Time delayed chaotic systems and their synchronization

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Synchronization of chaotic systems can be used in communications and control. However, time delay phenomena frequently appear in applications. Time delay can defy synchronization in coupled systems with simple connection. This paper has presented a complex structure of coupled chaotic systems. With this structure it is shown that synchronization can exist in coupled chaotic systems with time delay.
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I. INTRODUCTION

Since chaotic synchronization was found, it has attracted the attention of many researchers. Basically, there are two types of synchronous chaotic systems. One is a unidirectional [1–3] system. A unidirectional system consists of a driving system and a response system. The information of the response system will not be sent back to the driving system. Another type is coupled chaotic systems [4–7] where one system sends driving signals to other systems and, in the meantime, receives driving signals from other systems. In this case, synchronization can exist in those subsystems that do not explicitly contain driving signals. The potential applications of synchronization in communications [8–14] and control [15] have been studied. However, in the applications, it may be in such situations: information exchanged among systems is time delayed information. For example, there exists time delay in very long distance communications. Time delay naturally raises one question: are the chaotic systems still synchronized when their driving signals are time delayed signals?

In general, time delay increases the dimensionality, and hence the complexity, of dynamical systems, and most efforts have focused on those domains where time delay is not a major factor [16]. Thus, the ubiquitous nature of time delay leads naturally to the subject of this paper: an exploration of the dynamic behavior in the synchronous chaotic systems with time delay.

Usually, time delay will defy synchronization in coupled chaotic systems because time delay changes the structures of coupled systems and it is hard to get synchronization in nonidentical subsystems. However, when special structures of coupled systems are established, perfect synchronization can be obtained.

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II. TIME DELAYED UNIDIRECTIONAL SYSTEMS

Setup of synchronization in unidirectional chaotic systems is simple. Consider a system of n first-order ordinary differential equations:

$$\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x}), \tag{1}$$

where $\dot{\mathbf{x}}=d\mathbf{x}/dt$; $\mathbf{x}$ and $\mathbf{f}$ are the vectors $\mathbf{x}=(\mathbf{x}_1(t),\ldots,\mathbf{x}_n(t))$, $\mathbf{f}=(\mathbf{f}_1(t,\mathbf{x}),\ldots,\mathbf{f}_n(t,\mathbf{x}))$. As shown in [1,2], the system (1) is subdivided into two subsystems $\mathbf{u}$ and $\mathbf{v}$. With these subsystems, a driving system is given by

$$\begin{aligned}\dot{\mathbf{u}}&=\mathbf{h}(t,\mathbf{u},\mathbf{v}),\\ \dot{\mathbf{v}}&=\mathbf{g}(t,\mathbf{u},\mathbf{v}),\end{aligned} \tag{2}$$

and a response system is given by

$$\begin{aligned}\dot{\mathbf{u}}'&=\mathbf{u},\\ \dot{\mathbf{v}}'&=\mathbf{g}(t,\mathbf{u}',\mathbf{v}').\end{aligned} \tag{3}$$

Suppose that the driving signal in Eqs. (3) is a time delayed driving signal, i.e., the system (2) sends the driving signal $\mathbf{u}$ at time t and the system (3) receives the driving signal $\mathbf{u}$ at time $t-\delta$, where δ is delayed time. Usually, δ is small. Actually, it may be very common in practice that a response system needs small time to get information from a driving system, or the response system needs small time to respond to the driving system. For the purpose of convenience, denote time delayed components with subscript δ, where δ is the length of delayed time. For example, $\mathbf{u}_\delta$ represents the value of $\mathbf{u}(t)$ at time $(t-\delta)$, i.e., $\mathbf{u}(t-\delta)$. Using this notation, the response system with time delay is written as

$$\dot{\mathbf{v}}'=\mathbf{g}(t,\mathbf{u}_\delta,\mathbf{v}'). \tag{4}$$

The solution of $\mathbf{v}$ in the system (2) at time $t-\delta$ is denoted by $\mathbf{v}(t-\delta)$. Obviously, if the systems (2) and (3) are synchronized, the solution $\mathbf{v}'(t)$ of Eq. (4) synchronizes with the solution $\mathbf{v}(t-\delta)$ of Eq. (2).

