# Modeling of deterministic chaotic systems

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The success of deterministic modeling of a physical system relies on whether the solution of the model would approximate the dynamics of the actual system. When the system is chaotic, situations can arise where periodic orbits embedded in the chaotic set have distinct number of unstable directions and, as a consequence, no model of the system produces reasonably long trajectories that are realized by nature. We argue and present physical examples indicating that, in such a case, though the model is deterministic and low dimensional, statistical quantities can still be reliably computed. [S1063-651X(99)10302-7]

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Scientists and engineers rely heavily on models to understand natural phenomena. Usually, for a particular process, data from laboratory experiments or from observations are analyzed and, together with physical laws, a model of the process is formulated. In fact, this is done for a large variety of processes in fields such as physics, chemistry, biology, ecology, and engineering. The models are then used to understand the particular process, to make predictions, and to control its dynamics. There are two important classes of models. The first class is the deterministic dynamical systems and they evolve the relevant physical variables in time according to a set of prescribed rules. The second one is statistical models, which are models that involve some kind of stochastic process and, consequently, for these models, statistical averages regarding properties of the system are often obtained from the model. The conventional wisdom about statistical models is that they deal with situations where random noise is influential or systems that involve a large number of degrees of freedom such as those arising in statistical physics.

This paper deals with neither class; it deals with a special class of deterministic models, models that, in spite of being deterministic, yield only statistically relevant information in the form of averages about some quantities depending on their dynamical variables. A related question but one that captures the essential problem is to what extent predictions from deterministic models are expected to be valid. Problems with prediction arise when the deterministic system is chaotic; that is, when the system has sensitive dependence on initial conditions, or on small parameter variations, or on environmental noise, etc. To address the validity of deterministic modeling of chaotic systems, imagine that we construct two models of the natural system [1]: **A** and **B**, very close to each other: model **A**, $d\mathbf{x}/dt=\mathbf{f}(\mathbf{x},t)$; model **B**, $d\mathbf{x}/dt=\mathbf{f}(\mathbf{x},t)+\boldsymbol{\epsilon}(t)$, where $\boldsymbol{\epsilon}(t)$ is an arbitrarily small time dependent perturbation that is bounded, i.e., we exclude unbounded random perturbations such as Gaussian white noise in our consideration. Since no model is exact, they are at best a perturbed version of the natural system. For either model to reproduce and predict correctly the behavior of the natural system, trajectories of model **A** must stay close to some trajectories of **B**. If this is so, we say that there is model shadowability. If no trajectory of **A** is close to any trajectory of **B**, it is unlikely that the solution of either model **A** or **B** stays close to any trajectory of the natural system. In other words, there is no model which would produce trajectories that are realized by nature. In studying chaotic systems, previous work [1] has suggested that there is a hierarchy of difficulty levels obstructing model shadowability. Specifically, the levels of difficulty are (i) *minor modeling difficulties*: hyperbolic chaotic systems exhibiting sensitive dependence on initial conditions. For these systems, trajectories of model **A** can always be shadowed by trajectories of model **B** for an infinite time [2]; (ii) *moderate modeling difficulties*: chaotic systems with nonhyperbolic tangencies. For these systems, trajectories of model **A** are shadowed by trajectories of model **B** for a long but finite amount of time [3]; and (iii) *severe modeling difficulties*: nonhyperbolic chaotic systems with unstable-dimension variability [4–6]. For these systems, the shadowing times are surprisingly short [7].

In this paper, we argue that chaotic systems having severe modeling difficulties could still be modeled deterministically but such models are able to make relevant predictions that are only statistical in nature. The necessity for statistical predictions stems from the fact that any individual trajectory yields no reliable information about the state of the system. Instead, statistical quantities should be considered when analyzing and evaluating solutions of the model. Thus, although the system is intrinsically deterministic, no long-term deterministic information regarding the system's state can be obtained about its behavior through deterministic modeling.

