Energy distribution of maxima and minima in a one-dimensional random system

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We study the energy distribution of maxima and minima of a simple one-dimensional disordered Hamiltonian. We find that in systems with short-range correlated disorder there is energy separation between maxima and minima, such that at fixed energy only one kind of stationary point is dominant in number over the other. On the other hand, in the case of systems with long-range correlated disorder maxima and minima are completely mixed. [S1063-651X(99)14703-2]

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When the statistical properties of a system are studied, great attention is usually devoted to its ground state and to the first excited states. Moreover, in the case of random systems it is known that, in addition to the lowest-energy states, also metastable states are important, especially for the dynamical evolution of the system. As a consequence, the physical understanding of disordered models is most of the time founded on the properties of absolute as well as local *minima* of the Hamiltonian, while the role of stationary points of different nature is in general disregarded. In spite of this, the conviction is growing up that stationary points different from minima do have an importance, both from a dynamical and a static point of view.

Many different disordered systems display an off-equilibrium dynamical behavior which is suitable to be interpreted in terms of nontrivial structure of their phase space. Among these we find structural glasses [1], spin glasses [2,3], random manifolds [4], and neural networks [5]. In all these cases the geometric structure of the energy landscape is often invoked in order to give at least an intuitive picture of the relaxational dynamics. In this context it is clear that also unstable stationary points must be taken into consideration. For instance, the presence of flat directions in the phase space, marking a borderline between the last stable minima and the first unstable saddles, has been proposed as a possible explanation of slowness in glassy systems [6–8].

In light of these considerations, we believe it is important to understand the connections between the physical properties of a disordered system and the geometrical structure of *all* the stationary points of its Hamiltonian. Unfortunately, it is in general very difficult in an N-dimensional model to discriminate the stationary points according to their degree of instability. In this paper we will thus focus on a simple one-dimensional case and exactly compute the average energy distributions of maxima and minima. We will find a simple connection between the nature of the disorder that rules the physics of the system and the mutual distributions of the stationary points. Despite its simplicity, we expect the model studied here to capture at least some of the main features of more general problems.

Let us consider the one-dimensional random Hamiltonian [9,10],

$$H(x)=\tfrac{1}{2}mx^2+V(x),$$

where the position x is a real variable and the mass m is a parameter. $V(x)$ is a Gaussian random potential, with zero average and variance $\overline{V(x_1)V(x_2)}=G(x_1-x_2)$, with $G(x)=G(-x)$. The statics and the dynamics of this model have been studied both for the one-dimensional case [10] and for the more general N-dimensional case [11–13].

The number of stationary points of H is determined by the competition between the random potential and the harmonic mass term. This number is large for small m, whereas only one single minimum is present at large m. The physical properties of this model are encoded in the function G. In order to understand its meaning we consider the average displacement $[\Delta(d)]^2=\overline{[V(x_1)-V(x_2)]^2}=2G(0)-2G(d)$, where $d=(x_1-x_2)$ is the distance. Once introduced Δ it is natural to define two different classes of random potentials. If $\Delta(d)$ goes to a finite value $\Delta(\infty)$ for $d\to\infty$, then the memory is lost after a finite distance and V is called *short range* (SR). On the other hand, if $\Delta(d)\sim d^{\gamma}(\gamma>0)$, then the displacement grows indefinitely with d and the potential is *long range* (LR). In the SR case we can assume without loss of generality that $G(x)$ is a positive even function which is zero at infinity, so that $\Delta(\infty)=\sqrt{2G(0)}$. In the LR case we have to be more careful, since a diverging displacement would require $G(d)\to-\infty$ for $d\to\infty$, which is incompatible with the condition of having a positive kernel in the functional distribution of V. In order to correctly define the LR model we must put the system in a box of size L and define $G_L(x)$ through its Fourier transform,

$$G_L(x)=\frac{1}{\pi}\int_{1/L}^{\infty}dq\hat{G}(q)e^{iqx}. \tag{1}$$

The function $\hat{G}(q)$ must be positive and for the LR case must be *not* integrable in zero. In order to avoid any ultraviolet divergence we can assume both for the SR and LR cases $\hat{G}(q)$ to decay at infinity faster than any power. We can thus define these two classes of models simply in terms

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of the behavior of $\hat{G}$ at zero momentum. In this way the LR case is well defined: $\Delta_L(d)$ increases indefinitely with d and $\overline{V_L(x)^2}=G_L(0)$ diverges with L, as expected, since in a LR random potential the uncertainty on the height V of one single point x increases with the size L of the system, while it remains finite in a SR potential. Our analysis will not depend on the explicit form of $\hat{G}$.

