Yang-Lee zeros of one-dimensional quantum many-body systems

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We obtain a generic formula for the distribution of Yang-Lee zeros of one-dimensional quantum many-body systems solvable by the Bethe ansatz and the Yang-Yang thermodynamic formalism. We find that the zeros are located on the negative real axis of the complex fugacity plane and thus prove that no phase transition occurs in these one-dimensional systems, as proved by others using different methods. [S1063-651X(99)08101-5]

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I. INTRODUCTION

In 1952 Yang and Lee [1] proposed a general theory of phase transitions. They observed that for a real interacting gas, the pair interaction has a hard core. For such a system in a finite volume, the grand partition function can be expressed as a polynomial of the fugacity. They introduced the zeros of the polynomial and showed that in the thermodynamic limit, the zero distribution can touch the positive real axis to develop the singularity of the thermodynamic potential. The occurrence of the positive real roots corresponds to the singularity of the thermodynamic potential. Yang and Lee introduced a lattice gas model and showed that the zeros of the model are located on a unit circle in the complex fugacity plane. They also made a successful application to the ferromagnetic Ising model and showed that the zeros of the partion function of the ferromagnetic Ising model on any lattice are located on a unit circle in the complex magnetic field plane. Since then, the circle theorem has been extended to many ferromagnetic systems, such as the higher-spin Ising model [2,3], Ising models with multiple spin interactions, the quantum Heisenberg model [4], the classical XY and Heisenberg model [5], and some continuous spin systems [6].

In 1965 Fisher [7] studied the zeros of the partition function in the complex temperature plane for the square lattice Ising model. Since then, there have been many studies about the zero distribution in the complex temperature plane, including the Ising model on many lattices [8], the Potts model [9], and the Hubbard model [10]. Recently, we found that for the Ising model on square, triangular, and honeycomb lattices, introducing the zeros of the Ising partition function on the elementary cycle of these lattices, we can get some useful information [11].

There have been many studies on the zero distribution of the grand partition function in the complex fugacity plane since Yang and Lee's pioneering work. The circle theorem has been proved to be valid for hard-core binary lattice gases by Runnels and Lebowitz [12]. For the one-dimensional (1D) classical hard rod gas [13,14], the 1D gas with very weak repulsion of very long range [13], and the monomer-dimer system [15], the zeros are located on the negative real axis. For the classical many-body systems with repulsive interactions, the zeros are located on the real axis [16]. In addition, for some physical systems such as van der Waals gas [14,17], the lattice gas models with more complicated interactions [18], the hard hexagon model [19], and the 1D plasma model [20], the zeros are investigated. However, in the case of quantum gas, to our knowledge, the distribution of Yang-Lee zeros is not clear. This is the theme of this paper.

The first model of the 1D quantum many-body system solved, to our knowledge, is the quantum hard rod gas [21]. In 1963 Lieb and Liniger [22] studied a gas of 1D Bose particles with a repulsive δ-function interaction. Using the Bethe ansatz, they diagonized the Hamiltonian and obtained the ground state energy. In 1967 Yang [23] solved the 1D electron system with a δ-function interaction and discovered the Yang-Baxter equation. In 1968 Lieb and Wu [24] solved a lattice version of this model (the 1D Hubbard model). In 1969 Yang and Yang [25] developed a thermodynamic formalism for dealing with those interacting systems whose Hamiltonian can be diagonalized with the use of the Bethe ansatz. This formalism has been used to determine the thermodynamics of many problems: the 1D quantum many-body system [26], the 1D Hubbard model, the 1D Heisenberg model, and the Kondo model.

This paper is organized as follows. In Sec. II we discuss the Yang-Lee theory of the phase transition. In Sec. III the zeros of an ideal Fermi gas in any dimension are determined. In Sec. IV the Bethe ansatz and Yang-Yang thermodynamic formalism are reviewed. The general formula of Yang-Lee zeros is given. In Sec. V the Yang-Lee zeros of a quantum hard rod gas are determined. In Secs. VI and VII the Yang-Lee zeros of the Sutherland model and the Sutherland model with a hard core are determined. In Sec. VIII the Yang-Lee zeros of a Bose gas with a repulsive δ-function interaction are determined. In Sec. IX a summary of this paper is given.

