Influence of dissipation on stationary states

Thierry Biben
Laboratoire de Physique (URA 1325 du CNRS), Ecole Normale Supérieure de Lyon, 46 Allée d'Italie 69364 Lyon Cedex 07, France

Jarosław Piasecki
Institute of Theoretical Physics, University of Warsaw, Hoża 69, 00 681 Warsaw, Poland
(Received 10 August 1998)

Stationary states in model dynamical systems are studied with the aim of elucidating the role of dissipative interactions. First, the influence of coupling to the surrounding medium via friction forces (solid or fluid) is considered on the example of a mass falling in the gravitational field and interacting with a thermalizing base (the corresponding stochastic boundary condition at the base is derived). The nature of deviations from the Maxwell-Boltzmann distribution is discussed based on rigorous analytic approach combined with numerical analysis. The effect of inelastic binary collisions, relevant to the theory of fluidized granular matter, is studied for a column of N colliding masses subject to gravity and absorbing energy from the base. In the case, where the rebound velocity distribution is centered on a single value, remarkably simple periodic states are observed. Analytic construction of the stationary distribution is presented for $N=2$. For $N>2$, a rigorous necessary condition for the existence of periodic states is derived, showing the relevance of parameter $\gamma=(1-\alpha)N$, where α is the restitution coefficient of inelastic collisions. Computer simulations for two inelastically colliding masses in the presence of a thermalizing base are described, indicating important differences in the stationary distribution with respect to that caused by friction. [S1063-651X(99)14902-X]

PACS number(s): 05.20.Dd

I. INTRODUCTION

The aim of this paper is to study the influence of dissipation on the nature of stationary states occurring in open systems. The idea is first to establish the physical situation in the absence of dissipation, and then derive modifications induced by it. This program is followed here with simple examples relevant to the theory of fluidized granular matter.

The system under consideration is a one-dimensional column of masses falling in a uniform gravitational field, and being sent back into space owing to the energy provided by collisions with an underlying base. The source of dissipation may be due to the coupling to the surrounding medium through friction forces, or to the inelastic character of binary collisions between the particles.

Theoretical, numerical, and experimental studies have been already devoted to the dynamics of such a system. Unforced systems, in which the kinetic energy is dissipated without replenishment, have been studied both in the quasielastic limit [1] and in the perfectly inelastic regime [2]. One of the interesting features of one-dimensional granular systems is the so-called ''inelastic-collapse'' [3,4], where particles can collide infinitely often in finite time. Vibrated one-dimensional systems have also been studied theoretically, numerically, and experimentally in the limit $N\to\infty$, where N is the number of beads [5–7]. In these references, the relevance of parameter $\gamma=(1-\alpha)N$, where α is the restitution coefficient of inelastic collisions, have been pointed out and validated. The special case $N=1$, with purely inelastic collisions with the base, proved to be an interesting dynamical system exhibiting a large variety of periodic states [8]. Finally, a recent experimental work was devoted to a study of the rebound velocity distribution of a vibrated one-dimensional system [9], with the result that the rebound velocity distribution function is proportional to v times the Maxwell-Boltzmann distribution at high frequency.

However, the theory presented in most of the references cited above consisted essentially of analyzing and describing the results of experimental data or numerical simulations, and comparing them with predictions of some approximate kinetic or Langevin-like equations. Also, a type of hydrodynamic approach, involving numerous approximations, has been proposed [10].

In contradistinction to approximate approaches, the results reported here are based on a rigorous analysis of the dynamics, and thus contribute to a better understanding of the effects of dissipation. Although they could be obtained only in some relatively simple cases, they help in clarifying the status of intuitive ideas, in particular of the concept of granular temperature [11]. In some cases computer simulations turned out to be of great utility in finding a way to an exact solution of the equations of motion. We illustrate and complete the analytic results by pictures of the dynamics emerging from simulations.

In Sec. II a discussion of the stochastic boundary conditions at the energy providing base is presented. The form of the thermalizing rebound velocity distribution is derived therein. Then comes an analysis of the stationary distribution of a single mass subject, when falling in the gravitational field, to a solidlike or fluidlike friction (Secs. III and IV).

In Secs. V and VI we analyze stationary states of two or more masses falling freely in the gravitational field and dissipating their energy through inelastic binary collisions. Analytic results are described in Sec. V for the rebound distribution centered on a single characteristic velocity. The appearance of remarkably organized periodic states here yields a beautiful example for the reduction of the phase space volume. A necessary condition for the existence of

1063-651X/99/59(2)/2192(11)/$15.00 PRE 59

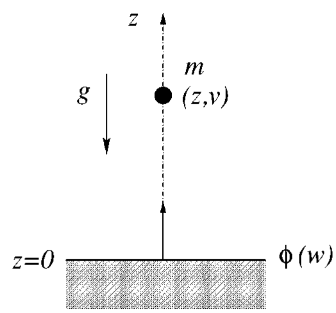

FIG. 1. Schematic representation of the system. $\phi(v)$ is the rebound velocity distribution.

these states is derived. Moreover, computer simulations provide useful information on the stability of periodic orbits. Section VI contains a discussion of numerical results in the case of a thermalizing base. The paper ends with concluding comments.

II. DISTRIBUTION OF THE REBOUND VELOCITY

Consider the one-dimensional motion of a particle of mass m falling in vacuum with acceleration g. Its position and velocity will be denoted by z and v, respectively (Fig. 1). At the level $z=0$, the particle encounters a base which sends it back into space with the rebound velocity w. The distribution of the stochastic variable w will be characterized by some probability density $\phi(w)$. It is thus assumed that no correlations occur between the velocities before and after collision with the base. In this situation, what will the stationary distribution of the particle look like? In order to answer this question we consider the kinetic equation for the probability density $f(z,v;t)$ for finding the particle in the state (z,v) at time t. It reads

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-g\frac{\partial}{\partial v}\right)f(z,v;t)$$
$$=\delta(z-0+)\left[\phi(v)\int_{-\infty}^{0}dw|w|f(0+,w;t)-|v|\theta(-v)f(0+,v;t)\right]. \quad (1)$$

On the left hand side of Eq. (1), density $f(z,v;t)$ is acted upon by the generator of a free motion with acceleration $-g$. On the right hand side one finds the usual balance between the gain and loss of velocity v through collisions with the base at $z=0$. The loss of memory at the encounters is shown by the fact that in the gain term the density $\phi(v)$ is multiplied by the total collision frequency

$$\nu(t)=\int_{-\infty}^{0}dw|w|f(0+,w;t). \quad (2)$$

Equation (1) can be rewritten in an equivalent integral form in terms of the trajectories of collisionless motion followed backward in time

$$z(-t)=z-vt-\tfrac{1}{2}gt^2, \quad v(-t)=v+gt. \quad (3)$$

If the freely accelerated motion starts from $[z(-t),v(-t)]$, the particle will reach the state (z,v) after time t. Equation (1) is thus equivalent to

$$f(z,v;t)=f(z(-t),v(-t);0)+\int_0^t d\tau\,\delta(z(-\tau))\phi(v(-\tau))\nu(t-\tau). \quad (4)$$

