Multispecies asymmetric simple exclusion process and its relation to traffic flow

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Using the matrix product formalism we formulate a natural p-species generalization of the asymmetric simple exclusion process, using the matrix product formalism. In this model particles hop with their own specific rate and fast particles can overtake slow ones with a rate equal to their relative speed. We obtain the algebraic structure and study the properties of the representations in detail. The uncorrelated steady state for the open system is obtained and in the $p \to \infty$ limit, the dependence of its characteristics on the distribution of velocities is determined. It is shown that when the total arrival rate of particles exceeds a certain value, the density of the slowest particles rises abruptly. [S1063-651X(99)07501-7]

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I. INTRODUCTION

The matrix product ansatz (MPA), introduced in the work of Hakim and Nadal [1], became popular in the study of one dimensional stochastic exclusion models after Derrida *et al.* [2] applied this technique to the simplest such model, namely, the totally asymmetric simple exclusion process or ASEP [3]. For a recent review see [4]. They were able to find nontrivial representations for the relevant algebra and calculate many physical properties of this process. Since then this technique has been applied to many other interesting stochastic systems, both with periodic and with open boundary conditions. Janowsky and Lebowitz [5] have considered the case where there is a single impurity (a fixed blockage) on the lattice, where hopping rates are reduced, and have found that this single blockage has global effects, due to long range correlations. The case of a moving blockage and the formation of shocks has also been addressed in several works [6–8]. Finally MPA has been formulated for models with quasiparallel updatings [9–11].

In most of these works, a simplified model of traffic flow [12] is cited as a physical application, especially when impurities are present which when encountered decrease the rate of hopping of other particles [13]. Evans [14] has applied MPA to the case of several kinds of particles hopping with their own rates on a ring and has found that above a critical density a macroscopic number of holes are condensed in front of the slowest particle. In the work of Evans, however, particles cannot overtake, and the order of particles is unchanged during the process.

Turning to the mathematical aspect of the problem, a matrix product state can be understood as a generalization of ordinary factorizable states with a product measure in which numbers are replaced with noncommuting objects. This allows the original problem to be reformulated in terms of algebraic relations. The main advantage of this technique is that once a nontrivial representation is found, all the physical quantities such as the currents, densities, and correlation functions can in principle be obtained from the representation. This is of course not a simple task and one must overcome many difficulties of combinatorial type in calculating the explicit form of the matrix elements of products of operators.

Whether or not the zero energy state of a Hamiltonian with nearest neighbor interaction can always be formulated as a matrix product state has been answered in the affirmative by Kreb and Sandow [15]. Thus to any Hamiltonian of the above type corresponds an algebra in the MPA formalism. However, finding nontrivial representations of this algebra may be harder than the original problem itself. Therefore some authors [16,17] have tried to reverse the problem, that is, to begin with a quadratic algebra, with, one hopes, simple representations and then to find the relevant stochastic process. In [17] the algebras considered are apparently related to the partially asymmetric case. This restriction has a twofold disadvantage. First they obtain the concrete form of the algebraic relations only in cases for a low number of species, and second they do not obtain [17] ''a general framework for arbitrary N (number of species), which when specialized to $N=2$ gives the known results.''

Our aim in this paper is to start from a suitable algebra and proceed to find a natural generalization of the one-species ASEP. We will show that such a natural generalization exists. We will obtain the algebra, its representations, and many of its properties. In this p-species ASEP each particle of type j hops with rate v_j to its right neighbor site, and when it encounters a particle of type i with $v_j > v_i$ they interchange their sites with rate $v_j - v_i$, as if the fast particle overtakes the slow one. In any other case the attempted move of the original particle is rejected. The model seems to be relevant as a simplified model of one way traffic flow. We consider open boundary conditions where particles (or cars) enter the system (the highway) with a rate proportional to their speed and leave at the other side. The paper is organized as follows. In Sec. II we introduce the algebra and solve its consistency conditions. In Sec. III we obtain the Hamiltonian which corresponds to this algebra. In Sec. IV we obtain the uncorrelated steady state and finally in Sec. V we construct the infinite dimensional representations of the algebra and present some of its useful properties. In all the steps we compare our results with the one-species ASEP.