II. YANG-LEE THEORY OF PHASE TRANSITION

In the original Yang-Lee approach, the existence of zeros is guaranteed by the hard-core interaction

$$\begin{aligned} V(r) &= \infty \quad (r \leqslant a) \\ &\neq 0 \quad (r > a), \end{aligned} \tag{1}$$

where a is the radius of the hard core. For a given volume V, the maximum number M of particles that can be crammed into the volume is limited by the size of hard core, i.e., $M \sim V/a^3$. The grand partition function can be expressed as a polynomial of fugacity $z = \exp(\mu/T)$,

$$\Xi = \sum_{n=0}^{M} z^n Q_n / n! = \prod_{l=1}^{M} \left(1 - \frac{z}{z_l}\right), \tag{2}$$

where Q_n is the partition function of the system with n particles in the volume V. The roots z_l are never positive real. The root distribution can touch the positive real axis only in the thermodynamic limit and give the transition point. The singularity of the thermodynamic potential is connected with the positive real root.

The hard core approximation is quite reasonable for a real gas. However, in some cases, the hard core does not exist. As noted a long time ago by Hauge and Hemmer [13], for a 1D classical system with weak long-range repulsion and no hard core, the Yang-Lee zeros are still present and located on the negative real axis. From this example they speculated that the Yang-Lee theory is applicable even in the absence of the condition of a hard core. We consider that their speculation is viable. We shall show several examples where the Yang-Lee approach still works under a relaxed condition that as $r \to 0$, the two-body potential $V(r) \to +\infty$. For such a system one can envision an effective hard core whose radius is temperature dependent so that the Yang-Lee zeros exist.

III. IDEAL FERMI GAS

For simplicity, let us consider an ideal Fermi gas in any dimension. For an ideal Fermi or Bose gas, no direct interaction exists. However, quantum effects act as indirect interactions. As shown by Uhlenbeck and Gropper [27], the quantum gas can be treated as a classical interacting gas with a statistical interparticle potential $V_s(r)$,

$$V_s(r) = -T \ln[1 \pm \exp(-2\pi r^2/\lambda^2)], \tag{3}$$

where $\lambda = (4\pi/T)^{1/2}$ is the thermal wavelength. Throughout this paper, we use the units $\hbar = k_B = 1$ and $2m = 1$ (m is the mass of one particle). The sign $\pm$ corresponds to Bose and Fermi statistics, respectively. As $r \to 0$,

$$V_s(r) = \begin{cases} -T \ln(2\pi r^2/\lambda^2) \to +\infty & \text{(Fermi statistics)} \\ -T \ln 2 & \text{(Bose statistics).} \end{cases} \tag{4}$$

Therefore, we expect that for an ideal Fermi gas, Yang-Lee zeros should exist and for a Bose gas, the zeros should not exist.

Let us check this directly. For an ideal Fermi gas, the grand partition function is given by

$$\Xi = \prod_k \left[1 + \frac{z}{\exp(k^2/T)}\right], \tag{5}$$

where k denotes the quantum state with energy k^2. We can easily identify that the zeros exist on the negative real axis,

$$z_k = -\exp(k^2/T). \tag{6}$$

For an ideal Bose gas, the grand partition function is given by

$$\Xi^{-1} = \prod_k \left[1 - \frac{z}{\exp(k^2/T)}\right]. \tag{7}$$

No zeros exist in this case, as expected.

IV. YANG-LEE ZEROS OF A 1D SYSTEM

A. Bethe ansatz

We consider a 1D quantum gas of either fermions or bosons interacting via a two-body potential $V(|x_1 - x_2|)$. The potential is restricted such that no bound states exist. The two-body Schrödinger equation reads

$$\left[-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + V(|x_1 - x_2|)\right]\psi(x_1, x_2) = E\psi(x_1, x_2). \tag{8}$$

If we introduce the relative coordinate $r = (x_1 - x_2)/2$ and the center of mass coordinate $R = (x_1 + x_2)/2$, Eq. (8) becomes

$$\left[-\frac{\partial^2}{\partial r^2} + 2V(2r)\right]\psi(r) = k^2\psi(r), \tag{9}$$

where $E = k^2/2$. The condition that there are no bound states implies that in the $r \to \infty$ limit, the wave function asymptotically approaches

$$\psi(r) \to \sin\left[kr - \frac{1}{2}\theta(k)\right], \tag{10}$$

where $\theta(k)$ is the two-body phase shift and is odd in k.

