p}$ with $\boldsymbol{\pi}_1$ and $\boldsymbol{q}$ with $\boldsymbol{\pi}_2$. (This identification has no precise physical sense: the momentum associated with the first fermionic loop is $\boldsymbol{p}+\hbar\boldsymbol{k}$ as well as $\boldsymbol{p}$. The photon wave vector $\boldsymbol{k}$ being the Fourier dual of the relative position $\boldsymbol{r}$, this uncertainty on the momentum is nothing but the expression of Heisenberg's principle. Only the integrated quantity has a precise physical sense.) This shows that the perturbative expansion within QED which leads to Eq. (3.8) is isomorphic to an analogous perturbative expansion for the Darwin classical model in the phase space. (There is a subtle point to be taken into account here. In a perturbative expansion in powers of e^2, the nonperturbed quantities correspond to the *free* system. However, in the Hamiltonian description, the canonical momentum is an *intrinsic* object. Therefore the speeds of the particles in the presence of interactions are different from the speeds of the free particles, for a given set of canonical momenta. This must not be forgotten in the calculation of $\langle\boldsymbol{j}\cdot\boldsymbol{j}\rangle$). The effective electromagnetic interactions beyond Coulomb generated by the exchanged photons therefore correspond, at order $1/c^2$, to the transverse Darwin interactions, as expected.

Notice that the latter calculations have been performed within the Coulomb gauge. This point is crucial if we want to have the same structure in Eqs. (3.8) and (3.14). Indeed, although the present Feynman graph, once integrated, represents a physical quantity (and therefore gauge invariant), on the contrary the expression of the integrand depends on the gauge choice, for in the Hamiltonian method the relevant parameter is the canonical momentum, the velocities and current depending, at *fixed momenta*, on the gauge. The expression of the Feynman graph 1(a) in, say, the Feynman gauge, has therefore a different integrand than Eq. (3.8), but the difference of course vanishes after integration over the canonical momenta. The reader may remember that the construction of the Darwin Hamiltonian is made within the Coulomb (transverse) gauge; this explains the similarity between the structures involved in Eqs. (3.8) and (3.14).

2. TQFT at large distances and the classical field theory

At large distances, the Darwin models appear as rather questionable, already in their foundations. Indeed (see [in French] [16] for a detailed discussion), in order to construct the Darwin Lagrangian, one expresses the electromagnetic field created by the particles at the position of particle i at time t, in terms of the coordinates, velocities, and accelerations of the particle j at a retarded time $(t-\tau_{ij})$, defined by the well-known implicit equation

$$\tau_{ij}=\frac{|\boldsymbol{r}_i(t)-\boldsymbol{r}_j(t-\tau_{ij})|}{c}. \tag{3.15}$$

To truncate the equations of motion at a given order in $1/c$ is equivalent to taking, formally, a limit $[c\rightarrow\infty]$. Some trouble may now arise when looking at large-distance behaviors of correlation functions, for the limits $[r\rightarrow\infty]$ and $[c\rightarrow\infty]$, obviously, do not commute in Eq. (3.15). This shows that the Darwin models may provide spurious predictions at large distances. As a matter of fact, different Darwin models lead to contradictory behaviors at large distances.

In fact, at large distances, the electromagnetic field can be treated as a classical object, and the fully relativistic classical field theory (CFT) should provide the correct behaviors of interest. Let us consider a model of classical point particles, interacting with a classical electromagnetic field. This system can be described by a Hamiltonian $\mathcal{H}_{\text{CFT}}$, a function of the positions and momenta of the particles $(\boldsymbol{r}_i,\boldsymbol{\pi}_i)$, and generalized positions and momenta (q_α,p_α) for the transverse electromagnetic field. The index α is a multiple index $\alpha=(\boldsymbol{k},\boldsymbol{\varepsilon})$ running over a momentum $\boldsymbol{k}\in\mathbb{R}^3$ and two polarization vectors $\boldsymbol{\varepsilon}(\boldsymbol{k})$ and $\boldsymbol{\varepsilon}'(\boldsymbol{k})$, orthogonal to $\boldsymbol{k}$ and to each other. The classical canonical partition function reads