II. THE ALGEBRA

We begin with an algebra generated by the elements D_i, $i=1,\ldots,p$ and E, with the following relations:

$$D_i E = \frac{1}{v_i} D_i + E, \tag{1}$$

$$D_j D_i = \xi_{ji} D_j + \eta_{ji} D_i, \quad j>i \tag{2}$$

where the parameters v_i are finite real numbers and the parameters ξ_{ji} and η_{ji} are to be determined. In writing Eqs. (1) and (2) we have had in mind a totally asymmetric exclusion process. In order to be consistent with associativity we have to check that

$$(D_j D_i) E = D_j (D_i E). \tag{3}$$

Using Eqs. (1) and (2), this requirement determines ξ_{ji} and η_{ji} as

$$\xi_{ji} = \frac{v_i}{v_i - v_j}, \quad \eta_{ji} = \frac{-v_j}{v_i - v_j}. \tag{4}$$

For the present we take all the v_i to be different. Later we will relax this condition. We should also check that

$$(D_k D_j) D_i = D_k (D_j D_i), \quad k>j>i. \tag{5}$$

Using Eq. (2) we find that this relation imposes no new relations on the parameters. Since any monomials of the form $D_k D_j \cdots D_i$ with $k>j>\cdots>i$ can be reduced to a linear combination of generators, associativity is guaranteed for all the monomials of higher degree. The final form of the algebra reads

$$D_i E = \frac{1}{v_i} D_i + E, \tag{6}$$

$$D_j D_i = \frac{1}{v_i - v_j} (v_i D_j - v_j D_i), \quad j>i \tag{7}$$

where the parameters $v_1 \cdots v_p$ are free.

III. THE HAMILTONIAN AND THE PROCESS

We consider a one dimensional chain of N sites. The Hilbert space of each site is $p+1$ dimensional. The Hamiltonian is written as

$$H = h^1 + H^B + H^N = h^1 + \sum_{k=1}^{k=N-1} h^B_{k,k+1} + h^N, \tag{8}$$

where H^B is the bulk Hamiltonian and h^1 and h^N are boundary terms. In the matrix product formalism, the steady state of this Hamiltonian can be written as

$$|P\rangle = \langle W | \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \cdots \mathcal{D} \otimes \mathcal{D} | V \rangle, \tag{9}$$

where

$$\mathcal{D} = \begin{pmatrix} E \\ \dfrac{D_1}{p} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{D_p}{p} \end{pmatrix}$$

and $|V\rangle$ and $\langle W|$ are vectors in an auxiliary space. According to MPA, we have $H|P\rangle = 0$ if the following relations are satisfied:

$$h^B \mathcal{D} \otimes \mathcal{D} = X \otimes \mathcal{D} - \mathcal{D} \otimes X, \tag{10}$$

$$(h^N \mathcal{D} - X)|V\rangle = 0, \tag{11}$$

$$\langle W|(h^1 \mathcal{D} + X) = 0, \tag{12}$$

where we take X to be

$$X = \begin{pmatrix} -1 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_p \end{pmatrix},$$

with x_i as c-numbers. Anticipating the process from the form of the algebra, we write the bulk Hamiltonian as follows:

$$h^B = -\sum_{i=1}^{p} y_i (E_{0i} \otimes E_{i0} - E_{ii} \otimes E_{00})$$

$$- \sum_{j>i}^{p} y_{ij} (E_{ij} \otimes E_{ji} - E_{jj} \otimes E_{ii}). \tag{13}$$

Here the matrices E_{ij} act on the Hilbert space of one site and have the standard definition: $(E_{ij})_{k,l} = \delta_{ik} \delta_{jl}$. The first term expresses an ASEP for each species of particles (i) with rate y_i. The second term represents an interchange of particles of type ($j>i$), with rate y_{ij}. The parameters y_i and y_{ij} are to be determined.

The natural choice of the boundary terms should be such that particles of type (i) are injected at the left and extracted at the right with their specific rates, say, α_i and γ_i, respectively. So we take

$$h^1 = -\sum_{i=1}^{p} \alpha_i (E_{i0} - E_{00}), \tag{14}$$

$$h^N = -\sum_{i=1}^{p} \gamma_i (E_{0i} - E_{ii}). \tag{15}$$

Inserting Eq. (13) in Eq. (10) leads to the following equations:

$$\frac{y_i}{p} D_i E = \frac{D_i}{p} + x_i E, \tag{16}$$

$$\frac{y_{ij}}{p} D_j D_i = x_j D_i - x_i D_j, \quad j > i. \tag{17}$$

Comparison with Eqs. (6) and (7) shows that

$$x_i = \frac{v_i}{p}, \quad y_i = v_i, \quad y_{ij} = v_j - v_i. \tag{18}$$

Inserting the values of the parameters y_i and y_{ij} in Eq. (13) clearly highlights the physical process governed by the Hamiltonian. Denoting a vacant site by the symbol ϕ and a site occupied by a particle of type i, by the symbol A_i, the process defined by H^B is