Notice that the loss term due to collisions in Eq. (1) does not contribute to Eq. (4). Indeed, the conditions imposed by the $\delta(z-0+)$ distribution and the step function $\theta(-v)$ exclude the possibility of reaching the point $(z>0,v)$ through collisionless motion starting from $[z(-\tau)=0,\ v(-\tau)<0]$. Taking the $t\to\infty$ limit causes the term involving the initial condition to disappear. So, the stationary state $F(z,v)=\lim_{t\to\infty}f(z,v;t)$ satisfies the equation

$$F(z,v)=\nu(\infty)\int_0^\infty d\tau\,\delta(z(-\tau))\phi(v(-\tau)), \quad (5)$$

where $\nu(\infty)$ denotes the asymptotic value of the collision frequency (2). The δ distribution permits one to perform the time integration in Eq. (5). Then using the normalization condition

$$\int_0^\infty dz\int dv\,F(z,v)=1$$

we arrive at the final formula

$$F(z,v)=\nu(\infty)\frac{\phi(\sqrt{v^2+2zg})}{\sqrt{v^2+2zg}}, \quad (6)$$

with

$$\nu(\infty)=\frac{g}{\int dw|w|\phi(|w|)}. \quad (7)$$

Formula (6) is quite convenient to discuss the physically relevant boundary conditions. In particular, it tells us that in order to obtain the Maxwell-Boltzmann equilibrium distribution with temperature T, one has to choose

$$\phi(v)=\phi_T(v)\equiv\theta(v)\frac{mv}{k_BT}\exp\left(-\frac{mv^2}{2k_BT}\right), \quad (8)$$

where $\theta(v)$ is a unit step function.

It is interesting to note that the density of the form (8) has been found experimentally to reproduce faithfully the distribution of the rebound velocity of a particle suffering inelastic collisions with a sinusoidally oscillating base at sufficiently high frequency of oscillations [9]. In this case the introduction of an effective granular temperature was well founded, as the insertion of Eq. (8) into Eq. (6) predicts the Maxwell-Boltzmann equilibrium state

$$F_T(z,v)=\frac{gm}{k_BT}\sqrt{\frac{m}{2\pi k_BT}}\exp\left(-\frac{m}{2k_BT}[v^2+2zg]\right). \quad (9)$$

The thermalizing rebound velocity distribution ϕ_T will be adopted in the next two sections. The object of the investigation will be then the nature of deviations from the equilibrium distribution (9) caused by dissipative coupling of the particle to the surrounding medium.

III. DISSIPATION THROUGH SOLIDLIKE FRICTION

We begin by considering the simplest dissipation mechanism supposing that the motion in the gravitational field is accompanied by energy losses due to solidlike friction. This means that in addition to acceleration g the particle is constantly acted upon by a stopping force $[-\mathrm{sgn}(v)m\alpha]$, with $\alpha<g$. So, the kinetic equation (1) has to be modified since the magnitude of the acceleration depends now on the orientation of the velocity. It must be noted that we do not consider the static counterpart of solid friction in this study, but we will discuss this point below. The new expression for the kinetic equation is

$$\left(\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-[g+(\theta(v)-\theta(-v))\alpha]\frac{\partial}{\partial v}\right)f(z,v;t)$$
$$=\delta(z-0+)\left[\phi(v)\int_{-\infty}^{0}dw|w|f(0+,w;t)\right.$$
$$\left.-|v|\theta(-v)f(0+,v;t)\right]. \tag{10}$$

The integral equation determining the stationary state $F(z,v;\alpha)$ is again of the form (5),

$$F(z,v;\alpha)=\nu(\infty;\alpha)\int_{0}^{\infty}d\tau\,\delta(z(-\tau))\phi_T(v(-\tau)), \tag{11}$$

where $[z(-\tau),v(-\tau)]$ denote the collisionless trajectory traced back from the initial point (z,v) during the time interval $\tau>0$, and the stationary collision frequency $\nu(\infty;\alpha)$ is given by

$$\nu(\infty;\alpha)=\int_{-\infty}^{0}dw|w|F(z,w;\alpha).$$

The δ distribution in Eq. (11) permits to reformulate the dynamical problem in the following way: what must be the value of the rebound velocity $w=v(-\tau)>0$ at $z(-\tau)=0$ to find the particle after time τ at the level z with velocity v? If $v>0$, the situation is completely analogous to that of a frictionless motion, the role of acceleration g being played by $(g+\alpha)$. In this case,

$$w=\sqrt{2(g+\alpha)z+v^2},$$

and, in accordance with Eq. (6), the time integration in Eq. (11) yields a term proportional to

$$\frac{\phi_T(\sqrt{v^2+2z(g+\alpha)})}{\sqrt{v^2+2z(g+\alpha)}}. \tag{12}$$

However, when $v<0$, a modified behavior occurs. Indeed, the particle has first to attain $z_{\max}$ where its velocity vanishes, and then fall down with acceleration $(g-\alpha)$ to the altitude z, obtaining the required negative velocity. The energy conservation implies the relations

$$w^2=2(g+\alpha)z_{\max},$$
$$v^2=2(g-\alpha)(z_{\max}-z). \tag{13}$$

Solving these, we find

$$w=\sqrt{2(g+\alpha)z+\frac{g+\alpha}{g-\alpha}v^2}. \tag{14}$$

Hence, the time integration in Eq. (11) here gives a term proportional to

$$\frac{\phi_T\left(\sqrt{\dfrac{g+\alpha}{g-\alpha}v^2+2z(g+\alpha)}\right)}{\sqrt{\dfrac{g+\alpha}{g-\alpha}v^2+2z(g+\alpha)}}. \tag{15}$$

Now using the explicit form of the rebound density (8), we find the relation

$$F(z,v;\alpha)=2n(z;\alpha)\left[P_+\theta(v)\phi^M(v;T)\right.$$
$$\left.+P_-\theta(-v)\sqrt{\frac{g+\alpha}{g-\alpha}}\phi^M(v;T_\alpha)\right], \tag{16}$$

where $n(z;\alpha)$ is the normalized spatial density

$$n(z;\alpha)=\frac{m}{k_BT}(g+\alpha)\exp\left(-\frac{m}{k_BT}(g+\alpha)z\right), \tag{17}$$

ϕ^M denotes the Maxwell velocity distribution,

$$\phi^M(v;T)=\sqrt{\frac{m}{2\pi k_BT}}\exp\left(-\frac{mv^2}{2k_BT}\right), \tag{18}$$

and the α-dependent temperature T_α is given by

$$T_\alpha=\left(\frac{g-\alpha}{g+\alpha}\right)T. \tag{19}$$

The constants P_+ and P_- denote the probabilities for finding a particle with a positive and negative velocity, respectively. Clearly, $P_++P_-=1$. Moreover, whereas the particle starting from $z=0$ with some positive velocity $w>0$ reaches the highest point $z_{\max}$ after a time

$$t^{\uparrow}=\frac{w}{g+\alpha},$$

it subsequently falls down during the time interval

$$t^{\downarrow}=\frac{w}{\sqrt{g^2-\alpha^2}}.$$

This means that

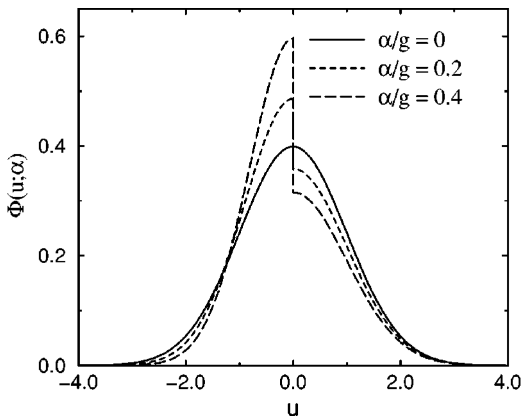

FIG. 2. Velocity distribution $\Phi(v;\alpha)$ for various values of the friction coefficient α, according to Eq. (21). It must be noted that this probability density is independent of the altitude z. The variable $u=v(m/k_BT)^{1/2}$ is the dimensionless velocity.

