Parametric coupling of a light wave and surface plasma waves

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A three-wave decay interaction in which a *p*-polarized light wave transmitted from vacuum into a plasma decays into two surface waves of transverse magnetic mode (surface polariton) is investigated. Nonlinear boundary conditions for the surface waves are formulated in terms of the surface charge and the body current, taking a full account of the rippling of the free boundary. The mode-coupling equations are derived and solved in the parametric approximation to obtain the threshold and the growth rate. We assess the relative strength of the kinematic and the dynamic nonlinearities in the parametric interaction. [S1063-651X(99)13202-1]

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I. INTRODUCTION

During the past few decades, extensive investigations have been made on nonlinear wave interactions and parametric instabilities in infinite plasmas [1–3]. However, such investigations for bounded plasmas are rather few. It is well known that surface waves can propagate along the interface of a plasma and vacuum (or dielectric) [4,5]. Surface waves can be used for plasma diagnostics [6] and for sustaining a plasma which can be used in plasma processing [7]. Furthermore, surface waves are relevant to laser fusion [8] and to astrophysical problems in the magnetosphere and in the solar corona [9].

In view of the various applications and occurrences of surface waves, it would be interesting to consider wave-wave interactions involving surface waves. Atanassov *et al.* [10] considered a nonlinear interaction of three high-frequency electrostatic surface waves which produce a low-frequency density perturbation. Aliev and Brodin [8] investigated the excitation of a surface wave and a volume plasma wave in an inhomogeneous plasma by a *p*-polarized pump wave. Brodin and Lundberg [11] considered the same problem including thermal effects. Lindgren *et al.* [12] developed a general theory of three-wave interactions in plasmas with sharp boundaries by using the diffuse charge distribution model.

In this work, we investigate the parametric decay of a light wave into two transverse magnetic (TM) mode surface waves in a plasma bounded by vacuum. We include the deformability of the boundary and assess the rippling effects of the boundary on the parametric interactions. A TM mode surface wave is a low-frequency electron surface wave with both the longitudinal and transverse components with respect to the propagation vector. The dispersion relation reads

$$\omega^2=c^2k^2+\frac{\omega_p^2}{2}-\tfrac{1}{2}\sqrt{\omega_p^4+4c^4k^4}, \tag{1}$$

where ω and k denote, respectively, the frequency and the wave number, c is the speed of light in vacuum, and ω_p is the electron plasma frequency $(=\sqrt{4\pi Ne^2/m})$. The light wave in the plasma that we consider is a dispersionless wave whose phase velocity is constant and is dealt with in detail in Sec. II. The decay of the light wave of frequency ω_1 and wave number k_1 into two TM mode surface waves of frequencies ω_2 and ω_3 and wave numbers k_2 and k_3 requires the resonance matching conditions

$$\begin{aligned}\omega_1&=\omega_2+\omega_3,\\ k_1&=k_2+k_3.\end{aligned} \tag{2}$$

We show later that the resonance conditions (2) can be met easily and there would be numerous events of the resonance interactions.

In this work, we assume that the three waves propagate in the z direction and that the unperturbed interface is the plane $x=0$, separating the plasma $(x>0)$ and vacuum $(x<0)$. We also assume that the plasma is a linear medium with respect to the high-frequency light wave, but nonlinear with respect to the low-frequency TM mode surface wave. In order to investigate the nonlinear surface wave, all the nonlinearities which could be responsible for the resonant interaction are retained in this work. Especially, nonlinear boundary conditions are used for the TM mode surface wave and special attention was paid to the rippling effect of the deformable free boundary. When the boundary is fixed and unmoving, the important nonlinearity is dynamic nonlinearity ($\mathbf{v}\cdot\nabla\mathbf{v}$ and $\mathbf{v}\times\mathbf{B}$ force). If the boundary is deformable, the kinematic nonlinearity plays a role. We assessed the relative importance of these two nonlinearities on the parametric decay instability.

The present analysis of decay instability of the TM mode surface waves may be of importance for many applications since laboratory plasmas are bounded. Furthermore, the model used in this work may have direct relevance to the heating of laser-produced plasma, ionospheric plasma of the boundary layer, and the solar corona since the excited surface waves should be damped by the particles, leading to surface heating.

The paper is organized as follows. In Sec. II, the high-frequency light wave which is transmitted to plasma from vacuum is described and its dispersion relation is derived. In Sec. III, nonlinear equations for the low-frequency TM mode surface wave are formulated and homogeneous solutions are obtained. In Sec. IV, nonlinear boundary conditions are set

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up allowing for the rippling effect of the moving boundary. In Sec. V, perturbation analysis is carried out for the nonlinear equation and the mode-coupling equation is derived. In Sec. VI, the coupling equation is solved in the parametric approximation to find the thresholds and growth rates. A discussion is furnished in Sec. VII.

II. HIGH-FREQUENCY LIGHT WAVE

We envisage that the pump wave of frequency ω_1, polarized parallel to the plane of incidence (x-z plane), is incident upon the interface $x=0$ between the plasma ($x>0$) and vacuum ($x<0$) from the vacuum side with an angle of incidence θ_0. The electric field of the pump wave can be written as

$$\mathbf{E}_0=E_0(\hat{\mathbf{x}}\sin\theta_0-\hat{\mathbf{z}}\cos\theta_0)e^{i(\omega_1/c)(x\cos\theta_0+z\sin\theta_0)-i\omega_1 t}. \tag{3}$$

Then the reflected and transmitted waves appear, which take the forms

$$\mathbf{E}_0'=E_0'(\hat{\mathbf{x}}\sin\theta_0+\hat{\mathbf{z}}\cos\theta_0)e^{i(\omega_1/c)(-x\cos\theta_0+z\sin\theta_0)-i\omega_1 t}, \tag{4}$$

$$\mathbf{E}_1=(\hat{\mathbf{x}}E_{1x}+\hat{\mathbf{z}}E_{1z})e^{-\alpha_1 x}e^{ik_1 z-i\omega_1 t}. \tag{5}$$