The N-body wave function is given by the Bethe ansatz

$$\psi(x_1, x_2, \ldots, x_N) = \sum_P A(P) \exp\left(i \sum k_{Pj} x_j\right), \tag{11}$$

where the coefficients $A(P)$ are determined solely by the two-body phase shift

$$A(\ldots, k', k, \ldots)/A(\ldots, k, k', \ldots) = -\exp[-i\theta(k - k')]. \tag{12}$$

The energy is given by $E = \Sigma_{i=1}^{N} k_i^2$. The periodic boundary condition is imposed so that the k's satisfy

$$kL = 2\pi I_k + \sum_{k'} \theta(k - k'), \tag{13}$$

where I_k is an integer if N is odd and $I_k + \frac{1}{2}$ is an integer if N is even. The numbers I are quantum numbers for the problem.

B. Yang-Yang thermodynamic formalism

In order to extend the Bethe ansatz to the finite temperature case, Yang and Yang [25] introduced the distribution functions for particles and holes

$$1 = 2\pi(\rho + \rho_h) + \int \theta' \rho(k') dk', \tag{14}$$

where $\theta' \equiv d\theta(k - k')/dk$. The energy and number of particles are, respectively, given by

$$E=L\int_{-\infty}^{\infty}\rho(k)k^2dk, \tag{15}$$

$$N=L\int_{-\infty}^{\infty}\rho(k)dk. \tag{16}$$

Yang and Yang envisioned the system as an ideal gas composed of particles and holes. So the entropy for given distribution ρ and ρ_h is given by

$$S=L\int_{-\infty}^{\infty}dk[(\rho+\rho_h)\ln(\rho+\rho_h)-\rho\ln\rho-\rho_h\ln\rho_h]. \tag{17}$$

The thermodynamics is obtained by minimizing the free energy density $f=E/N-TS/N$ with the particle density $n=N/L$ held constant, namely, $-\mu\delta N+\delta E-T\delta S=0$. Defining an auxiliary parameter $\rho_h/\rho\equiv\exp[\epsilon(k)/T]$, Eq. (14) becomes

$$1=2\pi\rho(k)\{1+\exp[\epsilon(k)/T]\}+\int\theta'\rho(k')dk'. \tag{18}$$

From the extremum condition, we obtain

$$\epsilon(k)=-\mu+k^2+\frac{T}{2\pi}\int_{-\infty}^{\infty}\theta'\ln[1+e^{-\epsilon(k')/T}]dk'. \tag{19}$$

The pressure P is given by

$$P=\frac{T}{2\pi}\int_{-\infty}^{\infty}dk\ln(1+e^{-\epsilon(k)/T}). \tag{20}$$

C. Yang-Lee zeros

The grand partition function of a 1D system is given by

$$\Xi=\exp(PL/T)=\exp\left[\frac{L}{2\pi}\int_{-\infty}^{\infty}dk\ln(1+e^{-\epsilon(k)/T})\right]$$
$$=\prod_k[1+e^{-\epsilon(k)/T}]\equiv\prod_k\left(1-\frac{z}{z_k}\right), \tag{21}$$

where $z=\exp(\mu/T)$. In the last equality of Eq. (21) we have identified the Yang-Lee zeros as

$$z_k=-ze^{\epsilon(k)/T}=-e^{k^2/T}e^{\lambda(k,T)/T}, \tag{22}$$

where we used Eq. (19) in the last equality and defined an interaction-dependent parameter $\lambda(k,T)$. We find that for a one-dimensional quantum many-body systems solvable by the Bethe ansatz and the Yang-Yang thermodynamic formalism, the Yang-Lee zeros are located on the negative real axis and are determined by the parameter $\epsilon(k)$.

The thermodynamic properties of the system are determined completely by the distribution of the Yang-Lee zeros. Since no positive real root exists, according to the Yang-Lee theory of phase transition, no phase transition exists in a one-dimensional system solvable by the Bethe ansatz and the Yang-Yang thermodynamic formalism. For some special cases, the nonexistence of phase transition has been proved by Yang and Yang [25] for $V(r)=2c\delta(r)$ and by Sutherland [26] for $V(r)=g/r^2$ using different methods. Thus we have proved the absence of a phase transition for a general one-dimensional quantum gas even with a long-range force. In the following we discuss several models.

