Wave chaos in terms of normal modes

A. L. Virovlyansky
Institute of Applied Physics, Russian Academy of Science, 46 Ul'yanov Street, 603600 Nizhny Novgorod, Russia

G. M. Zaslavsky
Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, New York 10012
and Department of Physics, New York University, 2-4 Washington Place, New York, New York 10003
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Wave propagation in a range-dependent waveguide can be considered as a classical physics problem similar to the quantum chaos problem situations. This analogy becomes especially strong when one uses the parabolic equation approximation. By projecting the wave field taken in the quasiclassical approximation onto eigenfunctions of the unperturbed boundary value problem, analytical description has been obtained for normal mode amplitudes in terms of geometrical optics relations. This approach provides a convenient way to study how chaotic behavior of ray trajectories reveals itself in a range dependence of mode amplitudes, and, hence, in the solution of the wave equation. An analog to nonlinear ray-medium resonance for normal modes has been investigated in details and the impact of this phenomenon on modal structure is discussed. It is argued that overlapping of different mode-medium resonances causes a complicated range dependence of mode amplitude in almost the same manner as the overlapping of ray-medium resonances leads to ray chaos. [S1063-651X(99)05902-4]

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INTRODUCTION

Now it is well known that ray trajectories in the range-dependent waveguide media generally exhibit chaotic motion [1–3]. This phenomenon analogous to chaotic dynamics of nonintegrable Hamiltonian systems in classical mechanics is of considerable interest in terms of the theory of wave propagation. In particular, in underwater acoustics this interest has grown in recent years in connection with the problems of ocean-acoustic tomography [4,5]. The point is that the chaotic behavior of trajectories limits our ability to make deterministic predictions using ray theory and, therefore, poses severe restrictions when solving inverse problems. In recent years such important features of the chaotic ray structure in underwater acoustic waveguides as extreme sensitivity to initial and environmental conditions, exponential divergence of neighboring rays, and exponential proliferation of eigenrays, have been established [5–8].

As an evident next step, one should study how chaotic ray dynamics reveals itself in wave phenomena. It is believed that although diffractive effects may smooth the sensitivity to initial and environmental conditions associated with ray chaos, the resulting wave picture will, nevertheless, be very complicated. This situation is called *wave chaos* in the analogy to *quantum chaos*: manifestation of the chaotic motion of a dynamical system in the behavior of the corresponding quantum system. The similarity between quantum and wave theories and, hence, between the problems of quantum and wave chaos, reduces to equivalence (at least from the formal viewpoint) when the wave theory is considered in the limit of small-angle propagation. In this case the wave field is governed by the parabolic equation [9–11] formally coinciding with the Schrödinger equation.

In this paper we address one aspect of the wave chaos problem, namely, complicated range variations of normal mode amplitudes. From the viewpoint of quantum mechanics, we are investigating how the chaotic motion of a classical system is connected to fluctuations of amplitudes of eigenfunctions in the corresponding quantum system.

The main idea of our analysis is the following. We project the ray theory solution of the parabolic equation onto normal modes of the unperturbed boundary value problems (taken in the WKB approximation) and evaluate the corresponding integrals using the stationary phase technique. As a result we have found an approximate analytical expression for mode amplitudes where the latter are expressed in terms of rays. This result not only simplifies the mode amplitude evaluation but also gives an additional insight into the relationship between the ray and mode representations of the wave field in a range-dependent environment. It presents a generalization of some results on ray-mode relations in a waveguide with weak inhomogeneities discussed in Refs. [12–15].

The connection between rays and modes becomes especially clear when the main relations are expressed through the angle-action variables [2,3]. According to the ray theory, the wave field at the observation point is formed by contributions from the so-called eigenrays, i.e., the rays that pass through that point. It turns out that the amplitude of the given mode at the given range is formed by contributions from the rays, which we call the eigenrays for the given mode. These eigenrays have the values of the action variables equal to that of the WKB mode, which are equal to the mode numbers up to a multiplicative constant [16–18].

The approach considered in this paper that provides a description of mode amplitude variations in terms of geometrical optics, presents a convenient tool for adapting the results obtained when studying chaotic ray behavior for the purpose of investigating a complicated range variations of the field modal structure.

Here we restrict our attention to a monochromatic wave field in a two-dimensional (2D) waveguide with a periodic range dependence. The mechanism of ray chaos in such a waveguide is analogous to that of chaotic behavior of a nonlinear oscillator driven by an external periodic force. Nonlinear ray-medium resonance plays a crucial role in the emergence of chaos [2,3]. According to the heuristic criterion proposed by Chirikov [19–21] chaos is a result of an overlap of different resonances between ray trajectories and medium inhomogeneities.

Since we express mode amplitudes through parameters of ray trajectories, the ray-medium resonance can be easily interpreted from the viewpoint of normal modes. It has been shown that a bunch of rays captured into an isolated resonance corresponds to a group of modes with amplitudes much stronger affected by inhomogeneities than those of other modes. We call this phenomenon the mode-medium resonance. It can be easily described on the basis of well-known relations for captured rays. Overlapping of different mode-medium resonances should yield the exponential growth (with range) of eigenrays contributing to a particular mode leading to a complicated range-dependence of the mode amplitude and to vanishing of mutual correlations between modes.

We do not broach here an important issue of breakdown of the semiclassical approximation, thus restricting our consideration to relatively short ranges. This topic will be considered elsewhere. Nevertheless, an ''available'' interval of distances can be large enough for many practical applications. For example, as it has been shown numerically in Ref. [22], the ray-based description of long-range sound transmission through ocean internal waves may capture important characteristics of the sound field surprisingly well even when ray trajectories exhibit chaotic behavior.

This paper is organized as follows. In Sec. II we derive the formulas expressing the mode amplitude variations through parameters of eigenrays contributing to this mode. This is done for two types of sources, i.e., for two types of starting fields at the initial cross section of the waveguide. We consider a starting field localized at a given point of the cross section (point source), and another starting field with the amplitude being a slow function of transverse coordinate and the phase corresponding to a quasiplane wave (distributed source). It is shown that an arbitrary starting field can be synthesized as a superposition of a number (this number maybe very large) of the sources of the second type. In Sec. III, we present the well-known basic analytical relations for description of the nonlinear ray-medium resonance in the scope of the perturbation theory, and discuss how they can be used for analysis of the modal field structure. It is shown that in the framework of our approach the same relations can be used to study the mode-medium resonance, i.e., the behavior of amplitudes of those modes which are in resonance with the perturbation, and, hence, are most strongly affected by inhomogeneities. It has been found that if a single resonant mode is excited, it is split into a bundle of Δm modes and a simple estimation for Δm is offered. In this section we also discuss how the overlapping of resonances gives rise to irregular range variations of the modal structure. To illustrate these ideas we present some results of numerical calculations of the modal structure in a simple model waveguide. The calculations of the wave field in the parabolic equation approximation have been performed using the UMPE code [27]. At the end of this section we argue that although under ray chaos conditions the mode amplitudes should typically become more and more random with range, there are some specific features of ray chaos that make us think that (i) some mode amplitudes may have nonrandom constituents at very long ranges; (ii) even if the mode amplitude has not such a constituent, it may reveal long-lasting correlations.