$$\frac{P_+}{P_-}=\frac{t^{\uparrow}}{t^{\downarrow}}=\sqrt{\frac{g-\alpha}{g+\alpha}}.$$

The above relations yield the following result for P_+:

$$P_+=\frac{\sqrt{g-\alpha}}{\sqrt{g+\alpha}+\sqrt{g-\alpha}}. \tag{20}$$

In this way we arrive at the final formula

$$F(z,v;\alpha)=n(z;\alpha)\Phi(v;\alpha)$$
$$=\frac{2n(z;\alpha)}{\sqrt{g+\alpha}+\sqrt{g-\alpha}}[\theta(v)\sqrt{g-\alpha}\phi^M(v;T)$$
$$+\theta(-v)\sqrt{g+\alpha}\phi^M(v;T_\alpha)]. \tag{21}$$

The structure of state (21) is quite simple, as there are no correlations between the position and velocity of the particle. This property of equilibrium (9) survives the action of the dissipation. However, the velocity distribution is changed in an important way. It is a linear combination of two halves of the Maxwell density (18): for positive velocities with temperature T, and for negative velocities with a different temperature T_α, defined in Eq. (19). One could thus introduce here an anisotropic temperature, depending on the orientation of the velocity, but it would not make much sense as the temperature is not a microscopic quantity. On the other hand, the so-called ''granular temperature'' defined by the relation $k_BT_{\rm gr}=\langle mv^2\rangle$ turns out to be the geometric mean of T and T_α:

$$T_{\rm gr}=T\sqrt{\frac{g-\alpha}{g+\alpha}}. \tag{22}$$

Clearly, for a sufficiently strong friction (α close to g) the temperature $T_{\rm gr}$ deviates substantially from both temperatures shown in Eq. (21), especially from that characterizing the ascending motion. By definition it represents the mean kinetic energy, but the use of the notion of temperature is rather misleading. To illustrate this last point, a graphical representation of the velocity distribution is given in Fig. 2.

Finally, let us note that although the velocity distribution Φ is not symmetric, it satisfies the condition

$$\langle v\rangle=\int v\Phi(v;\alpha)=0,$$

which reflects the vanishing of the particle current (at any altitude) in the stationary state.

As mentioned at the beginning of this section, the static part of solid friction has not been included in the model. This would lead to a static friction threshold $\alpha_{\rm stat}>\alpha$, where α is the dynamic coefficient introduced above. A particle initially at rest cannot move as long as the force acting on it stays below the threshold. In the model presented above we consider a single particle dissipating its energy through solidlike friction. The theory presented above correctly describes the stationary state provided $g>\alpha_{\rm stat}$. Conversely, when $g<\alpha_{\rm stat}$, the particle stops at the top of its trajectory and stays at rest forever. In this case, it is clear that the theory presented above should not apply. When several particles are present in the system the situation becomes more complicated. We can expect low-energy collisions to be strongly affected by a static friction threshold when one of the two colliding particles is initially at rest. But a simple model of instantaneous collisions cannot account for this effect since the force felt by the colliding particles is not defined in this case. The question of whether the system can reach the stationary solution where all particles are at rest is then raised when $g<\alpha_{\rm stat}$.

IV. EFFECTS OF FLUID FRICTION

Let us now suppose that, between collisions with the base, the falling mass dissipates its energy by suffering the fluid friction $(-\alpha v)$. The force exerted by the surrounding fluid is proportional to the magnitude of the velocity, and acts in the opposite direction. The generator of collisionless motion takes thus the form

$$\left[\frac{\partial}{\partial t}+v\frac{\partial}{\partial z}-\left(g+\frac{\alpha}{m}v\right)\frac{\partial}{\partial v}\right]. \tag{23}$$

It vanishes on the backward trajectory $[z(-t),v(-t)]$ given by

$$z(-t)=z+\frac{mg}{\alpha}t-\frac{m}{\alpha}\left(v+\frac{mg}{\alpha}\right)\left[\exp\left(\frac{\alpha}{m}t\right)-1\right],v(-t)$$
$$=\left(v+\frac{mg}{\alpha}\right)\exp\left(\frac{\alpha}{m}t\right)-\frac{mg}{\alpha}. \tag{24}$$

The Jacobian of transformation (24) equals $\exp(\alpha t/m)$. So, for the forward in time evolution we would obtain the factor $\exp(-\alpha t/m)$. It measures the contraction of the phase space volume accompanying the dissipative evolution. This must be taken into account when writing the kinetic equation. Indeed, in the absence of collisions with the base, the distribution $f(z,v;t)$ would be related to its initial value by

$$f(z,v;t)=\exp\left(\frac{\alpha}{m}t\right)f[z(-t),v(-t);t=0]. \tag{25}$$

The exponential factor compensates for the contraction of the phase space volume, preserving the normalization of the probability density $f(z,v;t)$. It follows that the kinetic equation in the presence of the fluid friction takes the form

$$\left[\frac{\partial}{\partial t}-\frac{\alpha}{m}+v\frac{\partial}{\partial z}-\left(g+\frac{\alpha}{m}v\right)\frac{\partial}{\partial v}\right]f(z,v;t)=\delta(z-0+)\left[\phi_T(v)\int_{-\infty}^{0}dw|w|f(0+,w;t)-|v|\theta(-v)f(0+,v;t)\right]. \quad (26)$$

As a consequence, Eq. (11) for the stationary state is to be modified, and reads

$$F(z,v;\alpha)=\nu(\infty;\alpha)\int_0^\infty d\tau\exp\left(\frac{\alpha}{m}\tau\right)\delta(z(-\tau))\phi_T(v(-\tau)). \quad (27)$$

In order to pursue the determination of the stationary state we have to insert trajectory (24) into Eq. (27). We then find the explicit formula

$$F(z,v;\alpha)=\int_0^\infty d\tau\exp\left(\frac{\alpha}{m}\tau\right)\delta\left\{z+\frac{mg}{\alpha}\tau-\frac{m}{\alpha}\left(v+\frac{mg}{\alpha}\right)\left[\exp\left(\frac{\alpha}{m}\tau\right)-1\right]\right\}\phi_T\left\{\left(v+\frac{mg}{\alpha}\right)\exp\left(\frac{\alpha}{m}\tau\right)-\frac{mg}{\alpha}\right\}\nu(\infty;\alpha), \quad (28)$$

where the collision frequency $\nu(\infty;\alpha)$ is to be calculated from the normalization condition.