The transmitted wave $\mathbf{E}_1$ is represented by Eq. (5) only when

$$\sin\theta_0>\sqrt{\varepsilon_1}, \tag{6}$$

where $\varepsilon_1=1-\omega_p^2/\omega_1^2$ is the dielectric constant of the plasma in response to the pump wave. Just in the case in which $\sin\theta_0=(1-\omega_p^2/\omega_1^2)^{1/2}$, the internal reflection occurs, and we have $\alpha_1=0$ and $E_{1z}=0$ [13]. The condition (6) restricts the pump frequency to the window defined by [9]

$$\omega_p<\omega_1<\frac{\omega_p}{\cos\theta_0}. \tag{7}$$

Using the continuity of the tangential components of the electric and magnetic fields and the normal components of the electric displacement, one finds

$$k_1=\frac{\omega_1}{c}\sin\theta_0, \tag{8}$$

$$\alpha_1=\frac{\omega_1}{c}\sqrt{\sin^2\theta_0-\varepsilon_1}, \tag{9}$$

$$E_{1x}=\frac{ck_1}{\omega_1\varepsilon_1}\frac{2E_0}{1+\dfrac{i\alpha_1\tan\theta_0}{k_1\varepsilon_1}}, \tag{10}$$

$$E_{1z}=-\frac{i\alpha_1}{k_1}E_{1x}. \tag{11}$$

Equation (8) is the dispersion relation of the high-frequency light wave propagating in the plasma transmitted from the vacuum with the angle of incidence θ_0. Equation (11) is obtained from $\nabla\cdot\mathbf{E}=\mathbf{0}$. The transmitted field $\mathbf{E}_1$ gives rise to the following magnetic field and electron velocity components oscillating with the frequency ω_1 in the plasma:

$$B_y(\omega_1)=\frac{\omega_1\varepsilon_1}{ck_1}E_{1x}e^{i\theta_1-\alpha_1 x}, \tag{12}$$

$$v_z(\omega_1)=\frac{-ie}{m\omega_1}E_{1z}e^{i\theta_1-\alpha_1 x}, \tag{13}$$

$$v_x(\omega_1)=\frac{-ie}{m\omega_1}E_{1x}e^{i\theta_1-\alpha_1 x}, \tag{14}$$

where

$$\theta_1=k_1 z-\omega_1 t, \tag{15}$$

and we used the linear electron equation of motion,

$$m\frac{\partial\mathbf{v}}{\partial t}=-e\mathbf{E}.$$

III. LOW-FREQUENCY EQUATIONS

To describe the nonlinear low-frequency surface wave, we use the following set of fluid and Maxwell equations:

$$\frac{\partial n}{\partial t}+\nabla\cdot[(n+N)\mathbf{v}]=0, \tag{16}$$

$$\left(\frac{\partial}{\partial t}+\nu\right)\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v}=-\frac{e}{m}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}\right), \tag{17}$$

$$\nabla\cdot\mathbf{E}=-4\pi en, \tag{18}$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{19}$$

$$\nabla\times\mathbf{B}=\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}-\frac{4\pi e}{c}(N+n)\mathbf{v}. \tag{20}$$

In the above, linear terms represent the low-frequency (ω_2 or ω_3) surface wave quantities, and the quadratic terms are beatings of the high-frequency waves with another low-frequency wave to excite the low-frequency wave under consideration. In Eq. (17), ν represents the collision frequency. We can discard the nonlinear current $n\mathbf{v}$ in Eqs. (16) and (20) because the density perturbation $n=0$ for the high-frequency light wave as well as for the low-frequency surface wave as shown later [see Eq. (60c)]. It should be noted that our low-frequency variables are the TM mode set (E_x,E_z,v_x,v_z,B_y), and the rest can be set to zero. The amplitude of the surface wave varies in the x direction while propagating in the z direction, and the y coordinate can be assumed to be ignorable ($\partial/\partial y=0$). Eliminating $\mathbf{B}$ between Eqs. (19) and (20), one obtains

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\omega_p^2}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\nu\right)^{-1}\right]\left[\frac{\partial^2}{\partial x^2}-A^2\right]E_z$$
$$=\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\nu\right)^{-1}\left(A^2Q_z-\frac{\partial^2 Q_x}{\partial x\partial z}\right), \tag{21}$$

$$\frac{\partial^2 E_z}{\partial x \partial z}+A^2 E_x=-\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\nu\right)^{-1} Q_x, \tag{22}$$

where

$$\mathbf{Q}=\frac{4\pi eN}{c^2}\left(\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}+\frac{e}{mc}\mathbf{v}\times\mathbf{B}\right), \tag{23a}$$

$$A=\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\frac{\omega_p^2}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\nu\right)^{-1}-\frac{\partial^2}{\partial z^2}\right\}^{1/2}. \tag{23b}$$

Some remarks are in order regarding the inverse operator $(\partial/\partial t+\nu)^{-1}$ and the operators A or A^{-1} used in the following. These operators act on the two-scale functions such as $f(z_1,t_1)e^{ikz_0-i\omega t_0}$, where z_0 and t_0 are fast space and time variables, and z_1 and t_1 are slow variables. The derivatives $\partial/\partial t$ and $\partial/\partial z$ are derivative-expanded in the fashion $\partial/\partial t=\partial/\partial t_0+\varepsilon(\partial/\partial t_1)$ and $\partial/\partial z=\partial/\partial z_0+\varepsilon(\partial/\partial z_1)$, where ε is the expansion parameter. Taking ν as a quantity of order ε, we can expand as

$$\left(\frac{\partial}{\partial t}+\nu\right)^{-1}=\left(\frac{\partial}{\partial t_0}\right)^{-1}-\varepsilon\left(\frac{\partial}{\partial t_0}\right)^{-2}\left(\frac{\partial}{\partial t_1}+\nu\right), \tag{24}$$

which obviously satisfies the relation $(\partial/\partial t+\nu)(\partial/\partial t+\nu)^{-1}=1$. Since $\partial/\partial t_0=-i\omega$ and $\partial/\partial z_0=ik$, these operators and their inverses are in fact algebraic quantities, and the meaning of the inverse operator in Eq. (24) is not ambiguous. An operator involving inverse (-1) or $(\frac{1}{2})$ power will be derivative-expanded by binomial theorem as was done in the above and the meaning of A or A^{-1} is not also ambiguous. Therefore, we can expand the operator A in Eq. (23b) as

$$A=A_0+\varepsilon A_1=\alpha+i\varepsilon\left(\frac{\partial\alpha}{\partial\omega}\frac{\partial}{\partial t_1}-\frac{\partial\alpha}{\partial k}\frac{\partial}{\partial z_1}\right)+\varepsilon\nu\left.\frac{\partial A}{\partial\nu}\right|_0, \tag{25}$$

where

$$\left.\frac{\partial A}{\partial\nu}\right|_0=\frac{-i\omega_p^2}{2c^2\omega\alpha}, \tag{26}$$

evaluating the derivative at $\nu=\partial/\partial t_1=\partial/\partial z_1=0$, and

$$\alpha(k,\omega)=\left(k^2+\frac{\omega_p^2-\omega^2}{c^2}\right)^{1/2}. \tag{27}$$