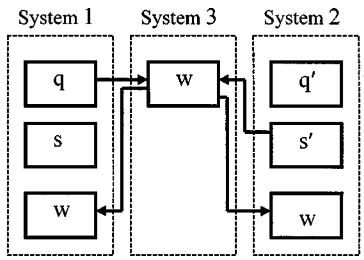

FIG. 1. The block diagram of synchronous coupled systems. These systems can be synchronized with time delay driving signals.

III. SYNCHRONIZATION OF TIME DELAYED COUPLED SYSTEMS

For the purpose of convenience, coupled systems are generated from the system (1). Divide the system (1) into three subsystems:

$$\dot{\mathbf{q}} = \Phi(t,\mathbf{q},\mathbf{r},\mathbf{s}),$$
$$\dot{\mathbf{r}} = \psi(t,\mathbf{q},\mathbf{r},\mathbf{s}), \tag{5}$$
$$\dot{\mathbf{s}} = \Theta(t,\mathbf{q},\mathbf{r},\mathbf{s}).$$

Now, $\mathbf{x}$ is decomposed into $\mathbf{q}$, $\mathbf{r}$, and $\mathbf{s}$. A coupled system can be created from the above subsystems. It consists of two systems. The first system is given by

$$\dot{\mathbf{q}} = \boldsymbol{\Phi}(t,\mathbf{q},\mathbf{r},\mathbf{s}),$$
$$\mathbf{r} = \mathbf{r}', \tag{6}$$
$$\dot{\mathbf{s}} = \boldsymbol{\Theta}(t,\mathbf{q},\mathbf{r},\mathbf{s}),$$

and the second system is given by

$$\mathbf{q}' = \mathbf{q},$$
$$\dot{\mathbf{r}} = \psi(t,\mathbf{q}',\mathbf{r}',\mathbf{s}'), \tag{7}$$
$$\dot{\mathbf{s}}' = \Theta(t,\mathbf{q}',\mathbf{r}',\mathbf{s}').$$

In general, despite of different initial conditions of the systems (6) and (7), under certain conditions the above two systems can be synchronized. However, when the information exchanged between the above two systems are time delayed signals, such as $\mathbf{q}_\delta$ or $\mathbf{r}_\delta$, they will no longer be synchronized if these systems are chaotic, because the subsystems, which do not explicitly contain driving signals, are not identical due to time delay.

In order to get synchronous coupled chaotic systems with time delay, a synchronous structure of coupled system, which has identical subsystems that do not explicitly contain driving signals, can be created, as described in Fig. 1. Two systems exchange information through the third system. They can be written as follows.

System 1,

$$\dot{\mathbf{q}} = \mathbf{h}(t,\mathbf{q},\mathbf{s},\mathbf{w}),$$
$$\dot{\mathbf{s}} = \mathbf{g}(t,\mathbf{q},\mathbf{s},\mathbf{w}). \tag{8}$$

System 2,

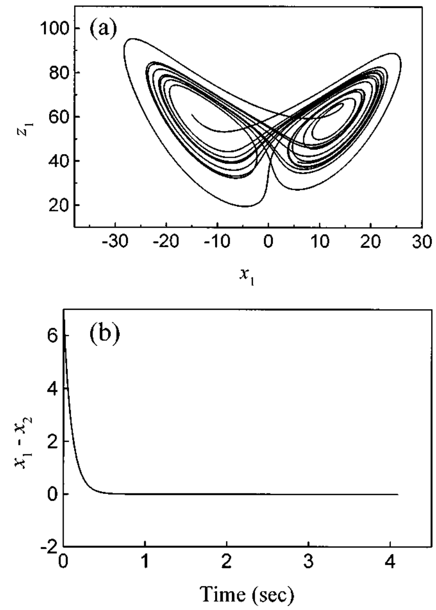

FIG. 2. (a) The trajectory of the system (11) in x_1 and z_1 phase plane. (b) The synchronization between the systems (11) and (12). $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$.

$$\dot{\mathbf{q}}' = \mathbf{h}(t,\mathbf{q}',\mathbf{s}',\mathbf{w}),$$
$$\dot{\mathbf{s}}' = \mathbf{g}(t,\mathbf{q}',\mathbf{s}',\mathbf{w}). \tag{9}$$

System 3,

$$\dot{\mathbf{w}} = \mathbf{p}(t,\mathbf{q},\mathbf{s}',\mathbf{w}). \tag{10}$$

Consider that the systems (8), (9), and (10) are in chaos and their initial conditions are different. Look at an example constructed from the Lorenz equation.