$$\mathcal{Z}=\int e^{-\beta\mathcal{H}_{\text{CFT}}}\prod_{i,\alpha}\frac{d\boldsymbol{\pi}_i\,d\boldsymbol{r}_i}{(2\pi\hbar)^3}\,\frac{dp_\alpha\,dq_\alpha}{2\pi\hbar}, \tag{3.16}$$

where the semiclassical counting rule is applied to the canonical volume elements $d\boldsymbol{\pi}_i\,d\boldsymbol{r}_i$ and $dp_\alpha\,dq_\alpha$. One can show [15] that it can be factorized into two contributions

$$\mathcal{Z}=\mathcal{Z}^{\text{matter}}_{\text{Coulomb}}\times\mathcal{Z}^{\text{class}}_{\text{trans}}. \tag{3.17}$$

The second factor is the classical partition function of the free transverse electromagnetic field. The first factor is the classical partition function of Coulomb matter, and it determines all the statistical properties of matter. Therefore, in classical field theory, all the electromagnetic forces beyond Coulomb do not affect equilibrium properties, and classical Coulomb matter is entirely decoupled from the classical transverse electromagnetic field. In particular, the current correlations identically vanish [15].

Nevertheless, the previous analysis fails at short distances since the classical partition function of the free transverse electromagnetic field is known to suffer from the so-called *ultraviolet divergence*. In fact, a classical treatment of a harmonic oscillator of pulsation ω in the framework of statistical mechanics is valid only if the energy gap $\hbar\omega$ is much smaller than the thermal energy $k_{\text{B}}T$, that is to say, $\beta\hbar c|\boldsymbol{k}|\ll 1$ since here $\omega=c|\boldsymbol{k}|$. We are therefore led to the conclusion that the classical field theory approach is justified only at interparticle distances $r\gg\beta\hbar c$. At these distances, the classical decoupling between matter and radiation holds, in agreement with the result obtained within the framework of QED. All the classical effects beyond Coulomb present in the Darwin model are therefore due to the quantum nature of the thermalization process of the electromagnetic field, as explained in Sec. III F.

E. Extrapolation to all the graphs

The phenomenon of thermal screening, characterized by the screening length $\lambda_{\text{photon}}=\beta\hbar c$, certainly occurs in all the Feynman graphs that enter in any equilibrium quantity.

For instance, we can compute the charge-charge correlation function, given by $\mathcal{J}^{00}$. The calculation is somewhat similar to that of the current correlations, and the result is similar: at intermediate distances, one recovers the Darwin correction (at a formal level at least, for it vanishes by parity after integration over the momenta), whereas at large distances, the transverse corrections do vanish exponentially fast in the classical limit.

Actually, if one writes any Feynman graph in the momentum representation, with the thermodynamical conditions (2.17), one may think of the fermionic loops as classical objects, with well defined canonical momenta and positions, interacting *via* effective interactions generated by the photon propagators. The fact that *classical* matter interacts only *via* Coulomb interactions at large distances (except for exponentially decaying terms $e^{-r/\lambda_{\text{photon}}}$) should be true at all orders in $1/c$, and not only at order $1/c^2$ as shown by the CFT analysis. The effective interactions beyond Coulomb have therefore a purely quantum nature at large distances, and should decrease algebraically, faster than $1/r$, with amplitudes proportional to *positive* powers of $\hbar$ since they are related to quantum fluctuations of the positions of the charges. These quantum fluctuations should induce algebraic tails in the spatial correlation functions, as already shown in [17–19] in the purely Coulomb case.