Likewise we have

$$A^{-1}=(A^{-1})_0+\varepsilon(A^{-1})_1$$
$$=\alpha^{-1}+i\varepsilon\left(\frac{\partial\alpha^{-1}}{\partial\omega}\frac{\partial}{\partial t_1}-\frac{\partial\alpha^{-1}}{\partial k}\frac{\partial}{\partial z_1}\right)+\varepsilon\nu\frac{i\omega_p^2}{2c^2\omega\alpha^3}. \tag{28}$$

In the vacuum, the low-frequency electric field is determined by

$$\left(\frac{\partial^2}{\partial x^2}-\Lambda^2\right)E_z=0, \tag{29}$$

$$\Lambda^2 E_x+\frac{\partial^2 E_z}{\partial x\partial z}=0, \tag{30}$$

where Λ is an operator defined by

$$\Lambda\left(\frac{\partial}{\partial t},\frac{\partial}{\partial z}\right)=\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)^{1/2}. \tag{31}$$

We first discuss the vacuum solutions. The x dependence of the function E_z in Eq. (29) is usually sought in the form of a power series, as is often done in the water wave or Rayleigh-Taylor instability problems [14,15]. In fact, the derivation of the power series solution is facilitated by regarding the operators $\partial/\partial t$ and $\partial/\partial z$ as algebraic quantities. Then Eq. (29) is solved by

$$E_z(x,z,t)=e^{x\Lambda(\partial/\partial t,\partial/\partial z)}W(z,t), \tag{32}$$

where the exponential function should read as a power series and the operator Λ acts on the unknown function W which is determined from the appropriate boundary conditions. Equation (30) yields

$$E_x(x,z,t)=-e^{x\Lambda}\Lambda^{-1}\frac{\partial W}{\partial z}. \tag{33}$$

The perturbation expansion of the operator Λ parallels that of A: setting $\omega_p=\nu=0$ in Eqs. (25) and (28) gives the corresponding formulas for Λ with the replacement of $\alpha\to\lambda$, where

$$\lambda=\left(k^2-\frac{\omega^2}{c^2}\right)^{1/2}. \tag{34}$$

Turning to the plasma equations (21) and (22), we note that the solution E_z as well as E_x consists of two parts, the homogeneous and the particular solutions:

$$E_z=E_{zH}+E_{zS}. \tag{35}$$

The homogeneous solution E_{zH} solves Eq. (21) with the right-hand side set to zero and thus can be written as

$$E_{zH}=e^{-xA(\partial/\partial t,\partial/\partial z)}F(z,t), \tag{36}$$

where $F(z,t)$ is an unknown function yet to be determined by the boundary conditions. The homogeneous solution of E_x is determined from Eq. (22):

$$E_{xH}=e^{-xA}A^{-1}\frac{\partial F}{\partial z}. \tag{37}$$

The particular solutions E_{zS} and E_{xS} of Eqs. (21) and (22) are determined with the right-hand sides constructed by an iterative procedure, i.e., by using the lower-order solutions for the quadratic terms. In our perturbation scheme, the lowest-order solutions are extracted from the homogeneous solutions.

IV. BOUNDARY CONDITIONS

The solutions of the equations formulated in the preceding section for the plasma and the vacuum regions should be matched on the interface

$$S(x,z,t)=x-\xi(z,t)=0, \tag{38}$$

through appropriate boundary conditions. First u, the normal velocity of the moving boundary $S(x,z,t)$, is expressed by

$$u=\frac{\partial\xi/\partial t}{\sqrt{1+\left(\frac{\partial\xi}{\partial z}\right)^2}}, \tag{39}$$

and the unit vector normal to the surface $S=0$ denoted by $\hat{\mathbf{n}}$ is

$$\hat{\mathbf{n}}=\frac{\hat{\mathbf{x}}-\hat{\mathbf{z}}\partial\xi/\partial z}{\sqrt{1+\left(\frac{\partial\xi}{\partial z}\right)^2}}. \tag{40}$$

The kinematic relation $u=\hat{\mathbf{n}}\cdot\mathbf{v}|_\xi$ gives the equation

$$v_x|_\xi=\frac{d\xi}{dt}=\frac{\partial\xi}{\partial t}+v_z\frac{\partial\xi}{\partial z}, \tag{41}$$

as expected. The boundary conditions on the moving interface have been obtained in Kruskal and Schwarzschild [16], and we have

$$[\hat{\mathbf{n}}\times\mathbf{E}]=\frac{u}{c}[\mathbf{B}], \tag{42}$$

$$[\hat{\mathbf{n}}\cdot\mathbf{E}]=4\pi\sigma, \tag{43}$$

where the bracket $[\mathbf{B}]=\mathbf{B}(\text{plasma})-\mathbf{B}(\text{vacuum})$ is evaluated at $x=\xi$. Other bracketed quantities also have a similar meaning. In Eq. (43), the surface charge density σ is given by Sedov [17],

$$\frac{\partial\sigma}{\partial t}+\hat{\mathbf{n}}\cdot\mathbf{J}+\nabla\cdot\mathbf{J}_\parallel^*=0, \tag{44}$$

where $\mathbf{J}_\parallel^*$ is the surface current density flowing on the interface and will be assumed to be zero. It would be instructive to derive Eq. (44) alternatively. By definition, we have

$$\sigma=-e\int_{-\delta}^{\delta}n\,dx,$$

where the surface charge layer is denoted by the range $(-\delta,\delta)$, and the limit $\delta\to 0$ is implied by assuming that the interface $x=0$ is fixed for simplicity. Using Eq. (16) gives

$$\frac{\partial\sigma}{\partial t}=e\int_{-\delta}^{\delta}\left\{\frac{\partial}{\partial x}[(n+N)v_x]+\frac{\partial}{\partial z}[(n+N)v_z]\right\}dx. \tag{44$'$}$$

The first integral is $e[(n+N)v_x]_{0^+}=J_x$. Equation (44) is a generalization of Eq. (44′). Here we note that the second integral of Eq. (44′) can give rise to higher-order surface currents [18,19], but we neglect those terms in this work for simplicity.