Let us denote by $\mathcal{N}_k^V(E,m)dE$, the number of stationary points of $H(x)$ with degree of instability equal to k ($k=0$ for minima, $k=1$ for maxima), which have energy between E and $E+dE$, for a given mass m. The superscript V indicates that this distribution corresponds to the sample V. Eventually we shall average over V. The distribution $\mathcal{N}_k^V(E,m)$ is given by

$$\mathcal{N}_k^V(E,m)=\int dx\,\delta(H')\,\delta(H-E)|H''|\,\delta(\theta(-H'')-k).$$

In order to handle the modulus and the θ function we use the following relations:

$$\theta(-H'')=\frac{1}{2\pi i}\lim_{\epsilon\to 0}[\ln(H''-i\epsilon)-\ln(H''+i\epsilon)],$$
$$|H''|=\lim_{\epsilon\to 0}(H''+i\epsilon)^{1/2}(H''-i\epsilon)^{1/2}. \tag{2}$$

Using an integral representation for the δ function, we can write,

$$I\equiv|H''|\,\delta[\,\theta(-H'')-k]=\int d\mu e^{ik\mu}(H''+i\epsilon)^{(1/2)+(\mu/2\pi)}\times(H''-i\epsilon)^{(1/2)-(\mu/2\pi)}.$$

The last two factors can be rewritten using the identity

$$(H''\pm i\epsilon)^{n_\pm}=\int d\overline{\chi}_\pm^b\,d\chi_\pm^b\exp\left(-\sum_{b=1}^{n_\pm}\overline{\chi}_\pm^b(H''\pm i\epsilon)\chi_\pm^b\right),$$

where $\overline{\chi}_\pm^b$ and $\chi_\pm^b$ are Grassmann variables and the analytic continuation $n_\pm\to(1/2\pm\mu/2\pi)$ must be done. As a next step we define the Grassmann vector [15]:

$$\psi_a\equiv(\chi_+^1\ldots\chi_+^{n_+},\chi_-^1\ldots\chi_-^{n_-}),$$

which allows us to write

$$I=\int d\mu e^{ik\mu}\int d\overline{\psi}_a d\psi_a\exp\left(-\sum_{a=1}^{n}\overline{\psi}_a(H''+i\epsilon_a)\psi_a\right),$$

where the vector ϵ_a is split into two parts: $\epsilon_a=\epsilon$ for $a\leqslant n_+$, $\epsilon_a=-\epsilon$ for $a>n_+$, and $n=(n_++n_-)\to 1$. Note that this replica approach can be easily generalized to N dimensions.

Let us introduce in the expression for $\mathcal{N}_k^V$ the Lagrange multipliers λ and ω, to represent respectively $\delta(H')$ and $\delta(H-E)$. The V-dependent part then becomes $\exp[(i\omega+i\lambda\partial_x-\overline{\psi}_a\psi_a\partial_x\partial_x)V(x)]$, which can be averaged over the Gaussian distribution of V. This produces a quartic term $(\Sigma_a\overline{\psi}_a\psi_a)^2$, that can be made quadratic by means of a Hubbard-Stratonovich transformation, introducing an auxiliary variable y. It is now possible to perform all the Gaussian integrals over (λ,x,ψ). This gives a term $(m+y+i\epsilon)^{1/2+\mu/2\pi}(m+y-i\epsilon)^{1/2-\mu/2\pi}$, which, using again relations (2), can be written as $|m+y|\exp[-i\mu\theta(-m-y)]$. Integrating over μ we finally obtain the average distributions $\mathcal{N}_k(E,m)\equiv\overline{\mathcal{N}_k^V(E,m)}$,

$$\mathcal{N}_0(E,m)=\int_{-m}^{+\infty}dy F(y,E,m), \tag{3}$$

$$\mathcal{N}_1(E,m)=\int_{-\infty}^{-m}dy F(y,E,m), \tag{4}$$

with

$$F(y,E,m)=\frac{|m+y|}{\sqrt{m}}\,\frac{e^{-(y^2/2a_2)}}{\sqrt{2\pi a_2}}\times\int\frac{d\omega}{2\pi}\,\frac{\exp[-\frac{1}{2}(a_0-a_1^2/a_2)\omega^2+i\omega E+i\omega y a_1/a_2]}{\sqrt{m+ia_1\omega}}, \tag{5}$$

where $a_0=G(0)$, $a_1=-G''(0)$, and $a_2=G''''(0)$. The difference $\mathcal{D}(E,m)=\mathcal{N}_0(E,m)-\mathcal{N}_1(E,m)$ between minima and maxima has a much simpler expression,

$$\mathcal{D}(E,m)=\frac{1}{\sqrt{m}}\int\frac{d\omega}{2\pi}\sqrt{m+ia_1\omega}\,e^{-(1/2)a_0\omega^2+i\omega E}.$$