$$A_i \phi \to \phi A_i, \quad \text{with rate } v_i, \tag{19}$$

$$A_j A_i \to A_i A_j, \quad j > i \text{ with rate } v_j - v_i. \tag{20}$$

In order for all the rates to be positive we restrict the range of v_i's as

$$v_1 \leqslant v_2 \leqslant v_3 \cdots \leqslant v_p. \tag{21}$$

Inserting Eqs. (14) and (15), respectively, in Eqs. (11) and (12) we find the following explicit matrix equations:

$$\left[\begin{pmatrix} 0 & -\gamma_1 & -\gamma_2 & -\gamma_3 & \cdot & \cdot & -\gamma_p \\ 0 & \gamma_1 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \gamma_2 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \gamma_3 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \gamma_p \end{pmatrix} \begin{pmatrix} E \\ \dfrac{D_1}{p} \\ \dfrac{D_2}{p} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{D_p}{p} \end{pmatrix} - \begin{pmatrix} -1 \\ \dfrac{v_1}{p} \\ \dfrac{v_2}{p} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{v_p}{p} \end{pmatrix} \right] |V\rangle = 0, \tag{22}$$

$$\langle W | \left[\begin{pmatrix} \alpha_1 + \alpha_2 + \cdots + \alpha_p & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\alpha_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\alpha_2 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -\alpha_3 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\alpha_p & 0 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} E \\ \dfrac{D_1}{p} \\ \dfrac{D_2}{p} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{D_p}{p} \end{pmatrix} + \begin{pmatrix} -1 \\ \dfrac{v_1}{p} \\ \dfrac{v_2}{p} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{v_p}{p} \end{pmatrix} \right] = 0. \tag{23}$$

These equations then yield

$$D_i |V\rangle = \frac{v_i}{\gamma_i} |V\rangle, \tag{24}$$

$$\langle W| E = \langle W| \frac{v_i}{p \alpha_i}, \tag{25}$$

and

$$\frac{v_1 + v_2 + \cdots + v_p}{p} = 1. \tag{26}$$

Equation (26), which means that the average speed of the particles is unity, is not really a condition on the rates, since by a rescaling of time one can always obtain this condition. Hereafter we will write this condition simply as $\langle v \rangle = 1$. Equation (25), however, implies that $v_i / p \alpha_i$ is independent of i. Thus we set $\alpha_i = (1/p) \alpha v_i$.

In order that Eq. (24) be consistent with Eq. (7), we require that

$$\left(D_j D_i - \frac{v_i D_j - v_j D_i}{v_i - v_j} \right) |V\rangle = 0. \tag{27}$$

After using Eq. (24) and rearranging terms, this condition shows that

$$\gamma_i - v_i = \gamma_j - v_j.$$

Thus we set

$$\gamma_i = v_i + \beta - 1.$$

Therefore we have

$$D_i|V\rangle = \frac{v_i}{v_i + \beta - 1}|V\rangle, \quad \langle W|E = \langle W|\frac{1}{\alpha}. \quad (28)$$

The meaning of the parameters α and β is now clear. From Eq. (26) we find

$$\alpha_1 + \alpha_2 + \cdots + \alpha_p = \alpha, \quad \frac{\gamma_1 + \gamma_2 + \cdots + \gamma_p}{p} = \beta.$$

Thus α and β are, respectively, the total rate of injection and the average rate of extraction of particles. In order for the individual rates to be positive we restrict further the range of speeds as

$$1 - \beta \leq v_1 \leq v_2 \leq v_3 \cdots \leq v_p. \quad (29)$$

The above results clearly justify this model as a natural generalization of the one-species ASEP.

We conclude this section with formulas for the current operators. A simple way to obtain these is to directly refer to the process (19) and (20) and write the equation for the one point function $\langle \tau_k^{(i)} \rangle$ which is the average density of particles of type (i) at site k. Determining from Eqs. (19) and (20) various ways of increasing and decreasing this density and denoting the probability of site k to be vacant by ϵ_k we find

$$\frac{d}{dt}\langle \tau_k^{(i)} \rangle = v_i \langle \tau_{k-1}^{(i)} \epsilon_k \rangle - v_i \langle \tau_k^{(i)} \epsilon_{k+1} \rangle$$

$$+ \sum_{j<i} (v_i - v_j)\langle \tau_{k-1}^{(i)} \tau_k^{(j)} \rangle - \sum_{j<i} (v_i - v_j)\langle \tau_k^{(i)} \tau_{k+1}^{(j)} \rangle$$

$$- \sum_{j>i} (v_j - v_i)\langle \tau_{k-1}^{(j)} \tau_k^{(i)} \rangle$$

$$+ \sum_{j>i} (v_j - v_i)\langle \tau_k^{(j)} \tau_{k+1}^{(i)} \rangle. \quad (30)$$