Using $u=\hat{\mathbf{n}}\cdot\mathbf{v}$ in Eq. (44) yields

$$\frac{\partial\sigma}{\partial t}=e(n+N)\frac{\partial\xi/\partial t}{\sqrt{1+\left(\frac{\partial\xi}{\partial z}\right)^2}}. \tag{45}$$

The y component of Eq. (42) reads

$$-[E_z]=\frac{\partial\xi}{\partial z}[E_x]+\frac{1}{c}\frac{\partial\xi}{\partial t}[B_y]. \tag{46}$$

Equations (43) and (45) yield

$$[E_x]-4\pi eN\xi=[E_z]\frac{\partial\xi}{\partial z}. \tag{47}$$

In writing Eq. (47), we neglected terms of $O(\varepsilon^3)$. The surface elevation ξ has all three components of high and low frequencies. Equations (46) and (47) are general jump conditions valid for both high- and low-frequency waves. The last term of Eq. (46) can be omitted since $[B_y]=0$ in the linear approximation for both the light wave and the surface wave. If Eq. (46) is used in Eq. (47) for $[E_z]$, the last term of Eq. (47) turns out to be of ε^3 order, and can be neglected. Thus we have

$$[E_z]=-\frac{\partial\xi}{\partial z}[E_x], \tag{46$'$}$$

$$[E_x]=4\pi eN\xi. \tag{47$'$}$$

The linear equation (47′) is nothing but the continuity of the normal component of the electric displacement vector $\mathbf{D}=\varepsilon_d\mathbf{E}$ (ε_d is the dielectric constant) [20]. Equation (46′) is the nonlinear version for the jump of the tangential component of the electric field.

Using Eqs. (32) and (36), Eq. (46′) is rewritten as

$$F-W=\xi(AF+\Lambda W)-4\pi eN\xi\frac{\partial\xi}{\partial z}-E_{zS}(\xi)\equiv N_B(\xi), \tag{48}$$

where we expanded the exponential operators.

It seems more advantageous to have the homogeneous solutions explicitly appear in Eq. (47′). Differentiating Eq. (47′) with respect to t gives

$$\frac{\partial}{\partial t}\left\{e^{-\xi A}A^{-1}\frac{\partial F}{\partial z}+E_{xS}(\xi)+e^{\xi\Lambda}\Lambda^{-1}\frac{\partial W}{\partial z}\right\}=4\pi eN\left\{v_x(\xi)-v_z(\xi)\frac{\partial\xi}{\partial z}\right\}, \tag{49}$$

where we used Eq. (41).

Inverting Eq. (17) yields

$$v_{xH}(\xi,z,t)=-\frac{e}{m}e^{-\xi A}A^{-1}\left(\frac{\partial}{\partial t}+\nu\right)^{-1}\frac{\partial F}{\partial z}, \tag{50}$$

$$v_{xS}(\xi,z,t)=-\left[\left(\frac{\partial}{\partial t}+\nu\right)^{-1}\left(\frac{e}{m}E_{xS}(x,z,t)+\frac{c^2}{4\pi eN}Q_x(x,z,t)\right)\right]_{x=\xi}, \tag{51}$$

where the subscripts H and S correspond, respectively, to the homogeneous and particular solutions. Differentiating Eq. (49) with respect to t again and using Eqs. (49) and (50), one obtains

$$\begin{aligned}\frac{\partial^2}{\partial t^2}&\left\{e^{-\xi A}A^{-1}\frac{\partial F}{\partial z}+e^{\xi\Lambda}\Lambda^{-1}\frac{\partial W}{\partial z}\right\}+\omega_p^2e^{-\xi A}A^{-1}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\nu\right)^{-1}\frac{\partial F}{\partial z}\\&=4\pi eN\frac{\partial}{\partial t}v_{xS}(\xi)-4\pi eN\frac{\partial}{\partial t}\left(v_z(\xi)\frac{\partial\xi}{\partial z}\right)-\frac{\partial^2}{\partial t^2}E_{xS}(\xi)+\omega_p^2\frac{\partial\xi}{\partial t}e^{-\xi A}\left(\frac{\partial}{\partial t}+\nu\right)^{-1}\frac{\partial F}{\partial z}\\&\equiv N_A(\xi).\end{aligned} \tag{52}$$

Equations (48) and (52) are the two equations which we analyze in the multiple scale perturbation scheme in the next section. The right-hand sides of Eqs. (48) and (52) contain the quadratic terms which are responsible for the parametric interactions. The lowest-order solutions of Eqs. (48) and (52) are the linear solutions which will be used to construct the right-hand sides by an iterative procedure.

V. PERTURBATION ANALYSIS

In this section we carry out a perturbation analysis of Eqs. (48) and (52) to derive coupled mode equations for the nonlinear wave interactions among the three waves satisfying the resonance conditions (2). In Fig. 1, we plotted the dispersion curves given by Eq. (8) of the high-frequency light wave (L) and Eq. (1) of the low-frequency TM mode surface waves (S). It is shown that the resonance conditions (2) can be easily met by constructing a parallelogram. The tips of the arrow falling on curve L give the values of ω_1 and k_1 while the tips of the arrows on curve S give the values of (ω_2,k_2) and (ω_3,k_3). In Fig. 1, the angle of incidence θ_0, which gives the slope of the straight line L, is taken to be 30° as a typical value.

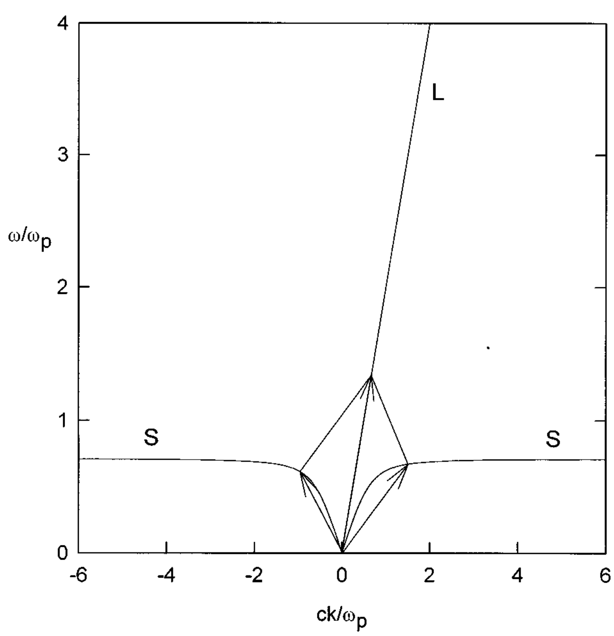

FIG. 1. Dispersion curves of light wave (L) and surface wave (S) and a resonant triad.