First of all, let us note that $F(z,v;\alpha)$ vanishes for $v<v_{\min}=-mg/\alpha$, as the argument of the δ distribution is always positive in this region. $v_{\min}$ is the asymptotic value of the velocity of the falling particle corresponding to mutual compensation of the gravitational and the friction forces. Clearly, this excludes the Gaussian distribution.

It will be convenient for the further analysis to use dimensionless variables

$$\zeta=\frac{mg}{k_BT}z, \quad u=\sqrt{\frac{m}{k_BT}}v, \quad x=\frac{\alpha}{m}\tau. \quad (29)$$

The normalized stationary distribution turns out to depend on the ratio of the thermal velocity $v_T=\sqrt{k_BT/m}$ and the minimum velocity $|v_{\min}|=mg/\alpha$. Putting

$$\epsilon=\frac{v_T}{|v_{\min}|}, \quad (30)$$

we find, from Eq. (28),

$$F(\zeta,u;\epsilon)=\nu(\infty;\epsilon)\int_0^\infty dx\, e^x\epsilon^2\delta(\epsilon^2\zeta+x-(\epsilon u+1)(e^x-1))\phi((u+\epsilon^{-1})e^x-\epsilon^{-1}), \quad (31)$$

where the rebound distribution takes the simple form [see Eq. (8)]

$$\phi(u)=u\exp\left(-\frac{u^2}{2}\right). \quad (32)$$

The probability density for finding the particle with velocity u equals

$$\Phi(u;\epsilon)=\int_0^\infty d\zeta F(\zeta,u;\epsilon)=\int_0^\infty dx e^x\theta((\epsilon u+1)(e^x-1)-x)\phi((u+\epsilon^{-1})e^x-\epsilon^{-1})\nu(\infty;\epsilon). \quad (33)$$

When $u>0$, the step function θ does not impose any restriction, and a straightforward calculation with the use of Eq. (32) yields the formula

$$\theta(u)\Phi(u;\epsilon)=\frac{\epsilon\nu(\infty;\epsilon)}{\epsilon u+1}\exp\left(-\frac{u^2}{2}\right). \quad (34)$$

Equation (34) explicitly shows the nature of deviation from the Maxwell distribution during the ascending motion.

Integrating formula (33) over the whole velocity space yields the relation

$$1=\nu(\infty;\epsilon)\int du\, x(u;\epsilon)\phi(u), \quad (35)$$

where $x(u;\epsilon)$ represents the unique solution of the equation

$$(\epsilon u+1)(e^x-1)=xe^x. \quad (36)$$

Relation (35) determines the normalizing factor $\nu(\infty;\epsilon)$.

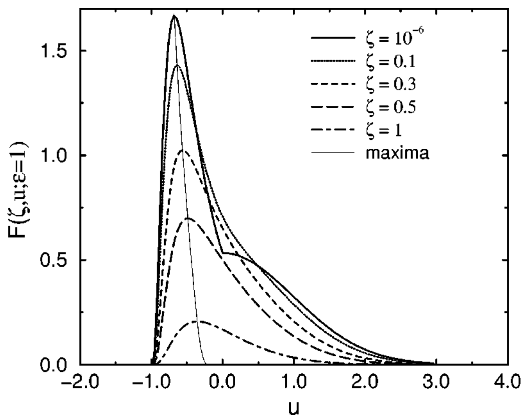

FIG. 3. Velocity distribution at different altitudes for $\epsilon = v_T/|v_{\min}|=1$. Here $v_T=(k_BT/m)^{1/2}$ is the thermal velocity, and $v_{\min}=-mg/\alpha$. The dimensionless position and velocity are defined by $\zeta=mgz/k_BT$, $u=v/v_T$.

Coming back to the general formula (31), we see that owing to the δ distribution the integration therein reduces to finding the point $x_o=x_o(\zeta,u;\epsilon)$, which solves the equation

$$\epsilon^2\zeta+x_o=(\epsilon u+1)(e^{x_o}-1). \tag{37}$$

Performing the integration in Eq. (31) thus yields the formula

$$F(\zeta,u;\epsilon)=\epsilon\nu(\infty;\epsilon)e^{x_o}\frac{\phi(u_o)}{u_o}\theta(x_o), \tag{38}$$

where $u_o=(u+\epsilon^{-1})e^{x_o}-\epsilon^{-1}$. One can check that the definition of u_o and Eq. (37) imply the relation $x_o=\epsilon(u_o-u-\epsilon\zeta)$. As a result, for the thermalizing rebound velocity distribution (32), $F(\zeta,u;\epsilon)$ takes the simple form

$$F(\zeta,u;\epsilon)=\epsilon\nu(\infty;\epsilon)\frac{\epsilon u_o+1}{\epsilon u+1}\exp\left(-\frac{u_o^2}{2}\right)\times\theta(u_o-u-\epsilon\zeta)\theta(u_o), \tag{39}$$

and u_o satisfies

$$\frac{\epsilon u_o+1}{\epsilon u+1}=\exp[\epsilon(u_o-u-\epsilon\zeta)]. \tag{40}$$

u_o represents the value of the rebound velocity sending the particle at altitude ζ with velocity u, and x_o is the collision time with the base. For each value of u and ζ, u_o can be extracted numerically from Eq. (40), whereas $\nu(\infty;\epsilon)$ is determined by the normalization condition. The results obtained in this way are shown in Figs. 3–5.

Figure 3 illustrates the behavior of the probability density $F(\zeta,u;\epsilon)$ for $\epsilon=1$, when the thermal velocity is equal to the minimum velocity. This value of ϵ clearly reflects the competition between the thermalizing base and the fluid friction. The variation of the velocity distribution with the altitude can be observed in Fig. 3. We can note the vanishing of $F(\zeta,u;\epsilon)$ when $u<u_{\min}=-1/\epsilon$, and the strong asymmetry of the velocity distribution. This asymmetry progressively reduces with increasing altitude, as shown by the solid line indicating the position of the maxima. However, it is clear that the velocity distribution stays strongly different from a Gaussian density.

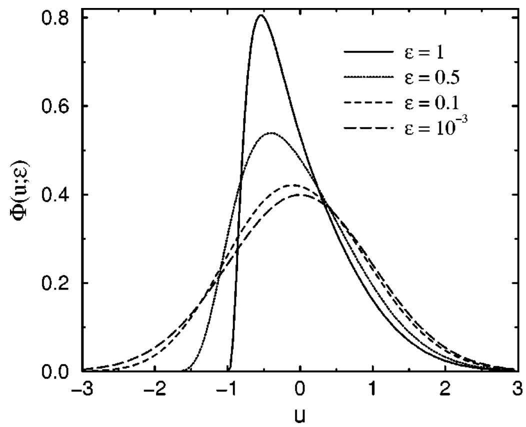

FIG. 4. Global velocity distribution for various values of ϵ.