We begin by describing the notion of hyperbolicity (or nonhyperbolicity), a fundamental property of chaotic systems which plays the key role in determining the validity of a model. The dynamics is hyperbolic on a chaotic set if, at each point of the trajectory, the phase space can be split into expanding and contracting subspaces and the angle between them is bounded away from zero. Furthermore, the expanding subspace evolves into the expanding one along the trajectory and the same is true for the contracting subspace. Otherwise the set is nonhyperbolic. For hyperbolic chaotic systems, one gets modeling shadowing for an infinitely long time [2], and one has only minor modeling difficulties. The situation with nonhyperbolic chaotic systems, on the other hand, is more complicated. To discuss model shadowability, it is insightful to classify nonhyperbolic systems into two types. For the *first type*, the splitting of the phase space into expanding and contracting subspaces is invariant along a trajectory except at the tangencies of the stable and unstable manifolds, where the angles between subspaces are zero. In this case, one gets modeling shadowing up to a time $\tau$ that scales with $\epsilon = \max[\epsilon(t)]$ as $\tau \sim \epsilon^{-\alpha}$, where $\alpha \lesssim \frac{1}{2}$ is the scaling exponent [3]. Thus, for this type of nonhyperbolic systems, modeling trajectories are expected to be reliable for a reasonably long time if the modeling error $\epsilon$ is small, and one has moderate modeling difficulties. The *second type* of nonhyperbolicity concerns a more drastic violation of the continuous splitting of the phase space into the expanding and contracting subspaces. For this case, the unstable periodic orbits embedded in the chaotic set have different number of unstable directions. Both sets of periodic orbits are expected to be densely mixed. Therefore, as the dynamics evolves, the trajectory of the system experiences different number of unstable (and, hence, stable) directions, and the neighborhood of each set can be visited for arbitrarily long times. This is called unstable-dimension variability [6], a phenomenon that is reflected and quantified by a finite-time Lyapunov exponent fluctuating about zero [4,8]. Modeling shadowability for this type of nonhyperbolic systems can be very short, for a time that scales with $\epsilon$ as $\tau \sim \epsilon^{-2m/\sigma^2}$, where $m$ and $\sigma$ are the mean and standard deviation of the fluctuating time-one Lyapunov exponent closest to zero [7]. In this case, one has severe modeling difficulties. The fundamental mechanism for modeling to fail so severely when the number of unstable directions changes along a trajectory was first described by Abraham and Smale [9].

We now give a physical example to address the issue of modeling. The system exhibits unstable dimension variability and hence, it has severe modeling difficulties. The system is the following four-dimensional map which physically describes the dynamics of a double rotor subject to periodic kicks [10,11]:

$$\mathbf{x}_{n+1} = \mathbf{M}\mathbf{y}_n + \mathbf{x}_n, \qquad \mathbf{y}_{n+1} = \mathbf{L}\mathbf{y}_n + \mathbf{G}(\mathbf{x}_{n+1}), \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} x(1) \\ x(2) \end{pmatrix} \in S^1 \times S^1, \quad \mathbf{y} = \begin{pmatrix} y(1) \\ y(2) \end{pmatrix} \in \mathbf{R} \times \mathbf{R},$$

and

$$\mathbf{G}(\mathbf{x}) = \begin{pmatrix} c_1 \sin x(1) \\ c_2 \sin x(2) \end{pmatrix}. \tag{2}$$

In Eqs. (1) and (2), $x_n(1,2)$ are the angular positions of the rotors at the instant of the $n$th kick, while $y_n(1,2)$ are the angular velocities of the rotors immediately after the $n$th kick, $\mathbf{L}$ and $\mathbf{M}$ are $2 \times 2$ constant matrices whose elements depend on the physical parameters of the rotors, $c_1$ and $c_2$ are two parameters that are proportional to the kicking strength $f$. In the following we choose the parameter setting in Ref. [11] so that $c_1 = c_2 = f$ [12].