We expand the various quantities in Eqs. (48) and (52) in the perturbation series as

$$F(z,t)=\varepsilon F^{(1)}(z_0,t_0,z_1,t_1)+\varepsilon^2F^{(2)}+\cdots. \tag{53}$$

The quantities W, ξ, v_x, B_y, E_{xH}, and E_{zH} are expanded in the fashion of Eq. (53). However, the particular solutions E_{xS} and E_{zS} are expanded as

$$E_{xS}=\varepsilon^2E_{xS}^{(2)}+\varepsilon^3E_{xS}^{(3)}+\cdots, \tag{54}$$

although we do not need the ε^3-order terms. The operators are already expanded in Sec. III.

By means of the above perturbation scheme, we can break down Eqs. (48) and (52) order by order. In the ε order, we have the relations

$$\mathcal{L}_0F^{(1)}=0 \tag{55}$$

with

$$\mathcal{L}_0=\left(\frac{\partial^2}{\partial t_0^2}+\omega_p^2\right)(A^{-1})_0\frac{\partial}{\partial z_0}+\frac{\partial^2}{\partial t_0^2}(\Lambda^{-1})_0\frac{\partial}{\partial z_0} \tag{56}$$

and

$$F^{(1)}=W^{(1)}. \tag{57}$$

Equation (55) admits the solution in the form

$$F^{(1)}(z,t)=\tilde{F}(z_1,t_1)e^{i\theta} \tag{58}$$

with $\theta=kz_0-\omega t_0$ and the linear dispersion relation [20,21],

$$D(k,\omega)=\frac{1}{\lambda}+\frac{1}{\alpha}\left(1-\frac{\omega_p^2}{\omega^2}\right)=0, \tag{59}$$

where α and λ have been introduced in Eqs. (27) and (34). Equation (59) can be solved to give Eq. (1). Corresponding to $F^{(1)}$, we have the following linear solutions in the plasma:

$$v_x^{(1)}=\frac{ek}{m\omega\alpha}e^{-x\alpha}\tilde{F}e^{i\theta}, \tag{60a}$$

$$v_z^{(1)} = -\frac{ie}{m\omega} e^{-x\alpha}\tilde{F}e^{i\theta}, \tag{60b}$$

$$n^{(1)} = 0, \tag{60c}$$

$$B_y^{(1)} = -i\frac{c}{\omega}\left(\alpha - \frac{k^2}{\alpha}\right)e^{-x\alpha}\tilde{F}e^{i\theta}, \tag{60d}$$

$$\xi^{(1)} = \frac{iek}{m\omega^2\alpha}\tilde{F}e^{i\theta}. \tag{60e}$$

Likewise the ε-order vacuum solutions of the surface wave are

$$E_z^{(1)} = e^{x\lambda}\tilde{F}e^{i\theta}, \tag{60f}$$

$$E_x^{(1)} = -\frac{ik}{\lambda}e^{x\lambda}\tilde{F}e^{i\theta}, \tag{60g}$$

$$B_y^{(1)} = -\frac{i\omega}{c\lambda}e^{x\lambda}\tilde{F}e^{i\theta}. \tag{60h}$$

It may be seen that $B_y^{(1)}$ and $E_z^{(1)}$ are continuous across $x=0$ in the linear approximation. When we evaluate $N_A(\xi)$ and $N_B(\xi)$ in Eqs. (48) and (52), we need the quantities at the boundary $x=\xi$. The exponential factor $e^{-\xi\alpha}=1-\xi\alpha+(\xi^2/2)\alpha+\cdots$ generates all the higher-order terms. Since we need only the quadratic terms, it is sufficient in this work to evaluate the boundary values at $x=0$ in Eqs. (60).

Next we move on to the ε^2-order equations. Equations (48) and (52) yield in the ε^2 order, taking ν as a quantity of ϵ order,

$$\mathcal{L}_0F^{(2)}+\mathcal{L}_1F^{(1)} = \nu\omega_p^2A_0^{-1}\left(\frac{\partial}{\partial t_0}\right)^{-1}\frac{\partial F^{(1)}}{\partial z_0}+\omega_p^2\xi^{(1)}\frac{\partial F^{(1)}}{\partial z_0} + \frac{\partial^2}{\partial t_0^2}\Lambda_0^{-1}\frac{\partial}{\partial z_0}N_B^{(2)}(\xi)+N_A^{(2)}(\xi), \tag{61}$$

$$W^{(2)} = F^{(2)} - N_B^{(2)}, \tag{62}$$

where

$$\mathcal{L}_1 = \left(\frac{\partial^2}{\partial t_0^2}+\omega_p^2\right)\left\{(A^{-1})_1\frac{\partial}{\partial z_0}+A_0^{-1}\frac{\partial}{\partial z_1}\right\}+2\frac{\partial^2}{\partial t_0\partial t_1} \times(A_0^{-1}+\Lambda_0^{-1})\frac{\partial}{\partial z_0}+\frac{\partial^2}{\partial t_0^2}\left((\Lambda^{-1})_1\frac{\partial}{\partial z_0}+\Lambda_0^{-1}\frac{\partial}{\partial z_1}\right), \tag{63}$$

$$N_A^{(2)} = 4\pi eN\frac{\partial}{\partial t_0}v_{xS}^{(2)}(\xi=0) - 4\pi eN\frac{\partial}{\partial t_0}\left\{v_z^{(1)}(\xi=0)\frac{\partial\xi^{(1)}}{\partial z}\right\} - \frac{\partial^2}{\partial t_0^2}E_{xS}^{(2)}(\xi=0)+\omega_p^2\left\{\frac{\partial\xi^{(1)}}{\partial t_0}\left(\frac{\partial}{\partial t_0}\right)^{-1}\frac{\partial F^{(1)}}{\partial z_0}\right\}, \tag{64}$$

$$N_B^{(2)}(\xi) = \xi^{(1)}(A_0F^{(1)}+\Lambda_0W^{(1)}) - 4\pi eN\xi^{(1)}\frac{\partial\xi^{(1)}}{\partial z_0} - E_{zS}^{(2)}(\xi=0). \tag{65}$$