V. QUANTUM HARD ROD GAS

It is easy to determine the phase shift $\theta(k)=ka$. From Eqs. (19) and (20) we obtain

$$\epsilon(k)=-\mu+k^2+aP, \tag{23}$$

$$P=\frac{T}{2\pi}\int_{-\infty}^{\infty}dk\ln\left[1+\exp\left(\frac{\mu-k^2-aP}{T}\right)\right], \tag{24}$$

$$N=\frac{L}{2\pi}(1-Na/L)\int_{-\infty}^{\infty}dk\frac{1}{\exp[(k^2+aP-\mu)/T]+1}. \tag{25}$$

We see that the quantum hard rod gas is an ideal Fermi gas, with the length $L-Na$ and the chemical potential $\mu-aP$. From Eq. (22) we identify the distribution of Yang-Lee zeros as

$$z_k=-e^{k^2/T}e^{aP/T}. \tag{26}$$

At high temperature, the gas becomes a Boltzmann gas (classical hard rod gas). The equation of state of the gas becomes $P(L-Na)=NT$. The zero distribution becomes $z_k=-e^{k^2/T}e^{Na/(L-Na)}$. Since $L\gg Na$, the zero distribution becomes $z_k=-e^{k^2/T}$. Hauge and Hemmmer [13] obtained a different distribution using another method. Both distributions give the same pressure and density. When a system does not have a phase transition, the distributions of Yang-Lee zeros are often found to be not unique, although these distributions give the same physical results [13,14,17].

VI. SUTHERLAND MODEL

The two-body potential is given by $V(r)=g/r^2$ $(g>0)$. Equation (9) becomes

$$\left[-\frac{\partial^2}{\partial r^2}+\frac{g/2}{r^2}\right]\psi(r)=k^2\psi(r). \tag{27}$$

With the change of variables $x=kr$ and $\psi(x)=x^{1/2}u(x)$, Eq. (27) is transformed into a Bessel equation

$$\frac{d^2}{dx^2}u+\frac{1}{x}\frac{d}{dx}u+\left(1-\frac{\alpha^2}{x^2}\right)u=0, \tag{28}$$

where $\alpha=\frac{1}{2}(1+2g)^{1/2}$. Since $\psi(x\to0)$ is finite, we have $u(x)=J_\alpha(x)$ and $\psi(r)=(kr)^{1/2}J_\alpha(kr)$. As $r\to\infty$,

$$\psi(r)\to\left(\frac{2}{\pi}\right)^{1/2}\sin\left(kr-\frac{\alpha\pi}{2}+\frac{\pi}{4}\right). \tag{29}$$

So we identify the phase shift $\theta(k)=(k/|k|)(\alpha\pi-\pi/2)$. Thus we obtain

$$\frac{d\theta(k)}{dk}=2\pi\gamma\delta(k), \tag{30}$$

where $\gamma=\alpha-\frac{1}{2}>0$. Substituting Eq. (30) into Eq. (19), we get

$$(e^{\epsilon/T})^{1+\gamma}(1+e^{\epsilon/T})^{-\gamma}=e^{(-\mu+k^2)/T}. \tag{31}$$

The particle distribution function is obtained from Eq. (18) as

$$\rho(k)=\frac{1}{2\pi[e^{\epsilon(k)/T}+1+\gamma]}. \tag{32}$$

Finally, the chemical potential μ is determined by

$$N=\frac{L}{2\pi}\int_{-\infty}^{\infty}dk\,\frac{1}{e^{\epsilon/T}+1+\gamma}. \tag{33}$$

At high temperature, the gas becomes a Boltzmann gas with $e^{\epsilon/T}=e^{(-\mu+k^2)/T}$. The distribution of Yang-Lee zeros becomes $z_k=-e^{k^2/T}$.