Note that inside any correlation function we have $\epsilon_k = 1 - \Sigma_i \tau_k^{(i)}$. Equation (30) can be rewritten as a continuity equation

$$\frac{d}{dt}\langle \tau_k^{(i)} \rangle = \langle J_k^{(i)} \rangle - \langle J_{k+1}^{(i)} \rangle,$$

where

$$\langle J_k^{(i)} \rangle = v_i \langle \tau_{k-1}^{(i)} \epsilon_k \rangle + \sum_{j<i} (v_i - v_j)\langle \tau_{k-1}^{(i)} \tau_k^{(j)} \rangle$$

$$- \sum_{j>i} (v_j - v_i)\langle \tau_{k-1}^{(j)} \tau_k^{(i)} \rangle. \quad (31)$$

According to MPA

$$\langle J_k^{(i)} \rangle = \frac{\langle W|C^{k-2}J^{(i)}C^{N-k}|V\rangle}{\langle W|C^N|V\rangle},$$

where $C := E + (1/p)D := E + (1/p)(D_1 + D_2 + \cdots + D_p)$. The current operators can be read from Eq. (31) to be

$$J^{(i)} = v_i \frac{D_i}{p}E + \sum_{j<i} (v_i - v_j)\frac{D_i}{p}\frac{D_j}{p} - \sum_{j>i} (v_j - v_i)\frac{D_j}{p}\frac{D_i}{p}. \quad (32)$$

Using Eqs. (6) and (7), we obtain

$$J^{(i)} = \frac{v_i}{p}\left(\frac{1}{v_i}D_i + E\right) - \frac{1}{p^2}\sum_{j<i} (v_j D_i - v_i D_j)$$

$$- \frac{1}{p^2}\sum_{j>i} (v_j D_i - v_i D_j)$$

$$= \frac{1}{p}D_i + \frac{v_i}{p}E - \frac{1}{p^2}\left(\sum_{j\neq i} v_j\right)D_i + \frac{v_i}{p^2}\left(\sum_{j\neq i} D_j\right)$$

$$= \frac{1}{p}D_i + \frac{1}{p}v_i E - \frac{1}{p^2}(p - v_i)D_i + \frac{v_i}{p^2}(D - D_i)$$

$$= \frac{v_i}{p}C. \quad (33)$$

We thus obtain

$$\langle J^{(i)} \rangle = \frac{v_i}{p}\frac{\langle W|C^{N-1}|V\rangle}{\langle W|C^N|V\rangle}. \quad (34)$$

Therefore all the currents are simply proportional to the average current J, however, J itself has a highly nontrivial dependence on the hopping rates.

In the next section we will find the one dimensional representations of the algebra. This case corresponds to the steady state being characterized by a Bernoulli measure. Although very simple this steady state has a rather rich structure.

IV. ONE DIMENSIONAL REPRESENTATIONS AND THE UNCORRELATED STEADY STATE

In the one dimensional representation the operators D_i and E are represented by c-numbers Δ_i and e, respectively. From Eq. (28) we have

$$\Delta_i = \frac{v_i}{v_i + \beta - 1}, \quad e = \frac{1}{\alpha}. \quad (35)$$

Combining Eqs. (35) and (6) gives the condition $\alpha + \beta = 1$ on the average rates which is of the same form as in the one-species ASEP. This, together with Eq. (29), means that for an uncorrelated steady state to exist, the minimum speed v_1 should be greater than the average arrival rate α. The steady state is now given by $|P\rangle = |\rho\rangle^{\otimes N}$ where

$$|\rho\rangle = \frac{1}{c} \begin{pmatrix} e \\ \dfrac{\Delta_1}{p} \\ \dfrac{\Delta_2}{p} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\Delta_p}{p} \end{pmatrix}, \quad c = e + \frac{1}{p}(\Delta_1 + \Delta_2 \cdots + \Delta_p) \equiv e + \frac{1}{p}\Delta.$$

(36)

The density and current of particles of type (i) and the total density and the total current are all site independent and are, respectively, given by [see Eq. (34)]

$$\rho(\alpha,i) = \frac{\Delta_i/p}{e + \Delta/p} \quad J(\alpha,i) = \frac{v_i/p}{e + \Delta/p},$$

$$\rho(\alpha) = \frac{\Delta/p}{e + \Delta/p}, \quad J(\alpha) = \frac{1}{e + \Delta/p}.$$

(37)