The $\mathcal{L}_1F^{(1)}$ term in Eq. (61) can be shown to reduce to

$$k\omega^2\left(\frac{\partial D}{\partial\omega}\frac{\partial F^{(1)}}{\partial t_1} - \frac{\partial D}{\partial k}\frac{\partial F^{(1)}}{\partial z_1}\right) \tag{66}$$

upon using the dispersion relation (59). If Eq. (61) reads as the ω_2-frequency equation, the $\mathcal{L}_1F^{(1)}$ term and the ω_2-frequency term on the right-hand side generated by the beats satisfying the resonance matching conditions (2) cause the secularity when one attempts to solve Eq. (61) for $F^{(2)}$. All these secularity-causing terms should be removed by requiring that

$$k\omega^2\left(\frac{\partial D}{\partial\omega}\frac{\partial}{\partial t_1} - \frac{\partial D}{\partial k}\frac{\partial}{\partial z_1}\right)F^{(1)} + \nu\left\{\left(\frac{\partial^2}{\partial t_0^2}+\omega_p^2\right)\frac{i\omega_p^2}{2c^2\omega\alpha^3} - \omega_p^2A_0^{-1}\left(\frac{\partial}{\partial t_0}\right)^{-1}\right\}\frac{\partial F^{(1)}}{\partial z_0} = \omega_p^2\left(\xi^{(1)}\frac{\partial F^{(1)}}{\partial z_0}\right)_\omega + N_A^{(2)}(\omega) - \frac{ik}{\lambda}\omega^2N_B^{(2)}(\omega), \tag{67}$$

where ω stands for either ω_2 or ω_3, the low frequencies. Equation (67) is the desired mode-coupling equation, and it remains for us to evaluate the coupling terms on the right-hand side. In writing Eq. (67), we tacitly assumed that the amplitude of the light wave is of order ε (weak pump). When one evaluates the coupling terms, one needs the elevation of the perturbed interface oscillating with frequency ω_1, which is obtained from the linear equation $\partial\xi^{(1)}(\omega_1)/\partial t = v_x^{(1)}\times(\omega_1)$ as, with the aid of Eq. (14),

$$\xi^{(1)}(\omega_1) = \frac{eE_{1x}}{m\omega_1^2}e^{i(k_1z_0-\omega_1t_0)}. \tag{68}$$

Omitting the details, we only present essential intermediate steps in the following. From Eq. (23a) we have

$$Q_x(\omega_2) = i\mu lE_{1x}\tilde{F}^*(\omega_3)e^{i\theta_2-(\alpha_1+\alpha_3)x}, \tag{69}$$

$$Q_z(\omega_2) = \mu gE_{1x}\tilde{F}^*(\omega_3)e^{i\theta_2-(\alpha_1+\alpha_3)x}, \tag{70}$$

where

$$\mu = \frac{e\omega_p^2}{m\omega_1\omega_3 c^2}, \tag{71}$$

$$l = (\alpha_1+\alpha_3)\left(\frac{\alpha_1}{k_1}+\frac{k_3}{\alpha_3}\right), \tag{72}$$

$$g = k_2\left(\frac{\alpha_1}{k_1}+\frac{k_3}{\alpha_3}\right), \tag{73}$$

and $\theta_2 = k_2 z_0 - \omega_2 t_0$, $\alpha_i = \alpha(\omega_i, k_i)$ $(i=1,2,3)$. Equations (21) and (22) can be solved to give

$$E_z(\omega_2) = \frac{-\mu g}{(\alpha_2^2-k_2^2)} E_{1x}\tilde{F}^*(\omega_3)e^{i\theta_2-(\alpha_1+\alpha_3)x}, \tag{74}$$

$$E_x(\omega_2) = -\frac{il\mu}{(\alpha_2^2-k_2^2)} E_{1x}\tilde{F}^*(\omega_3)e^{i\theta_2-(\alpha_1+\alpha_3)x}. \tag{75}$$

From Eq. (64), we obtain

$$N_A^{(2)}(\omega_2) = i\mu c^2\left\{k_3+\omega_2\left(\frac{\alpha_1 k_3^2}{k_1\omega_3\alpha_3}-\frac{k_1}{\omega_1}\right)\right\}. \tag{76}$$

After spending a considerable amount of algebra to calculate the whole right-hand side of Eq. (67), we finally obtain

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial\omega_2}{\partial k_2}\frac{\partial}{\partial z_1}+\Gamma_2\right)\tilde{F}(\omega_2) = \frac{ie\omega_p^2\eta_2}{m\omega_1\omega_3\omega_2^2 k_2\dfrac{\partial D_2(\omega_2,k_2)}{\partial\omega_2}} E_{1x}\tilde{F}^*(\omega_3), \tag{77}$$

where

$$\Gamma_2 = \frac{\nu(k_2^2+\alpha_2^2)\omega_p^2}{2\alpha_2^3\omega_2^3\dfrac{\partial D(\omega_2,k_2)}{\partial\omega_2}} \tag{78}$$

is the effective collision frequency

$$\eta_2 = -k_2+\frac{k_3}{\alpha_3}\frac{\omega_2}{\omega_3}\left(\alpha_1\frac{k_3}{k_1}+\alpha_2\frac{\omega_2}{\omega_1}\right)-\frac{k_2^2}{\alpha_2}\left(\frac{\alpha_1}{k_1}+\frac{k_3}{\alpha_3}\right) \tag{79}$$

and we used Eq. (59). If we assume that the interface is fixed and unmoving, the coupling coefficient η_2 reduces to the last term in Eq. (79). That is, the first two terms in η_2 come from the rippling of the interface. The evolution equation for the amplitude of the ω_3 frequency takes the form

$$\left(\frac{\partial}{\partial t_1}+\frac{\partial\omega_3}{\partial k_3}\frac{\partial}{\partial z_1}+\Gamma_3\right)\tilde{F}(\omega_3) = \frac{ie\omega_p^2\eta_3}{m\omega_1\omega_2\omega_3^2 k_3\dfrac{\partial D_2(\omega_3,k_3)}{\partial\omega_3}} E_{1x}\tilde{F}^*(\omega_2), \tag{80}$$

where Γ_3 and η_3 are obtained from Γ_2 and η_2, respectively, by interchanging the subscripts 2 and 3. Equations (77) and (80) will be solved in the parametric approximation in the next section.