I. ANALOG OF GEOMETRICAL OPTICS FOR MODE AMPLITUDES

A. Semiclassical approximation

We consider a scalar monochromatic wave field in a 2D medium with the phase velocity, C (in acoustics, for example, it is the sound speed), being the function of coordinates x and z but independent of time t. The field complex amplitude u satisfies the Helmholtz equation, which can be simplified under the assumption that the main direction of wave propagation coincides with the x axis and the wave grazing angles with respect to this direction are small. As discussed in Refs. [10,11,16], in this case the Helmholtz equation can be approximated by the parabolic equation for u,

$$2ik\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial z^2}+k^2\left(\frac{C_0^2}{C^2(x,z)}-1\right)u=0, \quad (1)$$

where C_0 is a reference phase speed, $k=2\pi f/C_0$ is the wave number in the reference medium with $C=C_0$, f is a carrier frequency. The time factor $e^{-i2\pi ft}$ is omitted throughout.

Using the notation

$$\frac{1}{2}\left(1-\frac{C_0^2}{C^2(x,z)}\right)=U(z)+\varepsilon V(x,z) \quad (2)$$

we rewrite Eq. (1) in the form

$$ik\frac{\partial u}{\partial x}=-\frac{1}{2}\frac{\partial^2 u}{\partial z^2}+k^2[U(z)+\varepsilon V(x,z)]u, \quad (3)$$

which coincides with the Schrödinger equation. Here the x variable plays the role of time and k^{-1} plays the role of the Planck constant.

The ''potential'' defined in Eq. (2) is split into two parts: the range-independent one, $U(z)$, and the range-dependent one, $\varepsilon V(x,z)$. Later, in order to simplify the analytical treatment of complicated ray trajectory behavior, we shall consider ε as a small parameter. However, in this section, this assumption is *not* used. Here we are studying the semiclassical solution of Eq. (3) which requires the wavelength, $2\pi/k$, to be small compared to the characteristic spatial scales of the total potential $U(z)+\varepsilon V(x,z)$ [2,11].

The Hamiltonian corresponding to Eq. (3),

$$H=\frac{p^2}{2}+U(z)+\varepsilon V(x,z), \quad (4)$$

is a function of coordinate z, momentum p, and timelike variable x. Solutions to the Hamilton equations,

$$\dot{z}=\frac{\partial H}{\partial p},\quad \dot{p}=-\frac{\partial H}{\partial z}, \tag{5}$$

define the ray trajectories that we denote as

$$z=z(x,p_0,z_0),\quad p=p(x,p_0,z_0), \tag{6}$$

where z_0 and p_0 are initial values of the coordinate and the momentum at $x=0$, respectively (see Appendix).

The ray eikonal, which is an analog to the mechanical action, is given by the relation [17,24]

$$S=\int (p dz - H dx), \tag{7}$$

where the integral runs over the ray trajectory. Considering the eikonal as a function of the range x and the initial and final coordinates, z_0 and z, respectively, of the ray trajectory, we have [24]

$$\frac{\partial S(x,z,z_0)}{\partial z}=p,\quad \frac{\partial S(x,z,z_0)}{\partial z_0}=-p_0. \tag{8}$$

Here $S(x,z,z_0)$ is the action corresponding to a ray trajectory starting at $x=0$ from the point z_0 and arriving in the point z at the range x; p_0 and p, which are also considered as functions of x, z, and z_0, represent the initial and final momenta for this trajectory.

The semiclassical approximation (or the geometrical optics approximation as it is called in wave theory) to the solution of Eq. (3) is given by the relation [2,17]

$$u(x,z)=\sum_{\nu} A^{\nu}(x,z)e^{ikS^{\nu}(x,z)-i(\pi/2)\mu^{\nu}}, \tag{9}$$

where each term represents a contribution to the total wave field from an eigenray, that is, a ray which passes through the point (x,z). The sum goes over all the eigenrays contributing at a particular receiver position. In the above formula the superscript ν numbers the eigenrays, A^{ν} and S^{ν} are the amplitude and the eikonal of the νth eigenray, respectively, and μ^{ν} is the Maslov index, or the integral number of times that the νth ray passes through caustics (at caustics the ray amplitude A^{ν} goes to infinity and the semiclassical approximation fails).

The explicit expression for the ray amplitude depends on the source exciting the wave field, or, in other words, on the initial conditions of Eq. (3) at $x=0$. We consider two important examples of such initial conditions. Since we are going to analyze the contribution from an individual eigenray, we shall omit the superscript ν in the remaining part of this subsection.

a. Point source. In this case

$$u(0,z)=\delta(z-z_0), \tag{10}$$

and the desired solution represents the Green's function of Eq. (3). The semiclassical approximation to this function is known (see, for example, Ref. [25]):

$$A=\sqrt{\frac{k}{2\pi i}}\sqrt{\left|\frac{\partial^2 S}{\partial z \partial z_0}\right|}. \tag{11}$$

We rewrite it using the relation

$$\frac{\partial^2 S}{\partial z \partial z_0}=-\left(\frac{\partial z}{\partial p_0}\right)^{-1}$$

that follows from Eq. (8). This yields

$$A=\sqrt{\frac{k}{2\pi i|\partial z/\partial p_0|}}. \tag{12}$$

Note that all the rays escape from the same point z_0 determined by the source position and each ray is ''labeled'' by its initial momentum p_0. So the eigenrays are determined by the relation

$$z=z(x,p_0,z_0), \tag{13}$$

which formally coincides with the first equality in Eq. (6) but is considered here as an equation in p_0. On the other hand, the same equation can be treated as a definition of p_0 as a function of x and z. Substituting the function $p_0=p_0(x,z)$ into Eq. (12) (after evaluating the derivative) determines the ray amplitude as a function of x and z.

b. Quasi-plane-wave source. The initial wave field is determined by the function

$$u(0,z)=a(z)e^{iks(z)}, \tag{14}$$

where $a(z)$ and $s(z)$ are two functions slowly varying with z: their characteristic scales are much greater than the wavelength, $2\pi/k$. At the same time we assume that, due to large k, the phase of Eq. (14) is a rapidly oscillating function. This type of source excites a quasi-plane-wave.