Naturally, there also appears effective interactions of order larger than $1/c^2$, even in the simple graph 1(a). These interactions may be proportional to *negative* powers of $\hbar$. Indeed, in the Darwin ''window'' $\lambda_{\text{dB}}\ll r\ll\lambda_{\text{photon}}$, expansions are made in powers of the small parameters

$$\sqrt{\frac{\beta\hbar^2}{m}}|\boldsymbol{k}| \quad\text{and}\quad \frac{1}{\beta\hbar c|\boldsymbol{k}|}. \tag{3.18}$$

At sufficiently high orders in $1/c$, negative powers of $\hbar$ will appear. Their classical limit $(\hbar\rightarrow 0)$ is therefore not defined. This is in part related to the impossibility of constructing a Hamiltonian, within the framework of a classical Darwin-like theory (i.e., where the electromagnetic field degrees of freedom are eliminated in favor of the coordinates of the particles), with effective interactions of order higher than $1/c^2$ that only depend on the positions and momenta of the particles (see [4] for a detailed discussion).

At last, at very short distances $r<\lambda_{\text{dB}}$, matter should be treated by quantum mechanics. In this regime, the first relativistic effective electromagnetic interactions between quantum electrons should keep the same form as in the Darwin window, with purely quantum terms as in the so-called Breit Hamiltonian [3,2].

F. Dynamical interpretation of thermal screening

The fact that the electromagnetic field behaves as a classical degree of freedom at large distance $r\gg\lambda_{\text{photon}}$ implies that its contributions to equilibrium static quantities are purely configurational, and do not depend on its dynamical properties. Then, the previous CFT result appears as a direct consequence of this remarkable property of classical statistical mechanics. At the level of the real-time evolution of the system, it is tempting to interpret this thermal screening as follows. The photons which carry the electromagnetic interactions between two electrons separated by a sufficiently large distance are thermalized by diffusion, absorption, and emission. Therefore, they are unable to keep memory of the dynamical configuration of the emitting charge, and the resulting generated effective interactions (beyond the residual Coulomb potential) vanish. However, we stress that, if the presence of matter is indeed crucial in the thermalization process of the photons, the screening length that appear in the previous decorrelation mechanism is entirely controlled by quantum-mechanical properties of the photons : it reduces to $\lambda_{\rm photon}$, at least at sufficiently low density. At distances $r \ll \lambda_{\rm photon}$, the dynamical contributions of the electromagnetic field to equilibrium quantities cannot be disentangled from purely static ones, as that of any quantum degree of freedom. The corresponding effective interactions then incorporate dynamical features, which turn out to be those predicted from a classical analysis of the real-time dynamics of the electromagnetic field. The occurrence of the Darwin potential in these interactions illustrates the general relation in quantum statistical mechanics between static quantities and their classical dynamical counterparts. Similarly to the case $r \gg \lambda_{\rm photon}$, and with the same restrictions, we can give a dynamical interpretation of the behavior at distances $r \ll \lambda_{\rm photon}$. At sufficiently short distances, the photons exchanged between two electrons do not have time to be thermalized, and the Darwin approach is legitimate.

G. Darwin regimes of validity

According to the previous results, we can anticipate a brief discussion on the regime of the validity of the Darwin approach. This discussion will be detailed in a forthcoming paper. A first condition is, of course, that matter is almost classical and weakly relativistic, i.e., as shown in Sec. II C:

$$k_B T \ll |\mu^*| \ll mc^2.$$

A second one is that, for a given spatial configuration of the particles, the relative distances must be in the Darwin window