In the low-temperature limit [26], Eq. (31) can be easily solved for ϵ to yield $\epsilon=-k_f^2+k^2$ for $|k|>k_f$ and $\epsilon=(-k_f^2+k^2)/(1+\gamma)$ for $|k|\leqslant k_f$, where $k_f^2=\mu$. Here k_f is called the Fermi momentum in the sense that $\rho(k)=\text{const}$ for $|k|\leqslant k_f$ and $\rho(k)=0$ for $|k|>k_f$. From Eq. (33) we obtain the chemical potential $\sqrt{\mu}=k_f=\pi(1+\gamma)N/L$. Therefore, the Yang-Lee zeros are given by $z_k=-e^{k^2/T}$ for $|k|>k_f$ and $z_k=-e^{k^2/T}e^{\gamma k^2/(1+\gamma)T}e^{\gamma k_f^2/(1+\gamma)T}$ for $|k|\leqslant k_f$.

Equation (31) should be solved numerically in general. However, it can be solved analytically when $\gamma=1/3,1/2,1,2,3$. We list the solutions in the following.

(i) γ=*1*. From Eq. (31), we get

$$e^{\epsilon/T}=\frac{b}{2}+\sqrt{b+\frac{b^2}{4}}, \tag{34}$$

where $b=e^{(-\mu+k^2)/T}$. Thus the zeros are given by

$$z_k=-e^{k^2/T}\left[\frac{1}{2}+\sqrt{\frac{1}{4}+e^{(\mu-k^2)/T}}\right]. \tag{35}$$

(ii) γ=*2*. The parameter $\epsilon(k)$ is given by

$$e^{\epsilon/T}=\frac{b}{3}+(R+D^{1/2})^{1/3}+(R-D^{1/2})^{1/3}, \tag{36}$$

where $D=Q^3+R^2=b^2/4+b^3/27>0$, $R=(27b+18b^2+2b^3)/54$, and $Q=-(6b+b^2)/9$. So the Yang-Lee zeros are

$$z_k=-e^{k^2/T}\left[\frac{1}{3}+\frac{1}{b}(R+D^{1/2})^{1/3}+\frac{1}{b}(R-D^{1/2})^{1/3}\right]. \tag{37}$$

(iii) γ=*1/2*. The parameter $\epsilon(k)$ is given by

$$e^{\epsilon/T}=\begin{cases}2\sqrt{3}b\cos(\tau/3), & b\geqslant 3\sqrt{3}/2\\ D_1+D_2, & b<3\sqrt{3}/2,\end{cases} \tag{38}$$

where $\tau=\cos^{-1}(3\sqrt{3}/2b)$ and $D_1,D_2=[b^2/2\pm(b^4/4-b^6/27)^{1/2}]^{1/3}$. So the Yang-Lee zeros are

$$z_k=\begin{cases}-2\sqrt{3}e^{k^2/T}\cos(\tau/3), & b\geqslant 3\sqrt{3}/2\\ -e^{k^2/T}[(D_1+D_2)/b], & b<3\sqrt{3}/2.\end{cases} \tag{39}$$

(iv) γ=*1/3*. The real solution of $x^3+4b^3x-b^6=0$ is $x=D_3+D_4$. Here $D_3,D_4=[b^6/2\pm(64b^9/27+b^{12}/4)^{1/2}]^{1/3}$. The parameter $\epsilon(k)$ is given by

$$e^{\epsilon/T}=\sqrt{\frac{x}{2}}+\left[-\frac{x}{4}+\frac{1}{2}(x^2+4b^3)^{1/2}\right]^{1/2}. \tag{40}$$

The distribution of Yang-Lee zeros is

$$z_k=-e^{k^2/T}\left[\sqrt{\frac{x}{2}}+\left(-\frac{x}{4}+\frac{1}{2}(x^2+4b^3)^{1/2}\right)^{1/2}\right]\Big/\,b. \tag{41}$$

(v) γ=*3*. The parameter $\epsilon(k)$ is given by

$$w^4-bw^3-3bw^2-3bw-b=0, \tag{42}$$

where $w=e^{\epsilon/T}$. The solution is too lengthy to list.