The detailed description of the semiclassical solution to Eq. (3) with the initial conditions (14) is given in Ref. [26]. In this case different rays start from different points z_0. The trajectory leaving the point z_0 has the initial moment

$$p_0=\overline{p}(z_0),\quad \overline{p}(z)=\frac{\partial s(z)}{\partial z}. \tag{15}$$

So in this example (as opposed to the previous one) each ray is labeled by its initial coordinate z_0. The eigenrays are defined by the equation [analogous to Eq. (13)]

$$z=z_s(x,z_0), \tag{16}$$

where

$$z_s(x,z_0)=z(x,z_0,\overline{p}(z_0)).$$

Solving Eq. (16) for z_0 one finds the starting points of the rays crossing the given observation point z at the given distance x. The initial momenta of these eigenrays are then found from Eq. (15).

The function $u(x,z)$ is again represented by the sum (9), but this time the expression for the amplitude of an individual ray takes on the form

$$A=\frac{a(z_0)}{\sqrt{|\partial z_s/\partial z_0|}}e^{iks(z_0)}, \tag{17}$$

where z_0 is the initial coordinate of the given eigenray. The values of z_0 and, hence, of A, in accordance with the above remark, can be considered as functions of x and z.

Equations (12) and (17) allow one to find the eigenray amplitudes as functions of x and z, which can then be substituted into Eq. (9). The eigenray phases as functions of x and z can be expressed through the function $S(x,z,z_0)$ whose definition has been discussed in the commentary to Eq. (8). In the case of the point source (10) the function $S(x,z,z_0)$ with z_0 being the coordinate of the source, gives the desired eigenray phase. In the case of the distributed source (14), the argument z_0 should be expressed through x and z using Eq. (16).

B. Mode representation of the wave field

Considering the range-dependent component of potential, i.e., $\varepsilon V(x,z)$, as a perturbation (it has already been mentioned that in this section we do not assume this perturbation to be small) we expand the wave field $u(x,z)$ into a sum of eigenfunctions of the unperturbed Sturm-Liouville eigenvalue problem [11,17]

$$-\frac{1}{2}\frac{d^2\varphi_m}{dz^2}+k^2U(z)\varphi_m=k^2E_m\varphi_m. \tag{18}$$

In wave theory, the eigenfunctions $\varphi_m(z)$ are usually called the normal modes. They are orthogonal and we normalize them in such a way that

$$\int_{-\infty}^{\infty}dz\,\varphi_m(z)\varphi_n(z)=\delta_{mn}. \tag{19}$$

The modes form a complete set, which means that we can represent an arbitrary function as a sum of normal modes. Thus, we write the wave field as

$$u(x,z)=\sum_m c_m(x)\varphi_m(z). \tag{20}$$

Our main goal in this section is to derive comparatively simple semiclassical expressions for the mode amplitude c_m. In so doing we use the semiclassical approximation to $u(x,z)$ given in Eq. (9) and project it onto normal modes. According to Eqs. (19) and (20) our task is reduced to the evaluation of the integrals

$$c_m=\int_{-\infty}^{\infty}dz\,u(x,z)\varphi_m(z). \tag{21}$$

Since we consider the semiclassical approximation to $u(x,z)$, it is natural to use the same approximation for $\varphi_m(z)$. The corresponding formulas for $\varphi_m(z)$ are usually referred to as the WKB approximations to the eigenfunctions [11,16,17]. These formulas are expressed through parameters of ray trajectories in the unperturbed waveguide medium with the Hamiltonian

$$H_0=p^2/2+U(z). \tag{22}$$

Along the ray trajectory the conservation law

$$H_0(p,z)=E \tag{23}$$

holds true with the constant E being an analog to the mechanical energy. Equation (23) yields the explicit expression for the momentum p as a function of E and z:

$$p(E,z)=\pm\sqrt{2[E-U(z)]}. \tag{24}$$

All the trajectories are periodic curves. The coordinates of their upper and lower turning points ($z_{\max}$ and $z_{\min}$, respectively) are functions of the "energy" E and determined by the equation

$$U(z)=E.$$

For simplicity we shall assume that the potential $U(z)$ is smooth, has the only minimum, and its walls tend to infinity as $z\rightarrow\pm\infty$.

An important characteristics of ray trajectories that are widely used in both classical mechanics and ray theory, is the so-called action variable I related to E by [24]

$$I=\frac{1}{2\pi}\oint dz\,p(E,z)=\frac{1}{\pi}\int_{z_{\min}}^{z_{\max}}dz\sqrt{2[E-U(z)]}, \tag{25}$$

where the integration goes over the period of the ray trajectory. Equation (25) defines the function $E(I)$. Now the turning point coordinates $z_{\min}$ and $z_{\max}$ can also be regarded as functions of I.

In the scope of the WKB approximation, the eigenvalues of the action variable I_m are determined by the quantization rule

$$kI_m=m+\tfrac{1}{2}. \tag{26}$$

Then the eigenvalues of the energy are given by the relation $E_m=E(I_m)$.

The mth eigenfunction $\varphi_m(z)$ between its turning points can be represented as follows [11,16]:

$$\varphi_m(z)=\varphi_m^+(z)+\varphi_m^-(z), \tag{27}$$

where

$$\varphi_m^{\pm}(z)=Q_m(z)e^{\pm i(kS_0(z,I_m)-\pi/4)}, \tag{28}$$

$$S_0(z,I)=\int_{z_{\min}}^{z}dz\sqrt{2[E(I)-U(z)]}, \tag{29}$$

$$Q_m(z)=\frac{1}{\sqrt{D(I_m)}\sqrt[4]{2[E_m-U(z)]}},\qquad D(I)=2\int_{z_{\min}}^{z_{\max}}\frac{dz}{\sqrt{2[E(I)-U(z)]}}. \tag{30}$$

The functions $S_0(z,I)$ and $D(I)$ represent important characteristics of the quasiclassical solution to Eq. (3) in the range-independent environment. The first one determines the contribution to the action from the first term in Eq. (7) taken along the part of the ray trajectory connecting the lower turning point to the coordinate z, and the second one is the period of the ray trajectory along the x axis.

Equations (26)–(30) give the WKB approximation to the mth eigenfunction in the unperturbed waveguide ($\varepsilon=0$). Below these expressions are used when expanding the quasiclassical solution to Eq. (3).