$$\sqrt{\frac{\beta\hbar^2}{m}} \ll r \ll \beta\hbar c. \tag{3.19}$$

When calculating the excess pressure or free energy in a Darwin plasma [6], contributions from the whole space intervene. Some of them are reliable (in the sense that they make physical sense for real matter), the others are not. The Darwin model is therefore more reliable if the Darwin window is large compared to the other characteristic length scales. In fact, the equilibrium state of our system depends on three dimensionless parameters: $\alpha = e^2/\hbar c$ (strength of relativistic electromagnetic interactions beyond Coulomb), $\lambda_{\rm dB}/a$ (strength of quantum effects on matter), and $\Gamma = \beta e^2/a$ (strength of Coulomb interactions in thermal units), where a is the interparticle mean distance $a = (3/4\pi\rho)^{1/3}$. A weakly relativistic and weakly degenerate state obviously corresponds to small values of α and $\lambda_{\rm dB}/a$ [notice that $k_B T/mc^2 = (1/\Gamma^2)\alpha^2(\lambda_{\rm dB}/a)^2$]. The corresponding Darwin window indeed becomes large, and we will show that the reliable Darwin contributions appear in expansions of the real quantities (described by thermal QED) with respect to $(\lambda_{\rm dB}/a)$ and α, at fixed values of Γ. At a formal level, this regime can be obtained by setting m and c to infinity for instance. In practice, an equivalent situation will be obtained at low temperatures and low densities that determine the thermodynamical regime of validity of the Darwin approach.

IV. CONCLUSION

By studying the current correlations in a plasma described by thermal quantum electrodynamics, we have exhibited a phenomenon of thermal screening of the transverse interactions on a scale $\lambda_{\rm photon} = \beta\hbar c$. We stress that this screening has a completely different nature than the Debye screening, for instance, which is a many-body classical effect. In our case, the thermal screening appears already in a single Feynman graph (the sole ''collective'' effects responsible for this screening are therefore the ones hidden in the thermalization) with two fermionic loops. Our study also enlightens the ambiguities linked to the definition of a classical (nonquantum) and nonrelativistic limit for electromagnetically interacting systems. This is related to the ill-defined behavior of the photon thermal wavelength $\beta\hbar c$ when one takes both limits $[\hbar \to 0]$ and $[c \to \infty]$. (Actually, $\hbar$ controls the quantum effects for matter, as well as the thermodynamics of radiation, that has no well-behaved classical limit. If one could distinguish two different ''$\hbar$,'' say a blue one for matter and a red one for radiation, the classical treatment of matter would appear as an expansion in powers of $\hbar_{\rm blue}$ for a fixed $\hbar_{\rm red}$, and in order to obtain the classical contributions of matter, we would keep the zeroth-order terms in $\hbar_{\rm blue}$. However, somebody set $\hbar_{\rm blue} = \hbar_{\rm red}$ for the real world.)

The thermal screening is responsible for the failure of the Darwin models at sufficiently large distances. The present study also suggests that some thermodynamical predictions of the Darwin models, involving contributions from the window $\lambda_{\rm dB} \ll r \ll \lambda_{\rm photon}$, are expected to be relevant pieces of the relativistic corrections to the Coulomb quantities for real plasmas. (This analysis is confirmed by the study of the excess pressure, which will be published in a forthcoming article.)

Several Darwin models have actually been studied in the literature, in particular by Krizan [5,11,7], and by Kosachev and Trubnikov [9,8] (for a review, see [20]). The difference between these models lies in the way of truncating the expression of the velocities in terms of the canonical momenta when deriving the various Darwin Hamiltonians from the same Lagrangian. Consequently, their model Hamiltonians differ from terms of order $1/c^4$ and higher [4]. Nevertheless, within each model the predictions at large distances differ considerably: whereas in Krizan's model (the same as the one exposed here) the correlations decrease as $1/r$ (with oscillations), in Trubnikov's model the screening is slightly more efficient, and they decrease faster, namely as $1/r^3$ (without oscillations). Furthermore, the excess free energies are different at order $1/c^3$. These crucial differences at lowest orders in $1/c^2$ between two models, the Hamiltonians of which differ only at order $1/c^4$, can be explained by the fact that the corresponding screened contributions involve convolutions of an indefinite number of Darwin potentials, and thus a sum of terms of arbitrary orders in $1/c$ [4,6].