VII. SUTHERLAND MODEL WITH A HARD CORE

Now we consider the Sutherland gas with a hard core. The two-body potential is given by $V(r)=+\infty$ for $r\leqslant a$ and $V(r)=g/r^2$ for $r>a$. From the preceding section we know that the wave function is a linear combination of both kinds of Bessel functions

$$\psi(r)=(kr)^{1/2}[b_1J_\alpha(kr)+b_2N_\alpha(kr)]. \tag{43}$$

The boundary condition is $\psi(r=a/2)=0$, so

$$b_1J_\alpha(ka/2)+b_2N_\alpha(ka/2)=0. \tag{44}$$

In the limit $r\rightarrow\infty$ we have

$$\psi(r)\rightarrow\left(\frac{2}{\pi}\right)^{1/2}\left[b_1\cos\left(kr-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)+b_2\sin\left(kr-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)\right]. \tag{45}$$

We define $\tan\omega=-N_\alpha(ka/2)/J_\alpha(ka/2)$. Then we can express the phase shift as

$$\theta(k)=-2\omega+(k/|k|)\left(\alpha\pi+\frac{\pi}{2}\right). \tag{46}$$

For $g=0$, we have $\alpha=1/2$ and thus $\tan\omega=-N_{1/2}(ka/2)/J_{1/2}(ka/2)=\cot(ka/2)$. Thus $\omega=(k/|k|)(\pi/2)-ka/2$ and $\theta=ka$.

For $g\geqslant-1/2$, no bound state exists. In this case, the gas has a classical limit. Therefore, we take $g\geqslant-1/2$. Let us investgate a simple case: $g=4$ and so $\alpha=3/2$. Thus $\tan\omega=-N_{3/2}(ka/2)/J_{3/2}(ka/2)$. With a little bit of algebra we obtain

$$\omega=(k/|k|)\frac{\pi}{2}-ka/2+\tan^{-1}(ka/2) \tag{47}$$

and thus

$$\theta=ka-2\tan^{-1}(ka/2)+(k/|k|)\pi. \tag{48}$$

Substituting Eq. (48) into Eq. (19), we get

$$\epsilon=-\mu+k^2+aP+T\ln(1+e^{-\epsilon/T}) -\frac{2T}{\pi a}\int_{-\infty}^{\infty}dk'\frac{1}{(2/a)^2+(k-k')^2}\ln[1+e^{-\epsilon(k')/T}]. \tag{49}$$

At high temperature, the gas becomes a Boltzmann gas. The distribution of Yang-Lee zeros becomes $z_k=-e^{k^2/T}$.

In the low-temperature limit, we define the Fermi momentum k_f such that $\epsilon(k=k_f)=0$. When $|k|>k_f$, $\epsilon(k)>0$; when $|k|<k_f$, $\epsilon(k)<0$. Equation (18) gives $\rho=0$ for $|k|>k_f$ and $\rho_h=0$ for $|k|<k_f$. Equations (18) and (19) become, for $|k|<k_f$,

$$2\pi\rho(k)=1-\int_{-k_f}^{k_f}dk'\rho(k')\left[a-\frac{4}{a}\frac{1}{(2/a)^2+(k-k')^2} +2\pi\delta(k'-k)\right] \tag{50}$$

and

$$\epsilon=-\mu+k^2+aP-\epsilon+\frac{2}{\pi a}\int_{-k_f}^{k_f}dk'\frac{1}{(2/a)^2+(k-k')^2}\epsilon(k'); \tag{51}$$

for $|k|>k_f$,

$$\epsilon=-\mu+k^2+aP+\frac{2}{\pi a}\int_{-k_f}^{k_f}dk'\frac{1}{(2/a)^2+(k-k')^2}\epsilon(k'). \tag{52}$$

Let us consider the case $k_f a\ll 1$. For $|k|<k_f$ we have

$$\frac{1}{(2/a)^2+(k-k')^2}\approx\frac{a^2}{4}-\frac{a^4}{16}(k-k')^2+O((k-k')^4). \tag{53}$$

Thus Eq. (51) gives

$$2\epsilon=-\mu+k^2\left(1+\frac{a^3P}{4}\right)+C_1, \tag{54}$$

where $C_1=k_f^5a^3/60\pi$. From Eq. (20) we obtain $P=(1/3\pi)k_f^3[1+(1/12\pi)(k_f a)^3]$. Using $\epsilon(k_f)=0$, we deduce that $\mu=k_f^2[1+(1/10\pi)(k_f a)^3]$. From Eq. (50) we get $4\pi\rho(k)=1-(a^3N/4L)k^2-(a^3/4)C_2$. Here $C_2=k_f^3/6\pi$. Using Eq. (16) we obtain

$$\frac{2\pi N}{Lk_f}=1-\frac{Na}{12L}(k_f a)^2-\frac{1}{24\pi}(k_f a)^3. \tag{55}$$