C. Ray field projection onto normal modes

Now we have the approximate analytical expressions for both the total wave field $u(x,z)$ and the eigenfunctions $\varphi(z)$ and can find the mode amplitudes c_m by evaluating the integrals (21).

Taking into account Eq. (27) we transform Eq. (21) to

$$c_m(x)=c_m^+(x)+c_m^-(x), \tag{31}$$

where

$$c_m^\pm(x)=\int dz u(x,z)\varphi_m^\pm(z). \tag{32}$$

Replacing $u(x,z)$ and $\varphi_m^\pm(z)$ with their semiclassical approximations given in Eqs. (9) and (28), respectively, yields

$$c_m^\sigma(x)=\frac{1}{\sqrt{D(I_m)}}e^{-i(\pi/2)\mu^\nu-i\sigma\pi/4}\times\sum_n\int dz\frac{A^\nu}{\sqrt[4]{2[E_m-U(z)]}}e^{ik(S^\nu+\sigma S_0(z,I_m))} \tag{33}$$

with σ denoting $+1$ or -1 (or simply $+$ or $-$, when it is used as a superscript).

Let us consider one of these integrals and evaluate it using the stationary phase technique [11,16]. Up to the factor k, the phase of the integrand is equal to

$$\Phi=S(x,z,z_0)+\sigma S_0(z,I_m),$$

where we have again omitted the superscript ν. According to Eqs. (8) and (29) the first derivative of Φ with respect to z can be represented in the form

$$\frac{\partial\Phi}{\partial z}=p(x,z,z_0)+\sigma\sqrt{2[E_m-U(z)]}.$$

At the stationary phase point the first derivative vanishes, yielding

$$p(x,z,z_0)=-\sigma\sqrt{2[E_m-U(z)]}. \tag{34}$$

This is a very important relation: it singles out the rays contributing to the mth mode at the range x. According to the commentary to Eq. (8), $p(x,z,z_0)$ represents the final momentum of the ray connecting the point $(0,z_0)$ to (x,z) or, in other words, the final momentum of an eigenray arriving at the point (x,z). For the point source the value of z_0 is the same for all eigenrays. For the distributed source, as it has been discussed in the previous subsection, the value of z_0 can be considered as a function of x and z. Thus, for the given x, in both cases, the left-hand side of Eq. (34) is a function of z and the equation should be solved for this variable. Each solution will determine a ray that we shall call the eigenray for the mth mode.

The second derivative of Φ at the stationary phase point is given by

$$\frac{\partial^2\Phi}{\partial z^2}=\frac{\partial p}{\partial z}+\frac{1}{p}\frac{\partial U}{\partial z}=\frac{1}{p}\frac{\partial H_0}{\partial z}, \tag{35}$$

with p being the momentum of the eigenray arriving at the point (x,z). In the above relation p is considered as a function of x and z. The same is true of the function $H_0=H_0(p(x,z),z)$. The contribution to the mode amplitude from an individual eigenray is given by

$$c_m^\sigma(x)=\sqrt{\frac{2\pi}{kD(I_m)|\partial H_0/\partial z|}}Ae^{ik\Phi_{\rm st}+i\alpha},$$

$$\alpha=(\gamma-\sigma-2\mu)\pi/4, \tag{36}$$

with $\Phi_{\rm st}$ being the value of Φ at the stationary phase point and γ being the sign of the derivative $\partial^2\Phi/\partial z^2$ at this point. The total value of the mode amplitude is obtained by simply summing up the contributions of each of the eigenrays.

Replacing A in Eq. (36) with the expressions given in Eqs. (12) and (17) we obtain two versions of the above formula for the point and distributed sources. As we already know, in the case of the point source each ray is defined by its initial moment p_0. Any characteristic of the ray [including its current coordinate z, current momentum p, and, hence, current value of $H_0(z,p)$] can be considered as a function of p_0 and x. Bearing this in mind, we easily find that for the point source

$$c_m^\sigma(x)=\sum_\nu\frac{1}{\sqrt{iD(I_m)|\partial H_0/\partial p_0|_{p_0=p_0^\nu}}}e^{ik\Phi_{\rm st}^\nu+i\alpha^\nu}, \tag{37}$$

where the sum goes over all the eigenrays contributing to the mth mode, and p_0^ν denotes the initial momentum of the νth eigenray.

Similarly, for the distributed source (14) any characteristic of a ray can be regarded as a function of the initial coordinate z_0 and x. Substituting Eq. (17) into Eq. (36) yields

$$c_m^\sigma(x)=\sum_\nu\sqrt{\frac{2\pi}{kD(I_m)|\partial H_0/\partial z_0|_{z_0=z_0^\nu}}}\times a(z_0^\nu)e^{iks(z_0^\nu)+ik\Phi_{\rm st}^\nu+i\alpha^\nu} \tag{38}$$

with z_0^ν being the initial coordinate of the νth eigenray.

Equations (37) and (38) provide the analytical description of mode amplitudes in a range-dependent environment through the parameters of ray trajectories, i.e., through solutions to the Hamilton equations (5).

D. Action-angle variables

The above result can be reformulated in terms of the so-called action-angle variables, which are often used to simplify the analysis of a quasiperiodic motion in classical mechanics [24] and a quasiperiodic ray trajectory behavior in waveguide media [2].

We begin with the range-independent waveguide $(\varepsilon=0)$. In this case, the canonical transformation from our (p,z) variables to the action-angle variables (I,θ) is given by the pair of equations [24]

$$p=\frac{\partial G(z,I)}{\partial z}, \quad \theta=\frac{\partial G(z,I)}{\partial I} \tag{39}$$

with the generating function

$$G(z,I)=\int^{z} dz\sqrt{2[E(I)-U(z)]}. \tag{40}$$

The function $E(I)$ is determined by Eq. (25). It should be emphasized that p and z are periodic functions of the angle variable θ, i.e.,

$$p(I,\theta)=p(I,\theta+2\pi), \quad z(I,\theta)=z(I,\theta+2\pi). \tag{41}$$

The Hamilton equations in terms of the new variables reduce to

$$\dot{I}=0, \quad \dot{\theta}=\omega(I), \tag{42}$$

where

$$\omega(I)=\frac{dH_0(I)}{dI}=\frac{2\pi}{D(I)} \tag{43}$$

is the spatial frequency of the trajectory oscillations along the x axis.

Note that the function G grows with range. At each period of oscillations its value increases by I [24].