Some authors argued about which model was the best to describe a real plasma. According to our study, it appears clearly that the large-distance behavior of the screened Darwin potential is not physically relevant, for this screening occurs, in both models, at a distance ξ which is much larger than $\lambda_{\text{photon}}=\beta\hbar c$ in the considered regime. [The ratio $\lambda_{\text{photon}}/\xi$ is equal to $(\lambda_{\text{dB}}/a)\sqrt{\Gamma}$ in the low density and low temperature regime. This parameter is therefore small in expansions with respect to (λ_{dB}/a) at fixed Γ.] This should end the old battle between the two opposite sides.

APPENDIX A: THE FEYNMAN GRAPH

The aim of this appendix is to provide some details about the calculation of the Feynman graph shown in Fig. 1(a). First, we write the fermionic propagator as an integral,

$$\mathcal{G}(\omega_l,\boldsymbol{p})=\frac{(\not{\mathbf{p}}+mc^2)}{(i\omega_l+\mu)^2-E^2(\boldsymbol{p})}=\int_{-\infty}^{+\infty}\frac{(\not{\mathbf{P}}+mc^2)\rho_{\text{ms}}(\mathbf{P})}{p^0-(i\omega_l+\mu)}\,\frac{dp^0}{2\pi}, \tag{A1}$$

where $\mathbf{p}=(i\omega_l+\mu,c\boldsymbol{p})$ and $\mathbf{P}\overset{\text{def}}{=}(p^0,c\boldsymbol{p})$, and the mass-shell density function is defined by

$$\rho_{\text{ms}}(\mathbf{P})\overset{\text{def}}{=}2\pi\varepsilon(p^0)\,\delta(\mathbf{P}^2-m^2c^4) \tag{A2}$$

and the ε function is just $\varepsilon(p^0)\overset{\text{def}}{=}p^0/|p^0|$.

We can then write the correlation function as

$$\tilde{\mathcal{J}}^{\mu\nu}(\boldsymbol{k})=\sum_{n\in\mathbb{Z}}\Pi^{\mu\alpha}(\omega_n,\boldsymbol{k})\mathcal{D}_{\alpha\delta}(\omega_n,\boldsymbol{k})\Pi^{\delta\nu}(\omega_n,\boldsymbol{k}), \tag{A3}$$

where $\Pi(\omega_n,\boldsymbol{k})$ is the first-order term (in e^2) of the photon self-energy, namely [using Eq. (A1)]

$$\Pi^{\mu\nu}(\omega_n,\boldsymbol{k})=\frac{e^2}{\beta}\sum_{l\in\mathbb{Z}}\text{tr}\int\left\{\gamma^\mu\frac{\not{\mathbf{P}}+mc^2}{i\omega_l+\mu-p^0}\gamma^\nu\times\frac{\not{\mathbf{P}}'+mc^2}{i(\omega_l-\omega_n)+\mu-p'^0}\right\}\rho_{\text{ms}}(\mathbf{P})\,\rho_{\text{ms}}(\mathbf{P}')\times\frac{d\boldsymbol{p}}{(2\pi\hbar)^3}\,\frac{dp^0}{2\pi}\,\frac{dp'^0}{2\pi}, \tag{A4}$$

where we have set

$$\mathbf{P}\overset{\text{def}}{=}(p^0,c\boldsymbol{p})\quad\text{and}\quad\mathbf{P}'\overset{\text{def}}{=}(p'^0,c\boldsymbol{p}-\hbar c\boldsymbol{k}). \tag{A5}$$