Thus we obtain the distribution of Yang-Lee zeros, for $|k|<k_f$,

$$z_k=-e^{k^2/T}e^{k^2(-1+a^3P/4)/2T}e^{(\mu+C_1)/2T}. \tag{56}$$

VIII. BOSE GAS WITH REPULSIVE δ-FUNCTION INTERACTION

The two-body potential is given by $V(r)=2c\,\delta(r)$ ($c>0$). The phase shift can be easily obtained, $\theta(k)=-2\tan^{-1}(k/c)$. Now Eqs. (18) and (19) become

$$2\pi\rho(k)(1+e^{\epsilon/T})=1+2c\int_{-\infty}^{\infty}dk'\rho(k')\frac{1}{c^2+(k-k')^2} \tag{57}$$

and

$$\epsilon=-\mu+k^2-\frac{Tc}{\pi}\int_{-\infty}^{\infty}dk'\frac{1}{c^2+(k-k')^2}\ln[1+e^{-\epsilon(k')/T}]. \tag{58}$$

Here $\epsilon(k)$ is a monotonically increasing function of k^2.

At high temperature the gas becomes a Boltzmann gas. The distribution of Yang-Lee zeros becomes $z_k=-e^{k^2/T}$.

In the low-temperature limit, we define the Fermi momentum k_f such that $\epsilon(k_f)=0$. So $\epsilon(k)>0$ for $|k|>k_f$ and $\epsilon(k)<0$ for $|k|<k_f$. Equations (57) and (58) give $\rho=0$ for $|k|>k_f$. For $|k|<k_f$, we have $\rho_h=0$ and thus

$$2\pi\rho(k)=1+2c\int_{-k_f}^{k_f}dk'\rho(k')\frac{1}{c^2+(k-k')^2} \tag{59}$$

and ϵ is given by

$$\epsilon=-\mu+k^2+\frac{c}{\pi}\int_{-k_f}^{k_f}dk'\frac{1}{c^2+(k-k')^2}\epsilon(k'). \tag{60}$$

If $k_f/c\ll 1$, then $1/[c^2+(k-k')^2]\approx 1/c^2$. Thus Eqs. (16), (59), and (60) give $\rho(k)=c/(2\pi c-4k_f)$ and $\epsilon=-\mu+k^2+D$. Here $D=(-2\mu k_f+\frac{2}{3}k_f^3)/(\pi c-2k_f)$, $\mu=k_f^2-D$, and $k_f=2\pi Nc/(4N+2Lc)$. The Yang-Lee zero distribution becomes $z_k=-e^{k^2/T}e^{D/T}$ for $|k|<k_f$.

IX. CONCLUSION

We have studied the Yang-Lee zeros for several 1D quantum many-body systems. In the original Yang-Lee theory, the existence of Yang-Lee zeros requires the condition of a hard core. We have given a few popular examples, where the Yang-Lee theory is valid with a relaxed condition that $V(r)\to+\infty$ as $r\to 0$. We studied the ideal Fermi gas in any dimension to find the zeros at $z_k=-e^{k^2/T}$, whereas for an ideal Bose gas, we confirmed that the zeros do not exist. We also considered the 1D quantum many-body systems solvable by the Bethe ansatz and the Yang-Yang thermodynamic formalism. We have found that the zeros are given by $z_k=-z\exp[\epsilon(k)/T]$. Here z is the fugacity and $\epsilon(k)\equiv T\ln(\rho_h/\rho)$. The zeros are located on the negative real axis. No phase transition exists according to the Yang-Lee theory of phase transitions. We conclude that these 1D quantum gases solvable by the Bethe ansatz and the Yang-Yang thermodynamic formalism, with finite or long-range force, do not have phase transitions. We should point out that these results are valid only for 1D systems.

In higher dimensions, quantum many-body systems have phase transitions in general. Hence the distribution of Yang-Lee zeros is quite different from the 1D case. In particular, in the thermodynamic limit, the distribution of Yang-Lee zeros will approach the positive real axis and give the transition point. The elementary excitations are quite different.

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