In the range-dependent waveguide $(\varepsilon\neq 0)$ we define the action-angle variables using *the same* relations [given in Eqs. (39) and (40)] as in the unperturbed waveguide. The Hamilton equations in the new variables take the form [2]

$$\dot{I}=-\varepsilon\frac{\partial V}{\partial\theta}, \quad \dot{\theta}=\omega(I)+\varepsilon\frac{\partial V}{\partial I}. \tag{44}$$

Since Eq. (24) remains valid in the range-dependent environment, Eq. (34) which defines the eigenrays contributing to the mth mode, takes the very simple form in terms of the action-angle variables

$$I=I_m. \tag{45}$$

Let us discuss this condition. First of all, note, that we can consider the action variable satisfying Eqs. (44) as a function of range, and initial values of the momentum p_0 and the coordinate z_0, i.e.,

$$I=I(x,p_0,z_0). \tag{46}$$

For the point source given in Eq. (10) the value of z_0 is the same for all the rays and after substituting Eq. (46) into Eq. (45) we get the equation in p_0 analogous to Eq. (13), defining the initial momenta of the eigenrays for the mth mode. Taking into account Eq. (43) we rewrite the expression (37) as

$$c_m^{\sigma}=\sum_n \frac{1}{\sqrt{2\pi i|\partial I/\partial p_0|_{p_0=p_{0n}}}}e^{ik\Phi_n^{\mathrm{st}}+i\alpha_n}. \tag{47}$$

For the distributed source, as it is stated in Eq. (15), the initial momentum, p_0, is a function of the initial coordinate, z_0, and an analog to Eq. (16) is obtained by substituting $I(x,\overline{p(z_0)},z_0)$ for I in Eq. (46). In this case the expression for the mode amplitude (38) can be rewritten as

$$c_m^{\sigma}=\sum_n \frac{a(z_{0n})}{\sqrt{k|\partial I/\partial z_0|_{z_0=z_{0n}}}}e^{iks(z_{0n})+ik\Phi_n^{\mathrm{st}}+i\alpha_n}. \tag{48}$$

The last two equations as well as Eqs. (37) and (38) reduce the mode amplitude evaluation to a procedure quite analogous to that generally used when evaluating the field amplitude at the given point. It includes solving the Hamilton (ray) equations, finding the eigenrays, calculating ray eikonals and some derivatives with respect to initial values of ray parameters.

E. Arbitrary starting field

Although the expressions we have derived so far yield the mode amplitudes for the two particular types of starting fields, Eq. (48) can be applied to treating an arbitrary initial condition $u(0,z)$. This topic is addressed in this subsection.

Decomposing an arbitrary starting field as a sum of eigenfunctions

$$u(0,z)=\sum_m c_m(0)\varphi_m(z),$$

we shall study its further evolution with range using linearity of initial equation (3) and evaluating the contribution to the total wave field from each term of the above sum on the basis of Eq. (48).

According to Eqs. (27) and (28), the starting field $u(0,z)=\varphi_m(z)$ represents a superposition of two terms defined in Eq. (14). Each term can be associated with a congruence of rays taking off from the part of the initial cross section $x=0$ lying between the mode turning points. The initial momenta of these rays are defined in Eq. (15), where $s(z)$ must be replaced with $S_0(z,I_m)-\pi/4$ for one of the congruences and with $-S_0(z,I_m)+\pi/4$ for another. So there will be two rays leaving each point with the initial momenta equal in absolute value and opposite in sign.

Initial values of the action variable I are equal to I_m for *all* the rays belonging to *both* congruences, while the initial values of the angle variable θ cover the whole interval from 0 to 2π.

So, representing the wave field corresponding to $u(0,z)=\varphi_m(z)$ as

$$\sum_n K_{mn}(x)\varphi_n(z),$$

we see that each mode amplitude K_{mn} at the given range x is determined by the right-hand side of Eq. (48) with $a(z)$ replaced by $Q_m(z)$ and $ks(z)$ replaced by $\pm[kS_0(z,m)-\pi/4]$. In this case the summation goes over all the ray trajectories with initial and final values of the action variable I equal to I_m and I_n, respectively. This result has been obtained earlier in Ref. [18] (see also Ref. [23]).

Now it is clear that the total wave field for an arbitrary initial function $u(0,z)$ is given by

$$u(x,z)=\sum_{m,n} c_m(0)K_{mn}(x)\varphi_n(z),$$

and the mth mode amplitude at the range x is equal to

$$c_m(x)=\sum_{n} c_n(0)K_{nm}(x).$$

An additional summation over the mode number that has been absent for the above considered two types of source is the price one has to pay for generality.

II. MODE-MEDIUM RESONANCE

A. Perturbation theory for ray trajectories

Having the comparatively simple expressions relating the mode amplitudes to rays, we can now discuss how the complicated ray trajectory dynamics reveals itself in the mode amplitude variations. In so doing, we restrict our attention to a waveguide with a weak periodic range dependence. It means that $V(x,z)=V(x+2\pi/\Omega_0,z)$ and ε in Eq. (4) is considered as a small parameter. In terms of angle-action variables (I,θ) the periodic perturbation can be represented in the form of the Fourier series

$$V=\frac{1}{2}\sum_{l,q} V_{l,q}(I)e^{i(l\theta-q\Omega_0 x)}+\text{c.c.}, \tag{49}$$

where the symbol c.c. denotes complex conjugation. The smallness of the perturbations allows us to use a simple analytical description of the nonlinear resonance between the ray trajectories and the Fourier harmonics of medium inhomogeneities [2,3].

It is known that analysis of the nonlinear resonance is crucial for the understanding of the mechanism of ray stochasticity in a periodically varying waveguides. In this section we show how this phenomenon affects the mode amplitudes, leading to their complicated range dependence.

In terms of angle-action variables, ray trajectories are governed by the Hamilton equations (44) with the function V given in Eq. (49). A group of ray trajectories are captured in a resonance if their action variables are close to I_0 satisfying the condition

$$l\omega(I_0)=q\Omega_0, \tag{50}$$

with l and q being two integers. For an analytical treatment of the rays trapped into the resonance, it is convenient to introduce the new canonical variables ΔI and ψ using the generating function

$$G_1(I,\psi,x)=-\psi(I-I_0)-I(\omega_0 x-x\phi_0/l),$$

with $\omega_0=\omega(I_0)$, and ϕ_0 being the phase of the resonant term $V_{l,q}(I)$ at $I=I_0$. The canonical transformation is determined by

$$\frac{\partial G_1}{\partial I}=-\theta,\quad \frac{\partial G_1}{\partial\psi}=-\Delta I,\quad H_1-H=\frac{\partial G_1}{\partial x},$$

where H_1 is a new Hamiltonian. This yields

$$\Delta I=I-I_0, \tag{51}$$

and

$$\psi=\theta-\omega_0 x+\phi_0/l. \tag{52}$$

In what follows, for simplicity we assume $\phi_0=0$.