We now perform the summation over the fermionic indices l and l'. Using integration in the complex plane, one can show that, for any meromorphic function g on $\mathbb{C}$ with vanishing residue at ∞ (i.e. decreasing faster than $1/z^2$ at infinity), and holomorphic on an open set containing the line $\{z\in\mathbb{C};\ \text{Re}z=\mu\}$, the following formula holds:

$$\frac{1}{\beta}\sum_{l\in\mathbb{Z}}g(z=i\omega_l+\mu)=\sum_a\ \text{Res}\{g(a)\}N_{\text{F}}(a-\mu), \tag{A6}$$

where a runs over the set of poles of g, and where the Fermi distribution is

$$N_{\text{F}}(x)\overset{\text{def}}{=}\frac{1}{e^{\beta x}+1}.$$

In particular, this formula gives

$$\frac{1}{\beta}\sum_{l\in\mathbb{Z}}\frac{1}{[i\omega_l+\mu-p^0]}\,\frac{1}{[i(\omega_l-\omega_n)+\mu-p'^0]}=-\frac{N_{\text{F}}(p^0-\mu)-N_{\text{F}}(p'^0-\mu)}{i\omega_n-p^0+p'^0}. \tag{A7}$$

Now we have to sum over the bosonic index n. Still using complex integration, we obtain the following expression, involving the Bose distribution $N_{\text{B}}(x)\overset{\text{def}}{=}1/(e^{\beta x}-1)$:

$$\begin{aligned}\sum_{n\in\mathbb{Z}}&\frac{1}{[i\omega_n-(q^0-q'^0)]}\,\frac{1}{[i\omega_n-(p^0-p'^0)]}\,\frac{1}{[\omega_n^2+\hbar^2c^2\boldsymbol{k}^2]}\\
&=\frac{N_{\text{B}}(q^0-q'^0)}{[q^0-q'^0+p^0-p'^0]}\,\frac{1}{(q^0-q'^0)^2-\hbar^2c^2\boldsymbol{k}^2}\\
&\quad-\frac{N_{\text{B}}(p'^0-p^0)}{[q^0-q'^0+p^0-p'^0]}\,\frac{1}{(p^0-p'^0)^2-\hbar^2c^2\boldsymbol{k}^2}\\
&\quad+\frac{N_{\text{B}}(\hbar c|\boldsymbol{k}|)}{2\hbar c|\boldsymbol{k}|}\,\frac{1}{(\hbar c|\boldsymbol{k}|-q^0+q'^0)(\hbar c|\boldsymbol{k}|+p^0-p'^0)}\\
&\quad-\frac{N_{\text{B}}(-\hbar c|\boldsymbol{k}|)}{2\hbar c|\boldsymbol{k}|}\,\frac{1}{(\hbar c|\boldsymbol{k}|+q^0-q'^0)(\hbar c|\boldsymbol{k}|-p^0+p'^0)}.\end{aligned} \tag{A8}$$

We can now use the formulas (A7) and (A8) together with Eqs. (A3) and (A4) to obtain the (exact) formula giving $\tilde{\mathcal{J}}(\boldsymbol{k})$ at the first order in α. This formula involves (by integrating the ρ distribution) summations on the signs of q^0, q'^0, p^0, and p'^0.

It should be noticed that the set $\{p^0>0$ and $p'^0>0\}$ gives a purely electronic contribution, $\{p^0<0$ and $p'^0<0\}$ gives a purely positronic contribution (therefore vanishing in the limit considered), while mixed contributions $\{p^0>0$ and $p'^0<0\}$ or $\{p^0<0$ and $p'^0>0\}$ give both electronic *and* positronic contributions, and therefore must not be forgotten.