The building up of a singularity at low altitude for $u=0$ is to be noticed in Fig. 3. This occurs because for $\zeta \ll 1$ the ascending velocities are practically unaffected by friction, and their distribution induced by the base is Gaussian, whereas the negative velocities have a completely different distribution, entirely controlled by friction. It is then not surprising that the two different halves join together at $u=0$ with different slopes. This kind of discontinuity is smoothed out after integration over the position space, and is not seen in the global probability density $\Phi(u;\epsilon)$, as shown in Fig. 4. Figure 4 shows the variation of the velocity probability density $\Phi(u;\epsilon)$ when the dimensionless friction ϵ varies from 0 to 1. In the absence of friction ($\epsilon=0$) we find the equilibrium Maxwell state. Not surprisingly, the effect of the fluid friction is to concentrate the velocity distribution around the minimum velocity $-1/\epsilon$. This behavior of $\Phi(u;\epsilon)$ at large values of ϵ contrasts with the previously encountered situation where the solid friction produced a discontinuity at $u=0$, but the two halves of $\Phi(u;\epsilon)$ preserved the Gaussian shape (see Fig. 2).

The density profiles $n(\zeta;\epsilon)$ are shown in Fig. 5. Away from $\epsilon=0$, they deviate substantially from the barometric exponential law. Let us recall that the barometric formula was still valid in the case of the solid friction [see Eq. (17)], provided one used the renormalized mass $m_{\text{eff}}=m(1+\alpha/g)$.

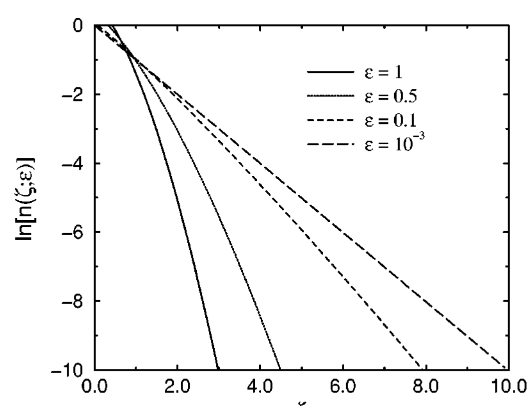

FIG. 5. Logarithmic plot of the density profiles.

The analysis presented up to now provided a description of how friction forces affected the stationary distribution, which for freely falling particles would have a Maxwell-Boltzmann shape. This problem is not purely academic. Indeed, solid friction usually occurs in experimental setups confining particles to move along a given axis. An estimation of the influence of the air drag on granular temperature, described in a thorough analysis of experimental data in Ref. [9], stressed the importance of the fluid friction.

V. DISSIPATION VIA INELASTIC BINARY COLLISIONS: THE CASE OF A SINGLE REBOUND VELOCITY

Dissipative coupling through macroscopic friction had the advantage of reducing the dynamic problem to the motion of a single particle in a *static* field of velocity dependent forces. The continuously acting friction induced a *systematic* reduction in the particle energy during its flight, which resulted in a strong anisotropy of the velocity distribution (see Fig. 3).

In a granular fluid material, dissipation occurs only at binary inelastic collisions between the particles. Their trajectories are thus piecewise parabolic, which reduces the anisotropy in the velocity distribution. We now turn to a study of the effects of inelastic binary collisions. The simplest possible system is one composed of two masses moving in a gravitational field. Their states will be denoted by (z_1,v_1) and (z_2,v_2), respectively. Particle 1 will be supposed to be closer to the base: $0\leqslant z_1\leqslant z_2$. This linear ordering is preserved by the dynamics.

Postponing the discussion of thermalizing boundary conditions [Eq. (8)] to Sec. VI we shall assume here that the rebound velocity is always taking the same characteristic value $w>0$. The corresponding probability density is thus $\phi(v)=\delta(v-w)$.

When a binary collision between masses 1 and 2 occurs, their velocities are instantaneously transformed according to the laws

$$
\begin{aligned}
v_1 &\rightarrow b_{12}(\alpha)(v_1)=v_1-\frac{1+\alpha}{2}v_{12},\\
v_2 &\rightarrow b_{12}(\alpha)(v_2)=v_2+\frac{1+\alpha}{2}v_{12},
\end{aligned}
\tag{41}
$$

where $v_{12}=v_1-v_2$.

The case of $\alpha=1$ is that of elastic collisions, where the particles simply exchange their velocities. The Jacobian of transformation (41) is equal to α. So, when $0<\alpha<1$, inelastic collisions contract the volume of the velocity space. Let us notice that the operator $b_{12}(\alpha^{-1})$ performs the inverse transformation to that defined by $b_{12}(\alpha)$ in Eq. (41). This means that the velocities (v_1,v_2) occur after collision if the precollisional velocities are $b_{12}(\alpha^{-1})(v_1),b_{12}(\alpha^{-1})(v_2)$.

We denote by $f_2(z_1,v_1;z_2,v_2;t)$ the two-particle probability density for finding the system at time t in the state $(z_1,v_1;z_2,v_2)$. The exact kinetic equation satisfied by f_2 reads

$$
\left(\frac{\partial}{\partial t}+L_\alpha(12)\right)f_2(z_1,v_1;z_2,v_2;t)=R(z_1,v_1;z_2,v_2;t).
\tag{42}
$$

Here the term

$$
\begin{aligned}
R(z_1,v_1;z_2,v_2;t)
&=\delta(z_1)\bigg[\delta(v_1-w)\int dw|w|\theta(-w)f_2(0,w;z_2,v_2;t)\\
&\quad+v_1\theta(-v_1)f_2(0,v_1;z_2,v_2;t)
\end{aligned}
$$

describes the collisions of particle 1 with the base at $z=0$. The Liouville operator $L_\alpha(12)$ equals

$$
L_\alpha(12)=L_0(12)+T_\alpha(12),
\tag{43}
$$

where $L_0(12)$ generates free motion in the gravitational field,

$$
L_0(12)=\left[v_1\frac{\partial}{\partial z_1}+v_2\frac{\partial}{\partial z_2}-g\left(\frac{\partial}{\partial v_1}+\frac{\partial}{\partial v_2}\right)\right],
\tag{44}
$$

and $T_\alpha(12)$ is the binary collision operator:

$$
T_\alpha(12)=\delta(z_{21})|v_{12}|[\alpha^{-2}\theta(v_{21})b_{12}(\alpha^{-1})-\theta(v_{12})].
\tag{45}
$$

The factor α^{-2} in Eq. (45) compensates for the contraction due to the inelasticity of collisions: one power of α^{-1} cancels the effect of the Jacobian of transformation (41), and another one provides the correct value for the precollisional relative velocity v_{12}/α. Of course, the stationary state $F(z_1,v_1;z_2,v_2)$ represents the time-independent solution of the kinetic equation (42).