Expressing the new Hamiltonian in terms of the new canonical variables we simplify it by (i) retaining only the resonant constituents of V [i.e., only the two complex conjugate Fourier harmonics with l and q satisfying Eq. (50)] taken at $I=I_0$, and (ii) approximating $\omega(I_0+\Delta I)$ by $\omega(I_0)+\omega'\Delta I$, where $\omega'=d\omega(I_0)/dI$. This yields

$$H_1(\Delta I,\psi,x)=H_0(I_0)-\omega_0 I_0+\tfrac{1}{2}\omega'\Delta I^2+\varepsilon V_0\cos(l\psi). \tag{53}$$

with $V_0=|V_{l,q}(I_0)|$. The applicability conditions of the approximations made when deriving Eq. (53) is discussed in Refs. [2,3]. Here we only note, that the main of them is given by the equation

$$\varepsilon\ll\left|\frac{d\omega(I_0)}{dI}\right|\frac{I}{\omega(I_0)}\ll\frac{1}{\varepsilon},$$

which is usually referred to as the condition of moderate nonlinearity.

The Hamilton equations obtained on the basis of Eq. (53),

$$\Delta\dot I=\varepsilon V_0\sin(l\psi),\quad \dot\psi=\omega'(I_0)\Delta I, \tag{54}$$

formally coincide with those for the nonlinear pendulum in classical mechanics and the last two terms in the right-hand side of Eq. (53) are analogous to kinetic and potential energies with the variables ΔI and ψ being analogues to the momentum and coordinate, respectively. It is clear that the values of ΔI corresponding to the finite motion belong to the interval $-\Delta I_{\max}<\Delta I<\Delta I_{\max}$, where

$$\Delta I_{\max}=2\sqrt{\varepsilon V_0/|\omega'|}. \tag{55}$$

From the viewpoint of rays, the above equation defines the width of the resonance in terms of action. Each trapped ray oscillates with some spatial frequency. The width of the resonance in terms of spatial frequency can be approximately estimated as

$$\Delta\omega=|\omega'|\Delta I_{\max}/2=\sqrt{\varepsilon V_0|\omega'|}. \tag{56}$$

The motion of rays with the action variable ΔI exceeding $\Delta I_{\max}$ is infinite and such rays are not captured into the resonance. The ray with $\Delta I=\Delta I_{\max}$ is a separatrix in the $(\Delta I,\psi)$ phase plane which forms a border between the two types of rays, trapped and not trapped [2,3].

The solutions of Eqs. (54) can be easily expressed in terms of the Jacobian elliptic functions [28] taking into account that a quantity

$$\mathcal{E}=\frac{\dot{\psi}^2}{2\omega'}+\varepsilon V_0\cos l\psi$$

remains constant along the ray trajectory. Here we assume ω' to be positive. For a negative ω' the following formulas can be easily modified. The ray trajectory is captured into the resonance if $\mathcal{E}<\varepsilon V_0$ and is not captured if $\mathcal{E}>\varepsilon V_0$. For a ray captured into the resonance we get

$$l\psi=2\arcsin[\rho sn(\kappa x+\xi,\rho)]+\pi, \tag{57}$$

$$\Delta I=\frac{2\rho\kappa}{l\omega'}cn(\kappa x+\xi,\rho), \tag{58}$$

where $sn(z,\rho)$ and $cn(z,\rho)$ are the Jacobian elliptic functions with a modulus

$$\rho=\sqrt{\frac{\mathcal{E}+\varepsilon V_0}{2\varepsilon V_0}},$$

and

$$\kappa=l\sqrt{\omega'\varepsilon V_0}.$$

The above relations combined with Eqs. (51) and (52) give explicit expressions for the action and angle variable, that is for I and θ. They depend on two constants $\mathcal{E}$ and ξ defined by initial conditions.

The expression for the eikonal given in Eq. (7) in terms of $(\Delta I,\psi)$ variables looks like

$$S=G+G_1+\int(\Delta I d\psi-H_1 dx).$$

Using the solution (57) and the approximate expression for the Hamiltonian H_1 given in Eq. (53), we rewrite it in the form

$$S=G-H_0(I_0)x-\theta(I-I_0)+(\mathcal{E}+2\varepsilon V_0)x$$
$$-4\varepsilon V_0\int_0^x\rho^2 sn^2(\kappa x+\xi,\rho)dx.$$

The explicit expressions for I and θ are determined by Eqs. (51), (52), (57), and (58). Note that the value of G increases with range. According to a remark made when introducing the action-angle variables, the value of G increases by I each time when the value of θ increases by 2π.

The trajectory not captured into the resonance is determined by

$$l\psi=2am(\kappa_1 x+\xi_1,\rho_1)+\pi \tag{59}$$

and

$$\Delta I=\frac{2\kappa_1}{l\omega'}\sqrt{1-\rho_1^2 sn^2(\kappa_1 x+\xi_1,\rho_1)}, \tag{60}$$

where $am(z,\rho_1)$ and $cn(z,\rho_1)$ are the Jacobian elliptic functions with a modulus $\rho_1=1/\rho$, and

$$\kappa_1=l\sqrt{\omega'(\mathcal{E}+\varepsilon V_0)/2}.$$

This solution also depends on two constants $\mathcal{E}$ and ξ_1. An approximate expression for the eikonal in this case is given by

$$S=G-H_0(I_0)x-\theta(I-I_0)+(\mathcal{E}+2\varepsilon V_0)x$$
$$-4\varepsilon V_0\int_0^x sn^2(\kappa_1 x+\xi_1,\rho_1)dx.$$

The above relations provide an approximate description of ray trajectories in the case of an isolated resonance. In the next subsection we discuss how they can be used for analysis of the modal structure.

B. From ray-medium resonance to mode-medium resonance

Let us consider the case when only the mth mode is excited at $x=0$. As has been discussed in Sec. I E this starting field can be represented as a superposition of two quasi-plane-waves and then treated on the basis of Eq. (48). In terms of geometrical optics at $x=0$, we have two congruences of rays with starting values of action variables equal to I_m.

A situation which we call *mode-medium resonance* occurs when the value of I_m satisfies Eq. (50). In this case the above rays are trapped into the resonance and if the latter is isolated, Eqs. (44) simplify to Eqs. (54) with $I_0=I_m$. Combining these equations with the results on ray-mode relations obtained in Sec. II provides an easy way to make some conclusions and estimations concerning the dependence of modal structure on range.