By integrating Eqs. (3) and (4) over the energy we get the total number of minima and maxima, $\mathcal{N}_0(m)$ and $\mathcal{N}_1(m)$, at a given value of the mass. Note that, as required by the Morse theorem [14], the *total* number of minima minus maxima is equal to one, that is $\mathcal{N}_0(m)-\mathcal{N}_1(m)=\int dE\mathcal{D}(E,m)=1$. The explicit expression for the total number of minima is

$$\mathcal{N}_0(m)=\tfrac{1}{2}+\tfrac{1}{2}\mathrm{erf}\left(\frac{m}{\sqrt{2a_2}}\right)+\frac{1}{m}\sqrt{\frac{a_2}{2\pi}}\exp\left(-\frac{m^2}{2a_2}\right).$$

$\mathcal{N}_0(m)$ is a smooth function of m which goes to one when m goes to infinity and starts increasing very steeply at masses smaller than $m\sim\sqrt{a_2}$. This value of the mass marks a crossover from the region where only one minimum exists to the region where many different minima (and maxima) appear. As expected this mass is the same critical mass as the

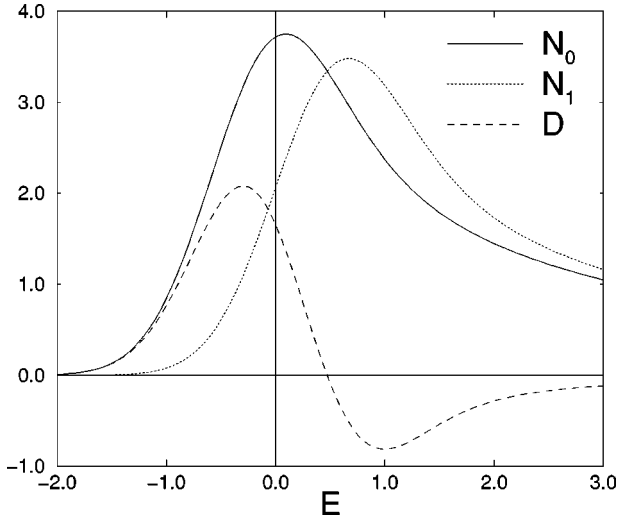

FIG. 1. SR potential: $\mathcal{N}_0(E,m)$, $\mathcal{N}_1(E,m)$, and $\mathcal{D}(E,m)$, functions of the energy E, at $m<m_c$. Here $\hat{G}(q)=e^{-q}$.

N-dimensional mean-field case, $m_c=\sqrt{a_2/3}$, where a glassy transition occurs [10,12]. In the following we will always consider $m<m_c$.

We analyze now our results, starting with the SR potential. In Fig. 1 we plot $\mathcal{N}_0(E,m)$, $\mathcal{N}_1(E,m)$ and $\mathcal{D}(E,m)$ functions of the energy E, for $m<m_c$. The first thing we notice is that the two curves of minima and maxima are quite separated one with respect to the other, so that their peaks do not overlap. As a consequence $\mathcal{D}(E,m)$ gives by itself a rather clear picture of the distribution of the different stationary points and at low energies it approximates well $\mathcal{N}_0(E,m)$. This is important because $\mathcal{D}$ is always very simple to compute, being $\mathcal{D}(E,m)=\int dx\,\delta(H')H''\delta(H-E)$. Thus, the computation of this quantity does not require the modulus, nor the θ function, which are in general very difficult to treat. In other words, in the SR case there is a partial decoupling between maxima and minima, which is sharper the lower the energy. As a consequence, at fixed energy only one kind of stationary points is dominant over the other and $\mathcal{N}_0(E,m)\sim\mathcal{D}(E,m)$, for low enough E. It is remarkable that this holds for the SR potential. Indeed, it has been proved in [7] that in the N-dimensional p-spin spherical spin glass, which belongs to the SR class [12], an identical phenomenon occurs: in that mean-field model, besides minima and maxima there are saddles of any order, but at fixed energy only one kind of stationary points is dominant over the other, so that the number of minima of the mean-field free energy (and therefore of states) can be safely calculated via the approximation $\mathcal{N}_0(E)\sim\mathcal{D}(E)$. We note that this same approximation has been used many times in the context of spin glasses, regardless of its grounding [16].