We determined the explicit analytic form of F owing to information which we obtained after visualizing the dynamics with the help of a computer. It turned out that after the disappearance of some initial effects, the two particles moved in a periodic orbit in a remarkably synchronized way, the binary collision always occurring at the same altitude. We could thus immediately conclude that the energy E_2 of particle 2 stayed constant all the time. Clearly, in the case of particle 1 a distinction was necessary between the energy $E_1^{\uparrow}=mw^2/2$ of ascending motion, and the energy $E_1^{\downarrow}$ with which it moved back to the base after colliding with particle 2. The collision law (41) implies the following transformations of energies:

$$
\begin{aligned}
E_1^{\downarrow}&=\frac{1-\alpha}{2}E_1^{\uparrow}+\frac{1+\alpha}{2}E_2-\frac{1-\alpha^2}{8}mw_{12}^2,\\
E_2&=\frac{1-\alpha}{2}E_2+\frac{1+\alpha}{2}E_1^{\uparrow}-\frac{1-\alpha^2}{8}mw_{12}^2,
\end{aligned}
\tag{46}
$$

where $w_{12}=w_1-w_2$ and (w_1,w_2) denote the precollisional velocities of particles 1 and 2, respectively. The second relation expresses the fact that the energy of particle 2 is not changed at collision. In fact, its velocity is just reversed, so [see again Eq. (41)] we also have

$$-w_2=w_2+\frac{1+\alpha}{2}w_{12}. \tag{47}$$

Finally, the time τ between consecutive collisions (period of the motion) must be the same for both particles. In the case of particle 2 we simply have $g\tau=-2w_2$. Particle 1 moves up during the time $\tau^{\uparrow}$ needed to attain the precollisional velocity w_1)

$$g\tau^{\uparrow}=w-w_1,$$

and then reaches back to the base after time $\tau^{\downarrow}$, during which the postcollisional velocity attains the value $-\sqrt{2E_1^{\downarrow}/m}$:

$$-\sqrt{\frac{2E_1^{\downarrow}}{m}}=w_1-\frac{1+\alpha}{2}w_{12}-g\tau^{\downarrow}.$$

For consistency, the equation

$$-2w_2=w-\frac{1+\alpha}{2}w_{12}+\sqrt{\frac{2E_1^{\downarrow}}{m}} \tag{48}$$

must thus hold.

Equations (46)–(48) suffice to determine entirely the parameters of the stationary state. We find

$$E_1^{\uparrow}=\frac{mw^2}{2},\quad E_1^{\downarrow}=E_1^{\uparrow}\left(\frac{1+3\alpha}{3+\alpha}\right)^2,$$
$$w_{12}=\frac{4}{3+\alpha}w,\quad E_2=E_1^{\uparrow}\left(\frac{1+14\alpha+\alpha^2}{(3+\alpha)^2}\right). \tag{49}$$

Taking into account the normalization condition and the fact that the relative velocity after collision is equal to $-\alpha w_{12}$, we eventually find the explicit formula for the stationary state

$$F(z_1,v_1;z_2,v_2)=\theta(z_1)\,\theta(z_{21})\,\frac{mg^2}{w}\left(\frac{3+\alpha}{1+\alpha}\right)\delta\left[E(z_2,v_2)-\frac{(1+14\alpha+\alpha^2)}{(3+\alpha)^2}\,\frac{mw^2}{2}\right]$$
$$\times\left\{\delta\left(v_{12}-\frac{4}{3+\alpha}w\right)\delta\left(E(z_1,v_1)-\frac{mw^2}{2}\right)+\delta\left(v_{12}+\frac{4\alpha}{3+\alpha}w\right)\delta\left(E(z_1,v_1)-\left(\frac{1+3\alpha}{3+\alpha}\right)^2\frac{mw^2}{2}\right)\right\}, \tag{50}$$

where the notation $E(z,v)\equiv[mgz+mv^2/2]$ has been used.

The probability density (50) is highly singular. It involves products of δ distributions confining the stationary state to a one-dimensional manifold in the four-dimensional phase space. It can be checked by a straightforward calculation that formula (50) provides the solution to the kinetic equation (42). For a two-particle system, state (50) turns out to be stable, attracting any initial condition.

In principle, analogous periodic states could exist for a column of $N>2$ masses, and we could indeed observe them. The particles keep oscillating within adjacent volumes in the position space in a remarkably synchronized way, suffering inelastic collisions with their nearest neighbors, always at the same fixed altitudes. The visualization of this motion is quite impressive. However, when $N>2$, one cannot expect to observe this simple periodic motion for any values of α. In order to clarify this interesting point we shall derive a rigorous necessary condition for the existence of N-particle single-period states described above.

We denote by z_j the altitude at which particles j and $(j+1)$ collide. Consider one period of motion of particle j. It starts moving upward at level z_{j-1} with velocity $v_j^{\uparrow}(z_{j-1})$. The time $\tau_j^{\uparrow}$ needed to attain the precollisional velocity for the encounter with particle $(j+1)$ equals

$$g\tau_j^{\uparrow}=v_j^{\uparrow}(z_{j-1})-v_j^{\uparrow}(z_j).$$

After collision, particle j acquires the velocity $v_j^{\downarrow}(z_j)$, and reaches the altitude z_{j-1} after time $\tau_j^{\downarrow}$ given by

$$g\tau_j^{\downarrow}=v_j^{\downarrow}(z_j)-v_j^{\downarrow}(z_{j-1}).$$

The equality of the periods of motion for neighboring particles thus imposes the constraint

$$v_j^{\uparrow}(z_{j-1})-v_j^{\uparrow}(z_j)+v_j^{\downarrow}(z_j)-v_j^{\downarrow}(z_{j-1})$$
$$=v_{j+1}^{\uparrow}(z_j)-v_{j+1}^{\uparrow}(z_{j+1})+v_{j+1}^{\downarrow}(z_{j+1})-v_{j+1}^{\downarrow}(z_j). \tag{51}$$

Now applying the inelastic collision law (41), we find the recurrence relation

$$w_{j,j+1}=\tfrac{1}{2}(w_{j+1,j+2}+w_{j-1,j}) \tag{52}$$

for relative precollisional velocities of particles j and $(j+1)$:

$$w_{j,j+1}=v_j^{\uparrow}(z_j)-v_{j+1}^{\downarrow}(z_j).$$

The recurrence equations (52) are simply solved by

$$w_{j,j+1}=\left(1-\frac{j}{N}\right)w_{0,1}. \tag{53}$$

The value of $w_{0,1}$ can be calculated by considering the period of motion of the lowest-lying particle 1. One can then readily check that Eq. (52) for $j=1$ reads

$$w_{1,2}=\frac{1}{2}(w_{2,3}+w_{0,1}),$$

with

$$w_{0,1}=\frac{2(w+\sqrt{2E_1^{\downarrow}/m})}{1+\alpha}. \tag{54}$$

The condition we are looking for can now be derived by considering the energies of the particles. We denote by $E_j^{\uparrow}$ and $E_j^{\downarrow}$ the energy of particle j during its upward and downward motion, respectively. The collision law (41) implies energy transformations of the forms

$$\begin{aligned} E_j^{\downarrow}&=\frac{1+\alpha}{2}E_{j-1}^{\uparrow}+\frac{1-\alpha}{2}E_j^{\downarrow}-\frac{1-\alpha^2}{8}mw_{j-1,j}^2,\\ E_{j+1}^{\uparrow}&=\frac{1+\alpha}{2}E_j^{\uparrow}+\frac{1-\alpha}{2}E_{j+1}^{\downarrow}-\frac{1-\alpha^2}{8}mw_{j,j+1}^2. \end{aligned} \tag{55}$$

Summing up these relations from $j=1$ to $j=(N-1)$, one finds

$$E_1^{\downarrow}=\frac{mw^2}{2}-\frac{1-\alpha^2}{4}\sum_{j=1}^{N-1}mw_{j,j+1}^2.$$