For example, it is clear that due to the resonance, at ranges of order of $1/\Delta\omega$ there will appear a bundle of rays with the action variables ΔI in the interval $|\Delta I|<\Delta I_{\max}$. This means [see Eqs. (26) and (55)] that starting with such ranges, the mth mode is split into a group of $2\Delta m$ modes with

$$\Delta m=\Delta I_{\max}k=2\sqrt{\varepsilon V_0/\omega'}k. \tag{61}$$

In the case of overlapping resonances it is natural to expect a further broadening of a group of modes.

To illustrate these points, we restrict our attention to a simple waveguide with a potential

$$U(z)=\begin{cases}Lz, & z\geqslant 0\\ \infty, & z<0\end{cases} \tag{62}$$

that is perturbed by a small additive term with

$$\varepsilon V(x,z)=\varepsilon Lz\sin\Omega x+\varepsilon_1 Lz\sin\Omega_1 x. \tag{63}$$

It can be easily shown that this potential can be treated by means of the relations from Sec. II for a smooth function $U(z)$. Only two minor revisions are necessary: (1) the constant $\frac{1}{2}$ in the right hand side of Eq. (26) must be changed to $\frac{3}{4}$; (2) the constant phase shift $-\pi/4$ in Eq. (28) and all subsequent formulas must be replaced by $-\pi/2$.

Using the explicit expressions for the main characteristics of unperturbed ray trajectories

$$E(I)=\left(\frac{I}{a}\right)^{2/3}, \quad a=\frac{2\sqrt{2}}{3\pi L},$$

$$S_0(z,I)=\pi[I-(I-a^{2/3}zL)^3],$$

$$D(I)=3\pi a^{2/3}I^{1/3}$$

we readily obtain the relations connecting (p,z) variables to action-angle variables (I,θ):

$$z=\frac{E(I)}{L}\left[1-\left(2n+1-\frac{\theta}{\pi}\right)^2\right], \quad \theta\in[2\pi n,2\pi(n+1)];$$

$$p=\sqrt{2E(I)}\left(2n+1-\frac{\theta}{\pi}\right), \quad \theta\in[2\pi n,2\pi(n+1)],$$

where $n=0,1,\ldots$.

The perturbation in terms of the action-angle variables for $\varepsilon_1=0$ takes the form

$$V=\frac{\varepsilon}{2i}\left(\frac{I}{a}\right)^{2/3}\sum_{q=-\infty}^{\infty} b_q(e^{i(q\theta+\Omega x)}-e^{i(q\theta-\Omega x)}), \qquad (64)$$

with

$$b_q=\begin{cases}2/3, & q=0\\ -2\pi^{-2}q^{-2}, & q\neq 0.\end{cases}$$

It is obvious that for a nonzero ε_1 we should add a similar sum with ε and Ω replaced by ε_1 and Ω_1, respectively.

Let us consider a mode-medium resonance with $l=1$. Its half-width in terms of the number of trapped modes, given in Eq. (61), for the present environmental model with $\varepsilon_1=0$ translates into

$$\Delta m=\sqrt{2\varepsilon}m. \qquad (65)$$

From the viewpoint of underwater acoustics Eqs. (62) and (63) present a strongly idealized model of an acoustic waveguide with a pressure release surface at $z=0$ and a range-dependent sound speed profile

$$C(x,z)=\frac{C_0}{\sqrt{1-2Lz(1+\varepsilon\sin\Omega x+\varepsilon_1 Lz\sin\Omega_1 x)}} \qquad (66)$$

in a half-space $z>0$. The selected values of the constants defining the unperturbed potential $U(z)$ are $C_0=1500$ m/s, $L=4.35\times10^{-5}$ 1/m. We also assume that the carrier frequency $f=250$ Hz.

As far as the perturbation is concerned, we assume the period of the first and second terms in the right-hand side of Eq. (63) to be equal to the cycle lengths of unperturbed rays corresponding to the 60th and 110th modes, respectively: $2\pi/\Omega=D(I_{60})=13.149$ km, and $2\pi/\Omega_1=D(I_{110})=16.310$ km. We consider three models of perturbation with parameters ε and ε_1.

	ε	ε_1
model 1	0.02	0
model 2	0.02	0.02
model 3	0.05	0.05.

In what follows it is assumed that only the 60th mode is excited at $x=0$, that is $u(0,z)=\varphi_{60}(z)$. The evolution of the wave field with range up to 1000 km has been numerically calculated using the UMPE code [27] for solving Eq. (3). To find mode amplitudes we have projected the solution onto eigenfunctions of the unperturbed boundary value problem (18). For this particular waveguide the eigenfunctions are expressed through the Airy function Ai(z) [28]:

$$\varphi_m(z)=g_m\mathrm{Ai}\left(-\frac{(2k^2)^{1/3}}{L^{2/3}}(E_m-Lz)\right),$$

where g_m is the normalization constant.

In model 1 we have an isolated resonance with $l=1$ centered at the 60th mode. The numerical calculation demonstrates that beginning from ranges of 30–50 km this single mode is split into a group of modes and at longer ranges the width of the group remains the same. Equation (65) gives an estimation of a half-width of this group $\Delta m=12$. Figure 1(a) shows mode amplitudes at a range of 400 km. The results agrees with the above estimation. A similar calculation for model 2 has given a mode amplitude distribution presented in Fig. 1(b). For the resonance centered at the 110th mode with $\varepsilon_1=0.02$, Eq. (65) gives the estimation $\Delta m_1=22$. So, we see that the resonances at the 60th and 110th modes do not overlap, and so, in model 2 the resonance at the 60th mode remains isolated. Figure 1(b) confirms this prediction: the presence of the second term in the perturbation causes only a small increase in the width of the excited mode group.

The situation changes drastically for model 3. This time Eq. (65) gives $\Delta m=19$ and $\Delta m_1=35$, which means that the resonances slightly overlap. Figure 1(c) shows that this overlapping leads to a great widening of the excited mode group. A similar effect was demonstrated in Ref. [23,29], where a different approach for interpretation of ray-medium resonance in terms of normal mode was considered.

The overlapping of resonances causes a strong mode coupling, which yields a rather complicated mode amplitude range dependence. This, in turn, leads to a complicated wave picture called wave chaos.

Our formalism provides a convenient tool for studying this phenomenon on the basis of results obtained for rays. The discussion of this topic is a subject of the next subsection.