It is known that the double-rotor map, Eq. (1), exhibits unstable dimension variability in certain parameter regimes [4,11]. In particular, at the parameter setting described in Ref. [11], there is unstable dimension variability and consequently obstruction to modeling shadowing near $f = 8.0$, while moderate modeling shadowing difficulties seem to occur in other parameter regimes, say, near $f = 9.0$ [4]. The reason, as given in Ref. [4], is that one of the Lyapunov exponents of Eq. (1) fluctuates about zero when it is computed in finite time. The fluctuations of finite-time Lyapunov exponents are a direct manifestation of unstable dimension variability [4,7]. We observe that unstable dimension variability typically occurs in a parameter region where there is a transition in the chaotic behavior of the system. In particular, for Eq. (1), near $f = 8.0$ there is a transition from one positive Lyapunov exponent to two positive ones in the attractor. That is, the system is low-dimensionally chaotic for $f < 8.0$, while it is hyperchaotic for $f \gtrsim 8.0$. This transition is shown in Fig. 1, where the four Lyapunov exponents of Eq. (1) are plotted as a function of $f$. In the computation, 1000 values of $f$ are chosen uniformly from the interval [6,10] and for each $f$, $10^7$ iterations (with $10^6$ initial iterations disregarded) are used. It can be seen that the system goes through a cascade of period-doubling bifurcations for $f < f_{c_1} \approx 6.746$ and becomes chaotic with one positive Lyapunov exponent at $f_{c_1}$. At $f = f_{c_2} \approx 8.0$, the second Lyapunov exponent becomes positive so that for $f > f_{c_2}$, the system is hyperchaotic. For parameter much above the transition, say near $f = 9.0$, the second Lyapunov exponent becomes so positive that the fluctuations in the finite-time Lyapunov exponent have only a negligible tail in the negative side. In this case, unstable dimension variability is less severe and the system becomes shadowable again. We note that in previous papers [4,7], it has been conjectured that unstable dimension variability is common in high-dimensional dynamical systems. Figure 1 suggests, however, that one should expect unstable dimension variability especially near the transition regions where the system becomes more (or less) chaotic.

We now demonstrate that when modeling difficulties are severe so that no model trajectory corresponds to any trajectory of the natural system which the model is supposed to describe, statistical prediction can still be done *reliably* through the use of the model. To argue this, we consider a family of models which is a slighly perturbed version of Eq. (1),

$$\mathbf{x}_{n+1} = \mathbf{M}\mathbf{y}_n + \mathbf{x}_n, \qquad \mathbf{y}_{n+1} = \mathbf{L}\mathbf{y}_n + \mathbf{G}(\mathbf{x}_{n+1}) + \epsilon\mathbf{H}(\mathbf{x}_{n+1}), \tag{3}$$

where $\epsilon \ll f$ is the magnitude of the model uncertainty, $\mathbf{H}(\mathbf{x}) = [\sigma_n^1 \sin x(1), \sigma_n^2 \sin x(2)]^T$, and $\sigma_n^1$ and $\sigma_n^2$ are two random variables uniformly distributed in [0, 1]. Following our previous notion, Eq. (1) is model **A** and Eq. (3) is model **B**. Now, suppose we want to compute a statistical quantity, say, the average energy $\langle E \rangle$ of the system, by using both models **A** and **B**. For the kicked double-rotor system (1) with the parameter setting chosen, $\langle E \rangle$ is given by