Then inserting results (53) and (54), and performing the summation, we arrive at the relation

$$\frac{w-\sqrt{2E_1^{\downarrow}/m}}{w+\sqrt{2E_1^{\downarrow}/m}}=\frac{1-\alpha}{1+\alpha}\left[\frac{2N}{3}-1+\frac{1}{3N}\right]. \tag{56}$$

Clearly, Eq. (56) cannot be satisfied unless the right hand side is a positive number smaller than 1. This condition can be written as

$$(1-\alpha)^2<6\gamma-2\gamma^2 \quad \text{where} \quad \gamma=(1-\alpha)N. \tag{57}$$

Our analysis caused a relevant parameter $\gamma=(1-\alpha)N$ to appear. Its role has already been recognized in the study of clustering transition in a column of beads [7]. Inequality (57) is automatically satisfied for $N=2$. However, already for $N=3$ not all values of α are admissible. Clearly, for large values of N, only at extremely small inelasticity (α very close to 1), one can expect the appearance of the simple periodic motion. Moreover, the stability of this state becomes fragile, as could be observed by visualizing the computer simulations. A more drastic inequality can be derived from the positivity of the collision times $\tau_j^{\uparrow}$ and $\tau_j^{\downarrow}$. The analytic expressions for $\tau_j^{\uparrow}$ and $\tau_j^{\downarrow}$ are

$$g\tau_j^{\uparrow}=w\frac{6-2(1-\alpha)(N-j+2)}{6\alpha N+(1-\alpha)(2N^2+1)}, \tag{58}$$

$$g\tau_j^{\downarrow}=w\frac{6+2(1-\alpha)(N-j-1)}{6\alpha N+(1-\alpha)(2N^2+1)}. \tag{59}$$

The period of motion $\tau=\tau_j^{\uparrow}+\tau_j^{\downarrow}$ is easily extracted from these last two expressions, and is obviously independent of the label j of the particle:

$$g\tau=w\frac{6(1+\alpha)}{6\alpha N+(1-\alpha)(2N^2+1)}. \tag{60}$$

Although, from Eqs. (59) and (60), τ and $\tau_j^{\downarrow}$ are always positive, the positivity of $\tau_j^{\uparrow}$ for all j leads to the condition

$$\alpha>\frac{N-2}{N+1}, \tag{61}$$

which can be rewritten in the equivalent form, introducing γ:

$$1-\alpha<3-\gamma. \tag{62}$$

When $N\to\infty$, at fixed γ, the periodic state ceases to exist when γ reaches the value $\gamma_c=3$. Interestingly, this value also corresponds to the collapse of the heap against the plane [3,7].

We have checked, on the basis of molecular dynamic simulations for $N<12$, that the periodic orbit presented above is stable when condition (61) is satisfied. Approaching the boundary $\alpha=(N-2)/(N+1)$, the stability decreases, and the periodic orbit eventually turns to a more complicated periodic state, or, possibly, to a nonperiodic state. Inside the stability region defined by condition (61), the periodic orbit presented above can exceptionally coexist with a more complicated periodic state. We could observe such a situation for $N=5$ and $\alpha\simeq0.65$. Below the boundary $\alpha=(N-2)/(N+1)$, the situation is more complicated. We can observe transitions between different periodic states, depending on the values of α and N. Interestingly, nonperiodic states are exceptions. In fact, nonperiodic states are difficult to identify since we can never be sure that we do not observe a transient regime, or that the period is larger than the simulation time. Even at large values of N, for which part of the system is clumped close to the wall, the motion of the particles remains periodic. An example of such a situation can be obtained for $N=10$ and $\alpha=0.7$ [see also Fig. 3 ($\gamma=3$) of Ref. [7]]. A full description of the phase diagram in terms of N and α is still lacking. It would certainly be interesting to identify the stationary states and investigate the stability conditions. However, such a study is beyond our scope. To summarize, we have shown in this section that for a single rebound velocity, stationary states play a key role in the dynamical behavior of the system. The simplest periodic orbit has been characterized explicitly, and we have obtained an existence condition that compares extremely well with simulation data. In Sec. VI, we show that for a thermalizing wall, the dynamical behavior is quite different.

VI. DISSIPATION VIA INELASTIC BINARY COLLISIONS: THE CASE OF A THERMALIZING BASE

As we have already mentioned, in a granular fluid material, dissipation occurs only at binary inelastic collisions which reduce the anisotropy in the velocity distribution as compared to the effects of friction. In the extreme case of completely inelastic collisions, the final velocity of the colliding pair coincides with that of their center of mass. So,

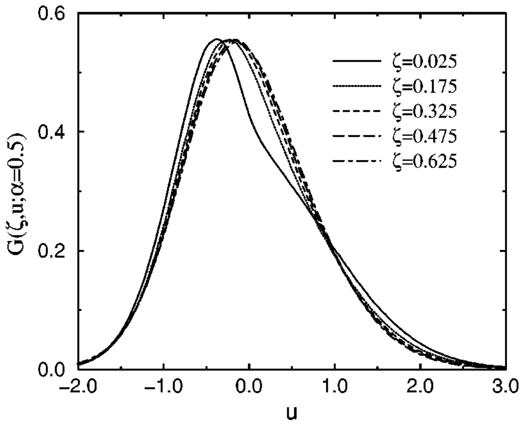

FIG. 6. Local velocity distribution $G(\zeta,u;\alpha)=F(\zeta,u;\alpha)/n(\zeta;\alpha)$ at various altitudes for $\alpha=0.5$.

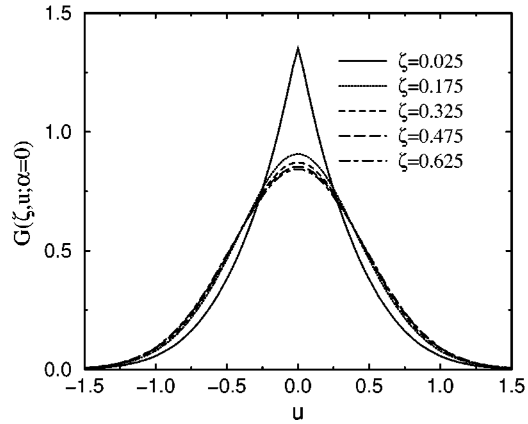

FIG. 7. Velocity distribution function $G(\zeta,u;\alpha)$ at various altitudes for $\alpha=0$.

after a short relaxation time all the particles will stick together. The energy dissipation can then occur only at collisions with the base. The system becomes equivalent to a single particle emitted at the base with an effective rebound velocity distribution accounting for dissipation. As no dissipation can occur during the free flight, the velocity distribution in this extreme case is again isotropic, although strongly non-Gaussian.

These points are illustrated in this section by considering the case of two particles acted upon by the gravitational field, suffering inelastic binary collisions, and interacting with a thermalizing base characterized by the rebound distribution (8). The results described below have been obtained numerically using the fact that the free evolution in the gravitational field could be simply determined from the equations of motion. If the restitution coefficient $\alpha=1$, the collisions are elastic and generate the Maxwell-Boltzmann distribution. When $\alpha<1$, the velocity distribution turns out to depend on the altitude, and it is convenient to consider the local velocity distribution $G(\zeta,u;\alpha)$ defined by

$$G(\zeta,u;\alpha)=F(\zeta,u;\alpha)/n(\zeta;\alpha). \tag{63}$$

F denotes the position and velocity dependent stationary state [the dimensionless variables have been defined in Eq. (29)]. As before, $n(\zeta;\alpha)$ denotes the density profile. Figure 6 shows (for $\alpha=0.5$) that the velocity distribution is strongly anisotropic at low altitudes, but rapidly converges to a more symmetrical shape. However, the anisotropy persists even for larger values of ζ as can be seen by monitoring the position of the maxima in Fig. 6.