Let us now turn to the LR potential. We stress that the model is defined as long range (and thus it is different from the SR case) only in the limit $L\to\infty$. From Eq. (1) we know that $a_0=G_L(0)=\overline{V(x)^2}$ diverges with L. As mentioned above, the physical meaning of this is that the uncertainty on the value of H in any point x diverges when L goes to infinity. As a consequence, the energy E is no longer a good variable to label the height of the stationary points. In order to keep everything well defined in the limit $L\to\infty$ it is thus necessary to measure the energy in units of the natural diverging scale, that is, $\sqrt{a_0}$. Therefore, we must define a rescaled energy $\mathcal{E}=E/\sqrt{a_0}$ and study the distributions of maxima and minima as functions of $\mathcal{E}$. Denoting these new distributions by P_0 and P_1, we have, $P_k(\mathcal{E},m)d\mathcal{E}\equiv\mathcal{N}_k(E,m)dE$. We stress that $\mathcal{E}$ is the only variable we can sensibly regard as the energy for the LR potential. Note, on the other hand, that this rescaling is irrelevant for the SR case, where a_0 is finite. Taking the limit $L\to\infty$ in Eq. (5) we find,

$$P_{0,1}(\mathcal{E},m)\to\mathcal{N}_{0,1}(m)\frac{1}{\sqrt{2\pi}}\,e^{-(1/2)\mathcal{E}^2}. \tag{6}$$

This equation shows that in the LR case the two distributions are just the *same* function, scaled by the total number $\mathcal{N}_0(m)$ or $\mathcal{N}_1(m)$. Maxima and minima are no longer separated in energy. Indeed, for $m\ll m_c$, we have $\mathcal{N}_0(m)/\mathcal{N}_1(m)\sim 1$ and the two curves collapse, one onto the other. The conclusion is that when the total number of stationary points is large in a LR system, maxima and minima are completely mixed together, so that at each given energy they are equally numerous. Thus, in stark contrast to the SR case, no decoupling of the stationary points occurs, no matter how low the energy.

A further step is necessary to prove that this mixing in the LR case is a *typical* behavior and not simply an artifact coming from the average. Indeed, it is possible to think of a system where sample by sample maxima and minima are well separated, but where the mixing described above appears only after averaging over different samples. As an example, we consider the family of Hamiltonians $H_w(x)=\sin(x)+w$, where w is a random variable with zero average and variance σ. It is clear that for each sample maxima and minima are perfectly separated, since $\mathcal{N}^w_{0,1}(E)=\delta(E\pm 1-w)$. However, averaging over w we obtain two distributions with separation between their averages equal to 2 and variance σ. Thus, if we rescale the energy by a factor $\sqrt{\sigma}$ and take the limit $\sigma\to\infty$, we would conclude that there is mixing between maxima and minima, which is sample by sample false.

In order to prove that the LR potential does not correspond to such an artifact, we consider the statistics of the extreme values of the Hamiltonian. Let us define the two distributions $A_0(E)\equiv\overline{\delta(E-E_{MIN})}$ and $A_1(E)\equiv\overline{\delta(E-E_{MAX})}$, where E_{MIN} and E_{MAX} are the energies of the absolute minimum and maximum of $H(x)$ (we consider as absolute maximum the highest local maximum). The separation between these two distributions is $\Delta A=\overline{E_{MAX}-E_{MIN}}=\langle E\rangle_{A_1}-\langle E\rangle_{A_0}$, and let S be their variance. Consider now the ratio $\Delta A/\sqrt{S}$. It is easy to see that in the artificial case described above this ratio goes to zero when the variance of the disorder σ goes to infinity, since ΔA is finite, whereas S diverges as σ. On the other hand, for the LR potential the ratio $\Delta A/\sqrt{S}$ remains finite in the limit $L\to\infty$. This is simple to prove exactly for $\gamma=1/2$, which corresponds to a Brownian random potential of size L [9]. For the general case the idea is that in a LR potential both ΔA and $\sqrt{S}$ diverge as L^γ [17], as can also be checked by means of numerical simulations. The divergence of ΔA implies that

the variance of the energy distribution of maxima and minima is *sample by sample* diverging as well with L. Therefore, unlike what happens in the case of the artifact, the average scenario of the long-range potential, where maxima and minima are completely mixed in energy, *is* the typical one.