It is better to consider the limiting case $p \to \infty$, that is, we assume that particle speeds are taken from a continuous probability distribution $P(v)$. Condition (26) is then transformed into

$$\langle v \rangle := \int v P(v) dv = 1.$$

Discrete quantities $(1/p)f(i)$ are transformed into the continuous functions $f(v)P(v)$ and sum into integrals. Instead of Eq. (36) we have

$$\rho(\alpha,v) = \frac{\Delta(\alpha,v)P(v)}{e + \Delta(\alpha)}, \quad J(\alpha,v) = \frac{vP(v)}{e + \Delta(\alpha)}, \quad (38)$$

$$\rho(\alpha) = \frac{\Delta(\alpha)}{e + \Delta(\alpha)}, \quad J(\alpha) = \frac{1}{e + \Delta(\alpha)}, \quad (39)$$

where

$$\Delta(\alpha,v) = \frac{v}{v - \alpha} \quad \text{and} \quad \Delta(\alpha) = \int_\alpha^\infty \frac{v}{v - \alpha} P(v) dv.$$

(40)

As an example, in the following, we take a distribution which at low speeds vanishes as some power of $v - \alpha$, so that α is indeed the minimum speed of particles. Requiring that the distribution has an exponential decay rate λ at high speeds and the average speed be unity gives

$$P_\lambda(v) = \frac{1}{\lambda^{m+1}\Gamma(m+1)}(v - \alpha)^m e^{-(v - \alpha)/\lambda}, \quad (41)$$

where $m + 1 = (1 - \alpha)/\lambda$. Here m is not necessarily an integer. A similar choice of $P(v)$ has also been made in [14]. Note that since m is to be positive for each choice of λ, we
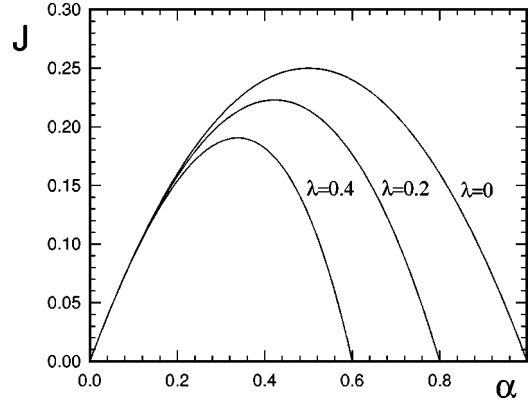

FIG. 1. The current versus the arrival rate of particles for different values of λ.

should have $0 < \alpha < 1 - \lambda$. This distribution is peaked at $v = 1 - \lambda$. Inserting Eq. (41) into Eq. (40) and doing the integral gives

$$\Delta(\alpha) = \frac{\lambda - 1}{\lambda + \alpha - 1} \quad (42)$$

from which we obtain

$$J(\alpha) = \frac{\alpha(\lambda + \alpha - 1)}{\lambda - 1 + \alpha\lambda}, \quad \rho(\alpha) = \frac{\alpha(\lambda - 1)}{\lambda - 1 + \alpha\lambda}. \quad (43)$$

Eliminating α between the above equations gives the current versus density:

$$J(\rho) = \rho\left(1 - \frac{\rho}{1 - \lambda(1 - \rho)}\right). \quad (44)$$

The curves $J(\alpha)$ and $J(\rho)$ are shown in Figs. 1 and 2 for various values of the widths of the distributions. Note that for zero width ($\lambda = 0$) the familiar results of the one-species ASEP are obtained, namely, $\rho = \alpha$ and $J = \rho(1 - \rho)$. For any finite value of λ, ρ is an increasing function of α but $J(\alpha)$ has a maximum at $\tilde{\alpha} := [(1 - \lambda)/\lambda](1 - \sqrt{1 - \lambda})$. The current at this optimal value of the arrival rate is
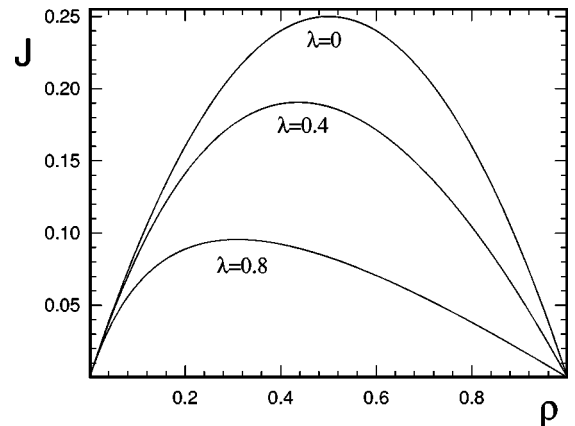

FIG. 2. The current versus the density for different widths of the distribution.

$$J_{\max}(\lambda) = (1-\lambda)\left(\frac{1-\sqrt{1-\lambda}}{\lambda}\right)^2. \qquad (45)$$

The maximum of J as a function of ρ occurs at $\tilde{\rho} = (1/\lambda)[\sqrt{1-\lambda} - (1-\lambda)]$. It is seen that the maximum current has its highest value of $\frac{1}{4}$ only when all the particles have the same hopping rates. It is interesting to note that even in such a simple uncorrleated steady state, a variance in hopping rates reduces the current.