A comparison with Fig. 3 emphasizes qualitative differences between the effects of fluid friction and those of inelastic binary collisions. The extreme situation where $\alpha=0$ has been simulated using a special algorithm to account for simultaneous collisions at the base. The results for $G(\zeta,u;\alpha=0)$ have been grouped in Fig. 7. We observe a symmetrical velocity distribution at any altitude, taking a rather singular shape at the base [even though the rebound density (8) leads to the Maxwell-Boltzmann distribution for $\alpha=1$]. Again, the velocity distribution rapidly reaches its asymptotic behavior for increasing ζ. We noticed that this asymptotic behavior was *numerically* consistent with a Maxwell-Boltzmann distribution with an effective temperature $T^{\mathrm{eff}}\simeq T/4$. Figure 8 shows the variations of the velocity distribution close to the base. The complex behavior of the maximum when α goes from 1 to 0 can be followed here. For $\alpha=1$, the distribution is Gaussian, so the maximum is at $u=0$; for intermediate values, the maximum corresponds to a negative value of u, and it goes back to zero when $\alpha=0$. Finally, the density profiles are presented in Fig. 9. The logarithmic plot indicates that the density follows reasonably well an exponential law in the limit of weak or strong inelasticity (α close to 1 or to 0). For intermediate values, deviations from the barometric law are observed.

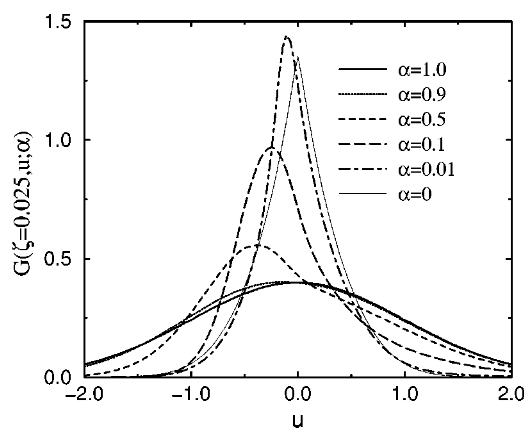

FIG. 8. Velocity distribution function $G(\zeta,u;\alpha)$ at $\zeta=0.025$ for different values of α.

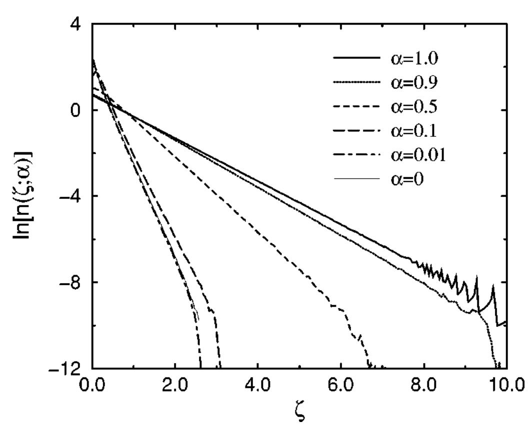

FIG. 9. Logarithmic plot of the density profiles.

VII. CONCLUDING COMMENTS

The main conclusions of this paper concern the influence of dissipative interactions on the structure of stationary states in a system of masses falling in a gravitational field, and obtaining the energy for the upward motion from a base. The action of the base was characterized by a given rebound velocity distribution.

The effect of solid friction reduced to split the Maxwell distribution into two halves with different temperatures, and to make appear an effective mass in the Boltzmann factor describing the density profile. The fluid friction perturbed the Maxwell-Boltzmann distribution even more, creating correlations between positions and velocities and introducing a cutoff in the velocity spectrum. The stationary non-Gaussian velocity distribution thus became strongly asymmetric.

The effects of inelastic binary collisions could be studied analytically in the case of a single rebound velocity. The appearance of stratified organization of the N-particle column of masses has been investigated. Each particle then oscillates between two fixed levels, the periods of motion being the same for all masses. The appearance of such an organization was shown to depend crucially on parameter $\gamma = N(1-\alpha)$ (α is the restitution coefficient).

For two particles coupled to a thermalizing base the interaction via inelastic collisions produces a much weaker asymmetry than the action of friction forces. The numerical analysis of this case predicts (away from the base) a Gaussian shape of the velocity distribution for $\alpha \to 0$ (limit of high inelasticity), with an effective temperature $T^{\mathrm{eff}} = T/4$, where T is the temperature at the base. The rigorous analysis based on the kinetic theory can be hopefully extended to this case.

The last comment concerns the one-dimensional nature of the theory presented above. In the model we consider, a particle can collide with two neighbors only, which clearly favors correlated moves in the system. We can mention, for example, the periodic states observed in the case of a single rebound velocity. In higher dimensions, momentum mixing and moves in horizontal directions should strongly reduce the correlations between successive collisions, and periodic states will probably not exist any longer. However, the predicted anisotropy of the velocity distribution function in the vertical direction has already been observed in molecular dynamics simulations of vibrated two-dimensional systems [12]. We then expect this last feature to be generic in higher dimensions.

ACKNOWLEDGMENTS

J.P. acknowledges financial support by the KBN (Committee for Scientific Research, Grant No. 2 P03 B 035 12) and the hospitality at the Laboratoire de Physique de l'École Normale Supérieure de Lyon.

[1] S. Mc Namara and W. R. Young, Phys. Fluids A **5**, 34 (1993).
[2] G. F. Carnevale, Y. Pomeau, and W. R. Young, Phys. Rev. Lett. **64**, 2913 (1990).
[3] B. Bernu and R. Mazighi, J. Phys. A **23**, 5745 (1990).
[4] S. Mc Namara and W. R. Young, Phys. Fluids A **4**, 496 (1992).
[5] E. Clément, S. Luding, A. Blumen, J. Rajchenbach, and J. Duran, Int. J. Mod. Phys. B **7**, 1807 (1993).
[6] S. Luding, E. Clément, A. Blumen, J. Rajchenbach, and J. Duran, Phys. Rev. E **49**, 1634 (1994).
[7] B. Bernu, F. Delyon, and R. Mazighi, Phys. Rev. E **50**, 4551 (1994).
[8] A. Metha and J. M. Luck, Phys. Rev. Lett. **65**, 393 (1990).
[9] S. Warr, W. Cooke, R. C. Ball, and J. M. Huntley, Physica A **231**, 551 (1996).
[10] E. B. Kremer and A. Ya. Fidlin, Dokl. Akad. Nauk SSSR **309**, 801 (1989) [Sov. Phys. Dokl. **34**, 1063 (1989)].
[11] H. M. Jaeger and S. R. Nagel, Rev. Mod. Phys. **68**, 1259 (1996).
[12] K. Helal, T. Biben, and J. P. Hansen, Physica A **240**, 361 (1997).