System 1,

$$\dot{x}_1 = \sigma(y - x_1),$$
$$\dot{z}_1 = x_1 y - \beta z_1. \tag{11}$$

System 2,

$$\dot{x}_2 = \sigma(y - x_2),$$
$$\dot{z}_2 = x_2 y - \beta z_2. \tag{12}$$

System 3,

$$\dot{y} = \rho x_1 - y - x_1 z_2. \tag{13}$$

For $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$ these systems are chaotic, as Fig. 2(a) shows. Figure 2(b) shows that the systems (11) and (12) are synchronized in spite of their different initial conditions. Now, consider that the driving signals exchanged among the systems (8), (9) and (10) are time delayed signals, i.e., the third system receives the signals $\mathbf{q}_\delta$ and $\mathbf{s}'_\delta$ at time $t-\delta$ instead of $\mathbf{q}$ and $\mathbf{s}'$ at time t. Such systems can be written as follows.

System 1,

$$\dot{\mathbf{q}} = \mathbf{h}(t,\mathbf{q},\mathbf{s},\mathbf{w}), \tag{14}$$

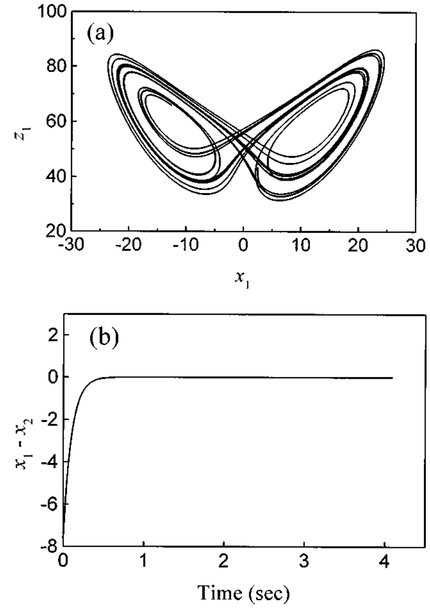

FIG. 3. (a) The chaotic trajectory of the system (17). (b) The synchronization between the systems (17) and (18). $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$, and the delayed time δ is 0.001 sec.

$$\dot{\mathbf{s}}=\mathbf{g}(t,\mathbf{q},\mathbf{s},\mathbf{w}).$$

System 2,

$$\dot{\mathbf{q}}'=\mathbf{h}(t,\mathbf{q}',\mathbf{s}',\mathbf{w}),\quad \dot{\mathbf{s}}'=\mathbf{g}(t,\mathbf{q}',\mathbf{s}',\mathbf{w}). \tag{15}$$

System 3,

$$\dot{\mathbf{w}}=\mathbf{p}(t,\mathbf{q}_\delta,\mathbf{s}'_\delta,\mathbf{w}). \tag{16}$$

An example is given as follows.
System 1,

$$\dot{x}_1=\sigma(y-x_1),\quad \dot{z}_1=x_1y-\beta z_1. \tag{17}$$

System 2,

$$\dot{x}_2=\sigma(y-x_2),\quad \dot{z}_2=x_2y-\beta z_2. \tag{18}$$

System 3,

$$\dot{y}=\rho x_{1\delta}-y-x_{1\delta}z_{2\delta}. \tag{19}$$

System 3 receives the time delayed signals $x_{1\delta}$ and $z_{2\delta}$ from the systems (17) and (18), respectively. Again the parameters are taken as $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$, and the delayed time δ is 0.001 sec for $x_{1\delta}$ and $z_{2\delta}$. Figure 3(a) shows that these systems are chaotic. Figure 3(b) shows that they are synchronized in spite of their different initial conditions.

Without loss of generality, the signal $\mathbf{w}$ in the systems (14) and (15) can also be a time delayed signal, and the lengths of time delay for the driving signals $\mathbf{q}$, $\mathbf{s}'$, and $\mathbf{w}$ can be different. Denote the length of time delay for $\mathbf{w}$ as δ_1; the length of time delay for $\mathbf{q}$ as δ_2; and the length of time delay for $\mathbf{s}'$ as δ_3. Thus, the system is written as follows.