The final quantity we consider is the average density of particles of different speeds, $\rho(\alpha,v)$ as a function of v. From Eqs. (38) and (41) we have

$$\rho(\alpha,v) \propto v(v-\alpha)^{m-1}e^{-(v-\alpha)/\lambda}. \qquad (46)$$

The interesting point is that when $m \geq 1$ (or $\alpha \leq 1-2\lambda$) the density vanishes at the lowest speed α and has a maximum at $v_{\max} := \frac{1}{2}[1-\lambda+\sqrt{(1-\lambda)^2 - 4\lambda\alpha}]$. However, when $m<1$ or ($\alpha > 1-2\lambda$) the distribution changes and the density of the slowest particles rises abruptly. This transition is one of the interesting features of this process and is somehow reminiscent of the Bose condensation noted in [14].

V. REPRESENTATIONS

As in the one-species ASEP the representations of this algebra are either one dimensional or infinite dimensional. To see this we first note that there is no nonzero eigenstate of E, say $|e\rangle$ such that $E|e\rangle = (1/v_i)|e\rangle$ for any i. Since if there is any such vector then by acting on both sides with D_i and using Eq. (5) we obtain $(1/v_i)|e\rangle = 0$ and since v_i is finite this means that $|e\rangle = 0$. Following [2] we now show that in any finite dimensional representation of the algebra all the generators mutually commute, for if the representation is finite dimensional then the matrices $E-1/v_1, \ldots, E-1/v_p$, having no zero eigenvalue, are all invertible and hence Eq. (5) gives $D_i = E(E-1/v_i)^{-1}$ which in turn means that the representation is commutative. Thus the representations of this algebra are either one dimensional or infinite dimensional. Note that in the one dimensional case the expression $D_i = E(E-1/v_i)^{-1}$ automatically satisfies Eqs. (6) and (7). Thus such one dimensional representations really exist, and were used in Sec. IV to find the uncorrelated steady state.

To find the infinite dimensional representations we assume that there is one single vector denoted by $|0\rangle$ such that

$$D_i|0\rangle = d_i|0\rangle, \quad i=1,\ldots,p \qquad (47)$$

where the parameters d_i are to be determined. We then consider the vector space W spanned by the formal vectors $\{|n\rangle := E^n|0\rangle, n=0,1,2,\ldots\}$. Clearly

$$E|n\rangle = |n+1\rangle, \quad \forall n. \qquad (48)$$

Iterating Eq. (5) we find

$$D_iE^n = v_i^{-n}D_i + v_i^{-n+1}E + v_i^{-n+2}E^2 + \cdots + v_i^{-1}E^{n-1} + E^n. \qquad (49)$$

Thus we obtain

$$D_i|n\rangle = v_i^{-n}d_i|0\rangle + v_i^{-n+1}|1\rangle + v_i^{-n+2}|2\rangle$$
$$+ \cdots + v_i^{-1}|n-1\rangle + |n\rangle. \qquad (50)$$

In order to check Eq. (6) in this representation it is enough to consider only the state $|0\rangle$, since all the other states are obtained by acting on this state using the algebraic relations which have been found to be consistent. Therefore we require that

$$\left(D_jD_i - \frac{1}{v_i-v_j}(v_iD_j - v_jD_i)\right)|0\rangle = 0. \qquad (51)$$

This fixes the parameters d_i to be $d_i = v_i/(\epsilon+v_i)$ where ϵ is a constant independent of i. The explicit matrix forms of the generators are then as follows:

$$E = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

$$ (52) $$

$$D_i = \begin{pmatrix} d_i & \dfrac{d_i}{v_i} & \dfrac{d_i}{v_i^2} & \dfrac{d_i}{v_i^3} & \dfrac{d_i}{v_i^4} & \cdot & \cdot \\ 0 & 1 & \dfrac{1}{v_i} & \dfrac{1}{v_i^2} & \dfrac{1}{v_i^3} & \cdot & \cdot \\ 0 & 0 & 1 & \dfrac{1}{v_i} & \dfrac{1}{v_i^2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

We make the following remarks.

(1) If $\epsilon = 0$, then all the $d_i = 1$ and using Eq. (50) one sees that $D_iD_j = D_jD_i \quad \forall i,j$ although $D_iE \neq ED_i$. Hereafter we restrict ourselves to this case. Note that this does not mean that the two point functions $\langle \tau_k^{(i)}\tau_l^{(j)}\rangle$ and $\langle \tau_k^{(j)}\tau_l^{(i)}\rangle$ are equal.