VI. PARAMETRIC INSTABILITY

In this section we consider the parametric instability, and the group velocity terms in Eqs. (77) and (80) will be neglected assuming that $\partial/\partial t_1 \gg (\partial\omega/\partial k)(\partial/\partial z_1)$. At the onset of a parametric instability this inequality is always valid. We also introduce a small frequency mismatch Δ in the resonance matching conditions; $\Delta = \omega_2+\omega_3-\omega_1$. Then Eqs. (77) and (80) are written as

$$\left(\frac{\partial}{\partial t_1}+\Gamma_2\right)\tilde{F}(\omega_2) = i\eta_2' E_{1x}\tilde{F}^*(\omega_3)e^{it_1\Delta}, \tag{81}$$

$$\left(\frac{\partial}{\partial t_1}+\Gamma_3\right)\tilde{F}(\omega_3) = i\eta_3' E_{1x}\tilde{F}^*(\omega_2)e^{it_1\Delta}, \tag{82}$$

where η_2' and η_3', the coefficients in Eqs. (77) and (80), are real. One can easily eliminate $\tilde{F}(\omega_2)$ in the above two equations to get

$$\frac{\partial^2 A}{\partial t_1^2}+(i\Delta+\Gamma_2-\Gamma_3)\frac{\partial A}{\partial t_1}-\Omega A=0, \tag{83}$$

where $A=\tilde{F}^*(\omega_3)e^{\Gamma_3 t_1}$ and

$$\Omega = \frac{|E_{1x}|^2\left(\dfrac{e\omega_p^2}{m\omega_1}\right)^2\eta_2\eta_3}{\omega_3^3\omega_2^3 k_2 k_3\dfrac{\partial D_2}{\partial\omega_2}\dfrac{\partial D_3}{\partial\omega_3}} \tag{84}$$

with $D_i = D(\omega_i, k_i)$ $(i=2,3)$.

We consider the threshold and growth rate of the instability. Setting $\partial/\partial t_1 = i(\omega_3-\omega)+\Gamma_3$, Eq. (83) gives the following dispersion relation:

$$(\omega-\omega_3)^2+(\omega-\omega_3)(i\Gamma_2+i\Gamma_3-\Delta)-\Gamma_2\Gamma_3-i\Gamma_3\Delta+\Omega=0, \tag{85}$$

which is the standard form for the dispersion relation of the parametric instability [22]. After separating this equation into real and imaginary parts, with $\omega-\omega_3 = x+iy$, and eliminating x, we get

$$(y+\Gamma_2)(y+\Gamma_3)\left\{1+\frac{\Delta^2}{(2y+\Gamma_2+\Gamma_3)^2}\right\}=\Omega. \tag{86}$$

If the value of Ω is not large enough, Eq. (86) can be satisfied by a negative value of y. For instability $(y>0)$, it is seen by inspection that Ω should be greater than at least $\Gamma_2\Gamma_3$ (when $\Delta=0$). The threshold value of Ω for instability is obtained by setting $y=0$ in Eq. (86),

$$\Omega_{\text{th}} = \Gamma_2\Gamma_3\left(1+\frac{\Delta^2}{(\Gamma_2+\Gamma_3)^2}\right). \tag{87}$$

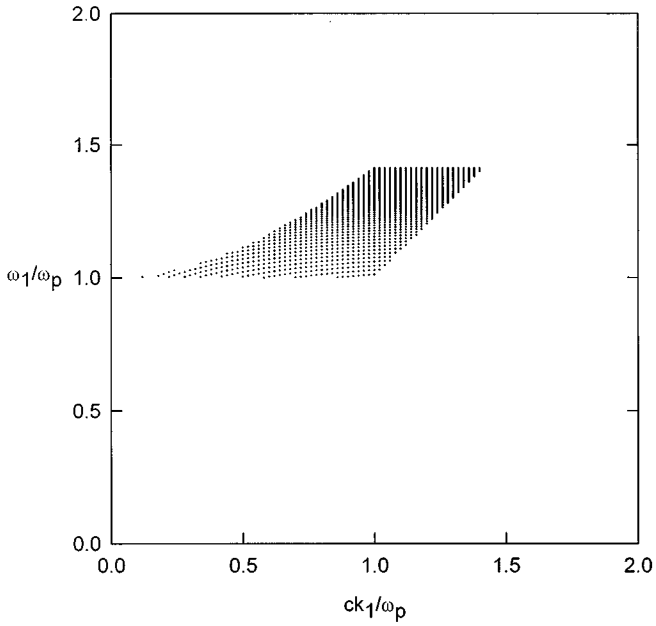

FIG. 2. The frequencies and wave numbers of the high-frequency light wave meeting the three-wave resonant conditions.

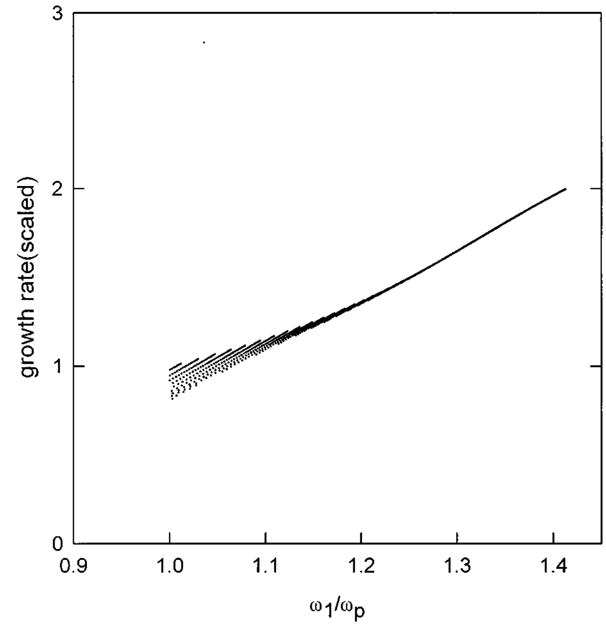

FIG. 3. Growth rate [right-hand side of Eq. (91)] versus light wave frequency.