System 1,

$$\dot{\mathbf{q}}=\mathbf{h}(t,\mathbf{q},\mathbf{s},\mathbf{w}_{\delta_1}),\quad \dot{\mathbf{s}}=\mathbf{g}(t,\mathbf{q},\mathbf{s},\mathbf{w}_{\delta_1}). \tag{20}$$

System 2,

$$\dot{\mathbf{q}}'=\mathbf{h}(t,\mathbf{q}',\mathbf{s}',\mathbf{w}_{\delta_1}),\quad \dot{\mathbf{s}}'=\mathbf{g}(t,\mathbf{q}',\mathbf{s}',\mathbf{w}_{\delta_1}). \tag{21}$$

System 3,

$$\dot{\mathbf{w}}=\mathbf{p}(t,\mathbf{q}_{\delta_2},\mathbf{s}'_{\delta_3},\mathbf{w}). \tag{22}$$

From [2], it can be seen that the necessary and sufficient condition of the synchronization between the systems (20) and (21) is that these subsystems are asymptotically stable. For example, consider the following systems.

System 1,

$$\dot{x}_1=\sigma(y_{\delta_1}-x_1),\quad \dot{z}_1=x_1y_{\delta_1}-\beta z_1. \tag{23}$$

System 2,

$$\dot{x}_2=\sigma(y_{\delta_1}-x_2),\quad \dot{z}_2=x_2y_{\delta_1}-\beta z_2. \tag{24}$$

System 3,

$$\dot{y}=\rho x_{1\delta_2}-y-x_{1\delta_2}z_{2\delta_3}. \tag{25}$$

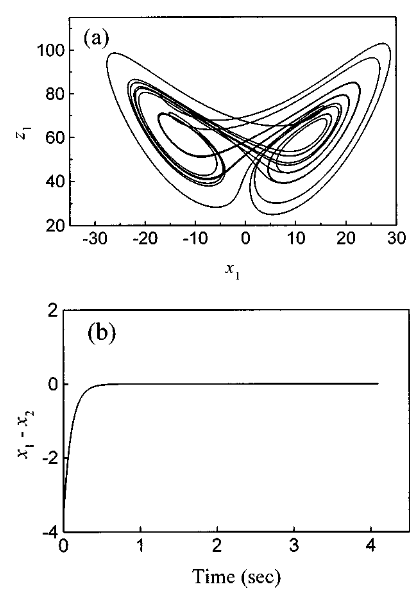

FIG. 4. (a) The trajectory of Eq. (23) in x_1 via z_1 phase plane. $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$; $\delta_1=0.004$ sec for y_{δ_1}; $\delta_2=0.001$ sec for $x_{1\delta_2}$; and $\delta_3=0.006$ sec for $z_{2\delta_3}$. (b) Synchronization of the systems (23) and (24).

The system (25) gets the time delayed signals $x_{1\delta_2}$ and $z_{2\delta_3}$ from the systems (23) and (24), and the systems (23) and (24) get the time delayed signal y_{δ_1} from the system (25). The parameters are taken as $\sigma=10$, $\rho=60$, and $\beta=\frac{8}{3}$. Now, let the delayed time be different: $\delta_1=0.004$ sec for y_{δ_1}; $\delta_2=0.001$ sec for $x_{1\delta_2}$; and $\delta_3=0.006$ sec for $z_{2\delta_3}$. In this case, the above systems are chaotic and synchronized, as shown in Fig. 4. However, to get the identical form of the subsystems (23) and (24), the delayed time δ_1 of y_{δ_1} must be the same for these two subsystems, which may not be true in practice.

IV. CONCLUSIONS

In this paper, synchronization in chaotic systems with time delayed driving signals has been discussed. Synchronization is unlikely to exist in time delayed coupled chaotic systems with simple connection. The complex structures of coupled chaotic systems have been presented in this paper. With these structures it has been shown that synchronization can exist in time delayed coupled chaotic systems. Several types of time delayed systems have been described. Since time delay phenomena frequently appear in communications and control, further study of synchronization of time delayed chaotic systems is worthwhile.

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