(2) When $\epsilon = 0$ we have $D_i = D_j$ for $v_i = v_j$, so that Eq. (7) becomes vacuous in this case and no singularity arises due to $v_i = v_j$ in this equation. Furthermore, one can safely eliminate one of these generators for the other one. This means that the $(p-1)$-ASEP algebra is naturally embedded in the (p)-ASEP algebra.

The ket and bra vectors $|V\rangle$ and $\langle W|$ are found to be

$$|V\rangle = \sum_{n=0}^{\infty} (1-\beta)^n|n\rangle, \qquad (53)$$

$$\langle W| = \sum_{n=0}^{\infty} (\alpha)^{-n}\langle n|. \qquad (54)$$

Having a representation at hand goes only half of the way in obtaining explicit expressions for physical quantities such as currents and correlation functions. To obtain these quantities, one should do lengthy calculations on the matrix elements of products of operators. Our work is not complete in the sense that we do not yet have the final explicit expression for these matrix elements. However, we explore the properties of the representations as far as we can, hoping that by using these, the rest of the problem will be solved on another occasion, by this author or by others.

A. Eigenvectors of D_i's

Let z be any complex number, with $|z| < v_1$. Define

$$|z\rangle = \sum_{n=0}^{\infty} z^n |n\rangle. \tag{55}$$

Then $|z\rangle$ is a common eigenvector of all D_i's:

$$D_i |z\rangle = \frac{v_i}{v_i - z} |z\rangle. \tag{56}$$

This can be easily proved by direct calculation. The states $|z\rangle$ can in fact be thought of as the coherent states of the algebra constructed as follows: $|z\rangle := [1/(1 - zE)]|0\rangle$ where $1/(1 - zE)$ is understood by its power series. We now define the dual vector $\langle z| = \sum_{n=0}^{\infty} z^{-n} \langle n|$. Then we have the following theorem, the proof of which is accomplished by noting that $\langle n|E = \langle n-1|$ and $\langle 0|E = 0$, and doing straightforward calculations.

Theorem
(a) $\langle z|E = (1/z)\langle z|$.
(b) The coherent states form a basis of the space W, and

$$|n\rangle = \int_{|z| \leqslant 1} \frac{1}{\pi} dz \, d\bar{z} \, z^{-n} |z\rangle.$$

(c) The coherent states $|z\rangle$ form an overcomplete basis for W. A complete basis is obtained by taking only the states with fixed $|z|$, with a completeness relation:

$$1 = \oint_{\text{fixed}|z|} \frac{dz}{2\pi i z} |z\rangle\langle z|.$$

(d) $\langle \omega|z\rangle = 1/(1 - z/\omega)$ for $|z| < |\omega|$.

Remarks
(1) It is now appropriate to recall that the states $|V\rangle$ and $\langle W|$ are in fact coherent states and so it is best to denote them, respectively, by $|1-\beta\rangle$ and $\langle\alpha|$.
(2) The calculation of any physical quantity is now reduced to the calculation of matrix elements of powers of C between the coherent states. For example, the average density $\langle n_k^i \rangle$ is given as

$$\langle n_k^i \rangle = \frac{1}{Z_N} \oint_{(1-\beta < |z| < \alpha, \ |z| < v_i)} \frac{dz}{2\pi i z}$$

$$\times \langle \alpha|C^{k-1}|z\rangle \langle z|C^{N-k}|1-\beta\rangle \frac{v_i}{v_i - z}, \tag{57}$$

where $Z_N = \langle \alpha|C^N|1-\beta\rangle$. Or the probability of a segment $[k,l]$ being empty is given by

$$P(k,l) := \frac{1}{Z_N} \oint_{1-\beta < |z| < \alpha,} \frac{dz}{2\pi i z} \langle \alpha|C^{k-1}|z\rangle$$

$$\times \langle z|C^{N-k-l}|1-\beta\rangle \frac{1}{z^{l-k}}. \tag{58}$$

Once the eigenvalues and eigenvectors of C are known, the problem can be solved completely. In the following we will give an implicit formula for these objects.