First we can evaluate the previous expression of $\tilde{\mathcal{J}}(\boldsymbol{k})$ in the regime where $\beta\hbar c|\boldsymbol{k}|\gg 1$ [which corresponds, *via* the Fourier transform, to the main contributions to $\mathcal{J}(\boldsymbol{r})$ at intermediate distances $\lambda_{\text{dB}}\ll r\ll\lambda_{\text{photon}}$]. The small dimensionless parameters of our expansions are therefore

$$\frac{1}{\beta \hbar c|\boldsymbol{k}|} \ll 1 \quad \text{and} \quad \beta \frac{\hbar^2 \boldsymbol{k}^2}{2m} \ll \beta \hbar \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{m} \ll 1.$$

In order to describe a weakly relativistic and weakly degenerate system, the following parameters are also considered as small:

$$\frac{1}{\beta m c^2} \ll 1 \quad \text{and} \quad e^{\beta\mu^*} \ll 1.$$

We therefore have $N_{\mathrm{B}}(\hbar c|\boldsymbol{k}|) \simeq 0$ while $N_{\mathrm{B}}(-\hbar c|\boldsymbol{k}|) \simeq -1$ up to exponentially small terms. Moreover, limit expressions of N_{B} must be taken, according to the sign of p^0 and p'^0. For instance, if p^0 and p'^0 are both positive, $N_{\mathrm{B}}(p^0 - p'^0)$ can be expanded as a series, the first term of which is $N_{\mathrm{B}}(\hbar \boldsymbol{p}\cdot\boldsymbol{k}/m) \simeq m/\hbar \boldsymbol{p}\cdot\boldsymbol{k}$ (higher-order terms should be taken into account in the global formula, due to a cancellation mechanism). If p^0 and p'^0 have opposite signs, we can write $N_{\mathrm{B}}(2mc^2) \simeq 0$ and $N_{\mathrm{B}}(-2mc^2) \simeq -1$, the omitted terms being *exponentially* small. Some denominators are expanded using

$$\frac{\hbar \boldsymbol{p}\cdot\boldsymbol{k}}{m} \ll \hbar c|\boldsymbol{k}| \quad \text{or} \quad \hbar c|\boldsymbol{k}| \ll mc^2.$$

(The second inequality comes from $\hbar^2 \boldsymbol{k}^2/m^2c^2 = [\beta\hbar^2\boldsymbol{k}^2/m] \times [1/\beta m c^2]$, which is the product of two small parameters. It also means that the photon energy $\hbar c|\boldsymbol{k}|$ is too small to create electron/positron pairs.) At last, the Fermi distribution $N_{\mathrm{F}}(p^0 - \mu)$ is zero up to exponentially small terms if $p^0 < 0$, and reduces to the Maxwell-Boltzmann distribution in the desired limit otherwise. After expanding the whole expression with respect to the small parameters, we find the expression (3.8) which is proportional to $1/\boldsymbol{k}^2$. This behavior corresponds to a slow decrease (in $1/r$) of $\mathcal{J}(\boldsymbol{r})$ at intermediate distances $\lambda_{\mathrm{dB}} \ll r \ll \lambda_{\mathrm{photon}}$.

We now evaluate $\tilde{\mathcal{J}}(\boldsymbol{k})$ in the regime where $\beta\hbar c|\boldsymbol{k}|$ is a small parameter. We perform expansions of $\tilde{\mathcal{J}}(\boldsymbol{k})$ with respect to the following small parameters:

$$\beta \frac{\hbar^2 \boldsymbol{k}^2}{2m} \ll \beta \hbar \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{m} \ll \beta \hbar c|\boldsymbol{k}| \ll 1$$

and

$$\frac{1}{\beta m c^2} \ll 1, \quad e^{\beta\mu^*} \ll 1.$$

We write $N_{\mathrm{B}}(\hbar c|\boldsymbol{k}|)$ as an algebraic series, the first term of which is $1/\beta\hbar c|\boldsymbol{k}|$ (higher order terms must also be taken into account). After expansion, the singular terms, proportional to $1/\boldsymbol{k}^2$, cancel out. At the order $\hbar^0$, the correlation function is therefore regular at $\boldsymbol{k} = \boldsymbol{0}$ and can be expanded in positive powers of $\boldsymbol{k}^2$. In the ''real'' world, we can therefore expect a rapidly decreasing behavior of the current correlation at large distances.

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