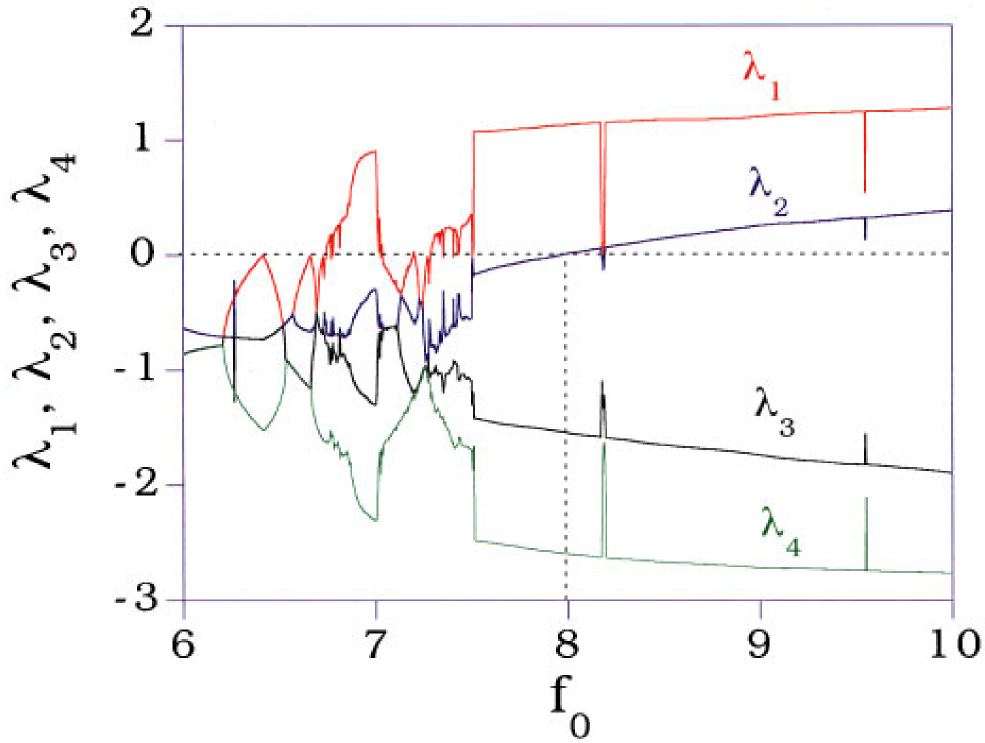


FIG. 1. (Color) Lyapunov spectrum of the kicked double-rotor map Eq. (1) as a function of the kicking strength $f$. Severe unstable dimension variability has been documented at $f=8.0$ near which there is a transition from low-dimensional chaos to hyperchaos.

$$\langle E\rangle=\frac{1}{N}\lim_{N\to\infty}\sum_{n=1}^{N}\left\{\tfrac{1}{2}[y_n^2(1)+y_n^2(2)]\right.$$
$$\left.+\frac{1}{\sqrt{2}}[\cos x_n(1)+\cos x_n(2)]\right\}. \qquad (4)$$

The expression for $\langle E\rangle$ is the same for the randomly perturbed double rotor in Eq. (3). We first compute, in the noiseless case ($\epsilon=0$), $\langle E\rangle$ versus the kicking strength $f$, as shown in Fig. 2 for 1000 values of $f$ in $6\leqslant f\leqslant 10$, where for each value of $f$, a trajectory of $10^7$ points on the attractor (with $10^6$ initial iterations disregarded) is used to compute

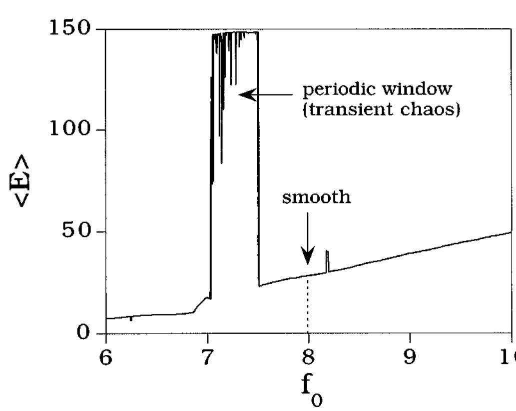


FIG. 2. The average energy $\langle E\rangle$ of the double rotor vs the kicking strength $f$ in a noiseless situation. The energy changes smoothly near $f=8.0$ where there is a severe modeling difficulty. The abrupt changes in $\langle E\rangle$ near $f=7.1$ and $f=7.5$ are due to a periodic window, and the fluctuations of $\langle E\rangle$ inside the window are due to transient chaos.