B. Eigenvalues of C

First we consider the case $p = 1$. For any coherent state $|z\rangle$ we have $E|z\rangle = (1/z)(|z\rangle - |0\rangle)$. For any two complex numbers z and ω define

$$|z,\omega\rangle = z|z\rangle - \omega|\omega\rangle. \tag{59}$$

Then it follows that

$$E|z,\omega\rangle = |z\rangle - |\omega\rangle. \tag{60}$$

Since for $p = 1$ we have $D|z\rangle = [1/(1-z)]|z\rangle$ we obtain

$$C|z,\omega\rangle = (D+E)|z,\omega\rangle = \frac{1}{1-z}|z\rangle - \frac{1}{1-\omega}|\omega\rangle. \tag{61}$$

Therefore the state $|z,\omega\rangle$ is an eigenstate of C if we have $z(1-z) = \omega(1-\omega)$, the solution of which is $\omega = z$ or $\omega = 1-z$. The second solution is acceptable and hence we have

$$C|z,1-z\rangle = \frac{1}{z(1-z)}|z,1-z\rangle. \tag{62}$$

Now let $p > 1$. Using Eqs. (56) and (60), we obtain

$$C|z,\omega\rangle = z\,\eta(z)|z\rangle - \omega\,\eta(\omega)|\omega\rangle, \tag{63}$$

where

$$\eta(z) = \frac{1}{z} + \frac{1}{p}\sum_{i=1}^{p} \frac{v_i}{v_i - z}. \tag{64}$$

The state $|z,\omega\rangle$ is an eigenvector of C if z and w lie on the curve $\eta(z) = \eta(\omega)$. We will then have $C|z,\omega(z)\rangle = \eta(z)|z,\omega(z)\rangle$.

Note that in the $p \to \infty$ z and ω are related by the following equation:

$$\frac{1}{z} + \int \frac{v\,P(v)}{v-z} = \frac{1}{\omega} + \int \frac{v\,P(v)}{v-\omega}. \tag{65}$$

In order to understand the connection with the eigenstate obtained by Derrida *et al.* [2] in the $p = 1$ case we change the basis of W as

$$|e_n\rangle = (E-1)^n |0\rangle, \quad |e_0\rangle = |0\rangle. \tag{66}$$

Using Eqs. (6) and (7) it is easily found that

$$E|e_n\rangle = |e_n\rangle + |e_{n+1}\rangle,$$

$$D|e_n\rangle = |e_n\rangle + |e_{n-1}\rangle,$$

$$C|e_n\rangle = |e_{n-1}\rangle + 2|e_n\rangle + |e_{n+1}\rangle.$$

Thus we have

$$|z\rangle = \frac{1}{1-zE}|0\rangle = \frac{1}{1-z-z(E-1)}|0\rangle$$

$$= \frac{1}{1-z}\left(\frac{1}{1-[z/(1-z)]E}\right)|0\rangle = \frac{1}{1-z}\left|\frac{z}{1-z}\right\rangle_d,$$
$$(67)$$

where by $|z\rangle_d$ we mean $|z\rangle_d = \Sigma_{n=0}^{\infty} z^n |e_n\rangle$. Therefore

$$|z,1-z\rangle = \frac{z}{1-z}\left|\frac{z}{1-z}\right\rangle_d - \frac{1-z}{z}\left|\frac{1-z}{z}\right\rangle_d$$

$$= \sum_{n=0}^{\infty}\left[\left(\frac{z}{1-z}\right)^{n+1} - \left(\frac{1-z}{z}\right)^{n+1}\right]|e_n\rangle. \quad (68)$$

Taking $z/(1-z) = e^{i\theta}$ gives

$$|z,1-z\rangle = \sum_{n=0}^{\infty} \sin(n+1)\theta|e_n\rangle, \qquad (69)$$

which is denoted by $|\theta\rangle$ in [2] with eigenvalue $1/z(1-z) = 2(1+\cos\theta)$. This concludes our treatment of the representations of the algebra.

VI. DISCUSSION

This work can be pursued further in the following directions.

(a) Finding a solution of the equation $\eta(z) = \eta(\omega)$ either for a low value of p (e.g., $p=2$) or in the $p\to\infty$ limit, and then expanding the state $|1-\beta\rangle$ in terms of the eigenstates of C. In this way one will obtain the current and the phase diagram of the system.

(b) Finding a solution of the mean field equations, either numerically or analytically. These equations can be written down from Eq. (31). In the $p\to\infty$ limit they are

$$\alpha v e_1 = J_k(v) = J_{k+1}(v) = \cdots = (v+\beta-1)n_N(v),$$

where

$$J_k(v) = v n_k(v) e_{k+1} + \int_0^v (v-v')n_k(v)n_{k+1}(v')dv'$$

$$- \int_v^{\infty} (v'-v)n_k(v')n_{k+1}(v)dv'.$$

Here $n_k(v)$ is the average density of particles of speeds v at site k and $e_k := 1 - \int_0^{\infty} n_k(v)dv$ is the probability of site k being vacant. In the one-species model it is known that the mean field analysis gives the phase diagram correctly. So it will be interesting to see for a typical probability distribution how the phase diagram will be modified.

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