In the following, we shall set $\Delta=0$ and take the threshold value as $\Gamma_2\Gamma_3$, and then the condition of instability is given as $\Omega>\Gamma_2\Gamma_3$, which can be written in the form

$$\frac{4e^2|E_{1x}|^2}{c^2m^2\nu^2}>\frac{k_1^2k_2k_3(k_2^2+\alpha_2^2)(k_3^2+\alpha_3^2)}{\eta_2\eta_3\alpha_2^3\alpha_3^3\sin^2\theta_0}. \tag{88}$$

In writing Eq. (88), we used Eq. (8). We numerically calculated the right side quantity for the resonant triads (Fig. 4). The maximum growth rate y_m above the threshold is obtained when $\Delta=0$ in Eq. (86):

$$y_m=\tfrac{1}{2}\{\sqrt{(\Gamma_2-\Gamma_3)^2+4\Omega}-(\Gamma_2+\Gamma_3)\}. \tag{89}$$

We assume that the value of Ω is slightly above the threshold $\Gamma_2\Gamma_3$, i.e., $\Omega=\Gamma_2\Gamma_3+\delta\Omega$ with $\delta\Omega\ll\Gamma_2^2,\Gamma_3^2$.

Expanding the square root in Eq. (89), we have

$$y_m\simeq\frac{\delta\Omega}{\Gamma_2+\Gamma_3}. \tag{90}$$

That is, the growth rate just above the threshold is inversely proportional to $(\Gamma_2+\Gamma_3)$. Using Eq. (78), we have

$$\frac{\nu}{2(\Gamma_2+\Gamma_3)}=\frac{1}{\omega_p^2}\left\{\frac{k_2^2+\alpha_2^2}{\alpha_2^3\omega_2^3\frac{\partial D_2}{\partial\omega_2}}+\frac{k_3^2+\alpha_3^2}{\alpha_3^3\omega_3^3\frac{\partial D_3}{\partial\omega_3}}\right\}^{-1}. \tag{91}$$

Equations (88) and (91) give, respectively, the dimensionless threshold and growth rate for the resonant triads which satisfy the matching conditions (2). We first solved the matching condition Eq. (2) numerically and plotted it in Fig. 2, where the high-frequency wave resonant frequencies and wave numbers are represented on the dimensionless (ω_1,k_1) plane. The restrictions on the dispersion of the high-frequency wave [Eqs. (6)–(8)] require that $\omega_1>ck_1$ and $\omega_p^2<\omega_1^2<\omega_p^2+c^2k_1^2$. The upper limit of ω_1 comes from the fact that the surface wave frequency is less than $\omega_p/\sqrt{2}$. Thus ω_1 cannot be greater than $\sqrt{2}\omega_p$. For the resonant triads for which the high-frequency wave belongs to the roots shown in Fig. 2, we plotted the right-hand sides of Eq. (91) (growth rate) and Eq. (88) (threshold), respectively, in Figs. 3 and 4. Figure 3 indicates that the higher the light wave frequency is, the faster the surface wave grows. This illustrates the tendency of the short wave components of the surface wave to be more easily excited than the long wave components. In the approximation employed to derive Eq. (91), the rippling effect of the boundary does not manifest itself in

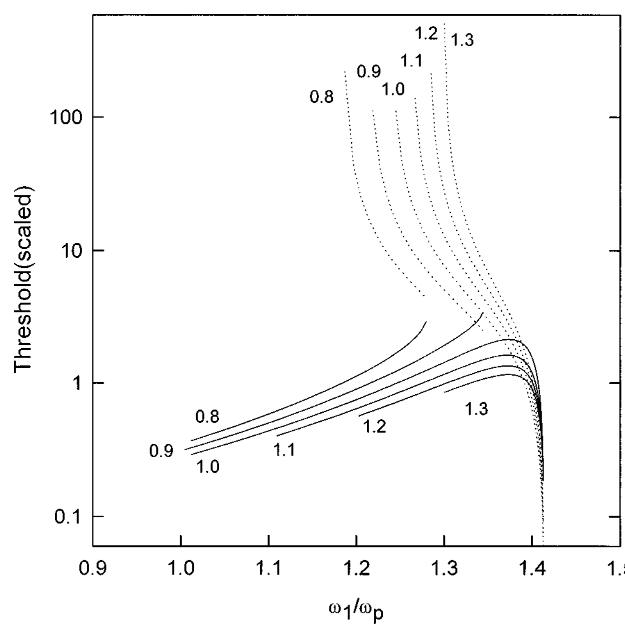

FIG. 4. Threshold [right-hand side of Eq. (88)] versus light wave frequency for different wave numbers. Solid (dotted) line represents the case where the boundary is free (fixed). The numbers on the curves represent the values of ck_1/ω_p.

the growth rate. In Fig. 4, we plotted the threshold both when the boundary is rippling (solid curve) and fixed (broken curve). It is interesting to note that the right-hand side of Eq. (88) is negative if the rippling effect of the boundary is neglected, provided that the light wave frequency is not big enough. In Fig. 4, we plotted only the positive values of the right-hand side of Eq. (88) as dotted lines whose values are obtained from η [Eq. (79)] retaining only the fixed boundary term. A negative value of threshold (Ω) means that, in view of Eq. (86), no instability occurs (even though the resonance conditions are met). If the rippling effect is included, all the resonant triads give rise to instability.

VII. SUMMARY

We investigated a decay instability in which a high-frequency light wave decays into two daughter waves of low-frequency TM mode surface wave. In deriving the coupled mode equations, we presented a method which is capable of dealing with the kinematics of the rippling free boundary. The nonlinear boundary conditions are formulated in terms of the surface charge and volume currents. The surface wave equation, whose solutions involve a power series in the perpendicular coordinate, can be conveniently solved in operatorial form, and two-scale analysis can be straightforwardly carried out. We solved the coupled mode equations in the parametric approximation and calculated the thresholds and growth rates. The dominant nonlinearities responsible for the parametric instability are the $\mathbf{v}\times\mathbf{B}$ force and the rippling effect of the free boundary. The nonlinear current is absent. If the boundary is assumed fixed, not all of the resonant triads give rise to instability. This feature of instability may be attributed to the absence of nonlinear currents. However, if the kinematics of rippling are included, parametrically unstable interactions are given rise to by all of the resonant triads.

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