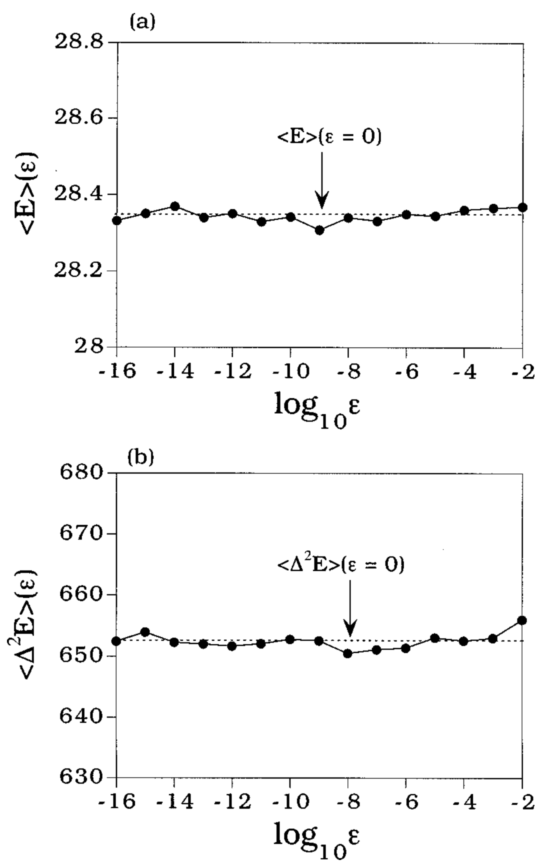


FIG. 3. (a) Computation of the average energy of the double rotor system from Eq. (1) (dashed line, $\langle E\rangle$) and from the perturbed Eq. (3) [closed circles, $\langle E\rangle(\epsilon)$], where $\epsilon$ is the magnitude of the model uncertainty. Apparently, we have $\langle E\rangle(\epsilon)\approx\langle E\rangle$ over 14 orders of magnitude in $\epsilon$. (b) The second energy moment $\langle(\Delta E)^2\rangle(\epsilon)$ vs $\epsilon$.

$\langle E\rangle$. This time is much longer than the modeling-shadowing time. We see that near $f=8.0$, $\langle E\rangle$ changes smoothly, implying robustness of statistical averages in parameter regimes where there is a severe modeling difficulty. For $f=8.0$, we obtain $\langle E\rangle\approx 28.3565$. Next, we compute $\langle E\rangle(\epsilon)$ from Eq. (3) for 16 values of $\epsilon$ in $\epsilon\in[10^{-16},10^{-1}]$, also by using trajectories of $10^7$ points. The result is shown in Fig. 3(a), where the dashed line indicates the value of $\langle E\rangle$ and the filled circles denote $\langle E\rangle(\epsilon)$. Apparently, we have $\langle E\rangle(\epsilon)\approx\langle E\rangle$ over many orders of magnitude in $\epsilon$, indicating that statistical prediction about the energy of the system is robust through the use of the model, despite unstable dimension variability. The fluctuations of $\langle E\rangle(\epsilon)$ around $\langle E\rangle$ are due to finite numerics and decrease as $1/\sqrt{N}$ as the length $N$ of the trajectory used in the computation increases. We can also consider other statistical invariants such as higher moments, the Lyapunov exponents and fractal dimensions. Computations reveal that those invariants are also robust in the sense that both models Eq. (1) and Eq. (3) yield the same values over many orders of magnitude of variation in the model uncertainty $\epsilon$. For example, Fig. 3(b) shows the average second moment of the energy $\langle(\Delta E)^2\rangle$ versus $\epsilon$, where behavior similar to that in Fig. 3(a) is observed.

In conclusion, we demonstrate that for physical systems with severe modeling difficulties, although it is not possible to make deterministic prediction reliably from the model, it is still possible to make statistical predictions. We expect unstable dimension variability to be common in high-dimensional chaotic systems. Alternatively, physical systems with severe modeling difficulties can be investigated by measuring some observables of the system and then using nonlinear time series analysis for the understanding, prediction, and control of their dynamics.

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[12] At this parameter setting, the matrices **L** and **M** are given by

$$\mathbf{L}=\begin{pmatrix} 0.241\,427\,724 & 0.272\,608\,938 \\ 0.272\,608\,938 & 0.514\,036\,662 \end{pmatrix},$$

$$\mathbf{M}=\begin{pmatrix} 0.485\,963\,338 & 0.213\,354\,401 \\ 0.213\,354\,401 & 0.699\,317\,739 \end{pmatrix}.$$