In this paper we have shown that the energy distribution of maxima and minima in a one-dimensional random system is radically different whether the disorder is long-range or short-range correlated. An important issue is the extension of this investigation to N dimensions, where we expect to find a qualitative similar behavior, at least at the mean-field level.

Indeed, as mentioned above, the p-spin spherical spin glass is a clear example of an N-dimensional mean-field SR system where at sufficiently low energies a decoupling between stationary points of different nature does occur. Moreover, a crucial feature of this SR model is that the asymptotic dynamical energy reached by the system is exactly the same energy where the stationary points decouple [7], and is larger than the equilibrium one, so that the dynamics never gets to the equilibrium landscape.

On the other hand, let us assume that for a LR N-dimensional model a generalization of Eq. (6) is valid, so that *all* the stationary points collapse on one single distribution, since saddles of any degree of instability will be bounded in energy by maxima and minima. This allows us to put forward the following hypothesis: if, as indicated by our results, in a LR system there is no separation at all between different stationary points, then no decoupling energy can exist, which means no energy level capable of trapping the system. In such a situation we would expect the dynamics to reach the minimum available energy, that is, the equilibrium energy.

An evidence of this conjecture can be found in the context of mean-field models for spin glasses. Here two very different classes of systems exist, a first class where the dynamical energy is larger than the equilibrium one and a second class where these two energies are the same. What has been noted in [12], is that the first class corresponds to systems with SR disorder (as the p-spin model), whereas to the second class belong LR disordered systems. This correspondence finds its natural explanation in the framework we have depicted above: the inability of a SR system to dynamically reach its equilibrium energy is due to the existence of an energy level below which stationary points of different nature are decoupled, while this cannot happen in the LR case. As we have tried to show in this paper, whether this decoupling occurs or not is information encoded in the energy distribution of *all* the stationary points.

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[1] C.A. Angell, Science **627**, 1924 (1995); F.H. Stillinger, *ibid.* **267**, 1935 (1995); S. Sastry, P.G. Debenedetti, and F.H. Stillinger, Nature (London) **393**, 554 (1998).

[2] M. Mézard, G. Parisi, and M.A. Virasoro, *Spin Glass Theory And Beyond* (World Scientific, Singapore, 1986).

[3] J.-P. Bouchaud, L.F. Cugliandolo, J. Kurchan, and M. Mézard, in *Spin-glasses and Random Fields*, edited by A.P. Young (World Scientific, Singapore, 1997).

[4] G. Blatter, M.V. Feigel'man, V.B. Geshkenbein, A.I. Larkin, and V.M. Vinokur, Rev. Mod. Phys. **66**, 1125 (1994).

[5] V. Dotsenko, *An Introduction to the Theory of Spin Glasses and Neural Networks* (World Scientific, Singapore, 1995).

[6] J. Kurchan and C. Laluox, J. Phys. A **29**, 1929 (1996).

[7] A. Cavagna, I. Giardina, and G. Parisi, Phys. Rev. B **57**, 11 251 (1998).

[8] F. Sciortino, S. Sastry, and P. Tartaglia, e-print cond-mat/9805040.

[9] J. Villain, B. Semeria, F. Lançon, and L. Billard, J. Phys. C **16**, 6153 (1983).

[10] M. Mézard and G. Parisi, J. Phys. I **2**, 2231 (1992); A. Engels, Nucl. Phys. B **410**, 617 (1993).

[11] M. Mézard and G. Parisi, J. Phys. A **23**, L1229 (1990); J. Phys. I **1**, 809 (1991).

[12] L.F. Cugliandolo, J. Kurchan, and P. Le Doussal, Phys. Rev. Lett. **76**, 2390 (1996); L.F. Cugliandolo and P. Le Doussal, Phys. Rev. E **53**, 1525 (1996).

[13] S. Franz and M. Mézard, Europhys. Lett. **26**, 209 (1994); Physica A **209**, 1 (1994).

[14] B. Doubrovine, S. Novikov, and A. Fomenko, *Geometrie Contemporaine* (Mir, Moscow, 1982).

[15] J. Kurchan (private communication).

[16] For a discussion of this problem see J. Kurchan, J. Phys. A **24**, 4969 (1991), and references therein.

[17] G.R. Grimmett and D.R. Stirzaker, *Probability and Random Processes* (Clarendon Press, Oxford, 1982).