C. From ray chaos to wave chaos

It is well known that nonlinear ray medium resonance plays an important role in the emergence of ray chaos. If there are at least two nonlinear resonances centered at spatial frequencies ω and $\omega+\delta\omega$, a chaotic motion of ray trajectories is possible. It takes place when the Chirikov's criterion [19–21],

$$\frac{\Delta\omega}{\delta\omega}>1, \qquad (67)$$

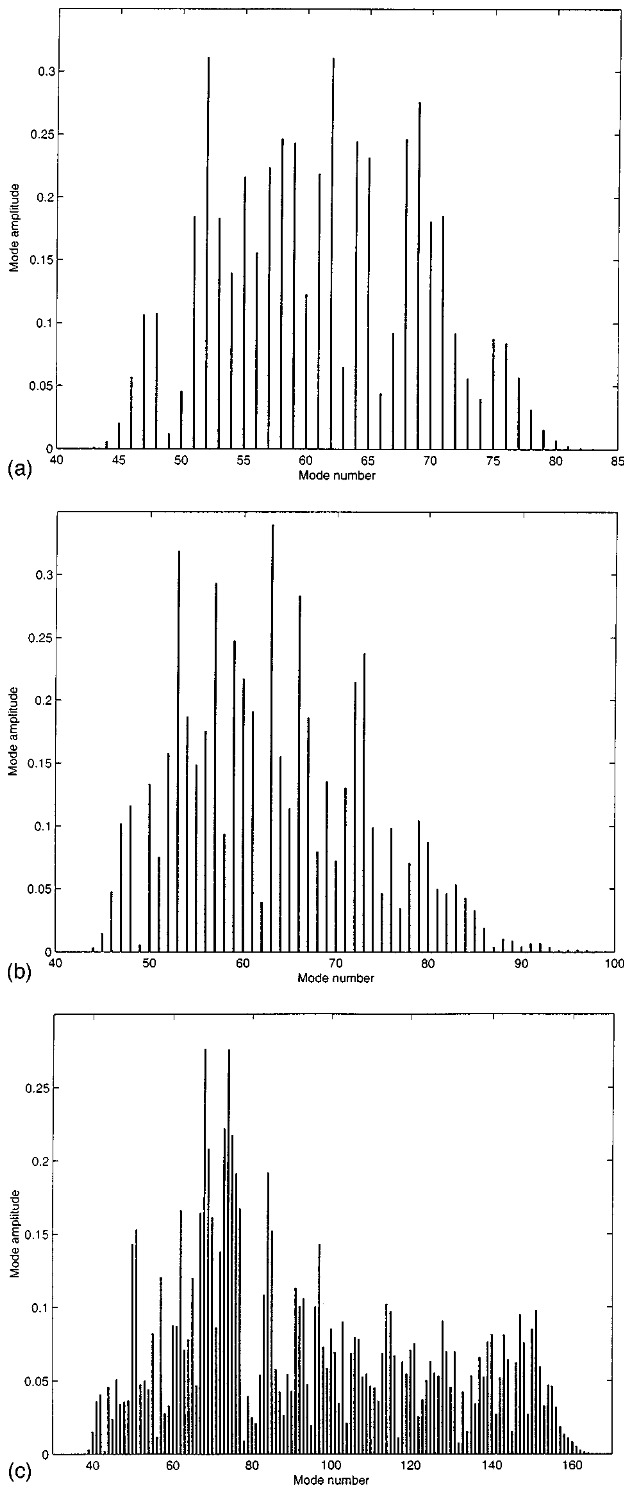

FIG. 1. Mode amplitudes in the 400-km range in model 1 (a), model 2 (b), and model 3 (c).

is met, i.e., when the resonances overlap leading to the stochastic instability of the system. If the overlapping is weak, only the rays with action variables close to that of the separatrix exhibit stochastic behavior and they form the so-called stochastic layer in the neighborhood of the separatrix. When the overlapping is stronger (for example, if we consider a greater value of ε) the width of the stochastic layer grows, different stochastic layers begin to overlap and more and more rays become chaotic.

The trajectories of two stochastic rays with close initial conditions diverge exponentially [7] and the number of eigenrays contributing at a given field point grows exponentially with range [8].

From the viewpoint of modes the latter is especially important. It is almost obvious that under chaotic conditions the number of eigenrays contributing to the given mode also grows exponentially with range, giving rise to a very complicated range dependence of mode amplitudes. This statement is qualitatively illustrated in Figs. 2–4. Figure 2 demonstrates how the behavior of the amplitude of the 61st mode, $|c_{61}(x)|$, changes from almost periodical in model 1 (it oscillates with the period of inhomogeneity) to quite irregular in model 3. The same is clearly seen in Fig. 3, where Fourier spectra of the 61st mode complex amplitude are presented. Figure 4 shows autocorrelation functions of three different modes. The absolute value of the autocorrelation function, defined as

$$Q(r)=\left|\frac{\int_0^{x_{\max}} c_m(x+r)c_m(x)^* dx}{\int_0^{x_{\max}} |c_m(x)|^2 dx}\right|,$$

has been calculated using the 1000-km realizations of mode complex amplitudes. In these figures the almost periodical range dependence in model 1 reveals itself in long lasting correlations of mode amplitudes. In model 2 and especially in model 3 the correlation becomes much weaker. Note different scales along the vertical axis in Figs. 4(a), 4(b), and 4(c).

It may seem that exponential proliferation of eigenrays contributing to the given mode leads to statistical independence of mode amplitude fluctuations under conditions of ray chaos. But we suppose that the problem of mode amplitude description is considerably more rich and complicated.

First of all, it should be pointed out that in the phase space of a chaotic Hamiltonian system there always exist so-called ‘‘stable islands’’ formed by regular periodic trajectories. Some of such regular rays will, generally, be eigenrays for some modes. Their contributions to modes cannot be considered as stochastic. So, we presume that under conditions of ray chaos there may be modes with amplitudes composed of two constituents: a chaotic one and a regular one.

There is another important phenomenon typical of chaotic dynamics, which may affect modal structure variations: so-called stickiness, i.e., the presence of such parts in a chaotic trajectory where the latter exhibits an almost regular behavior. Such a part corresponds to the situation when after wandering in the phase space the trajectory approaches a stable island and ‘‘sticks’’ to its border for some time. It should be pointed out that the latter can be fairly long [30]. In principle, one can presume that the stickiness may cause some long-lasting correlations of mode amplitudes.

However, the issues of mode amplitude correlations are beyond the scope of the present study. We plan to consider them in more detail elsewhere.

III. CONCLUSION

In this paper we have considered an approach permitting us to apply directly the results obtained when studying chaotic ray behavior in a range-dependent waveguide to the in-

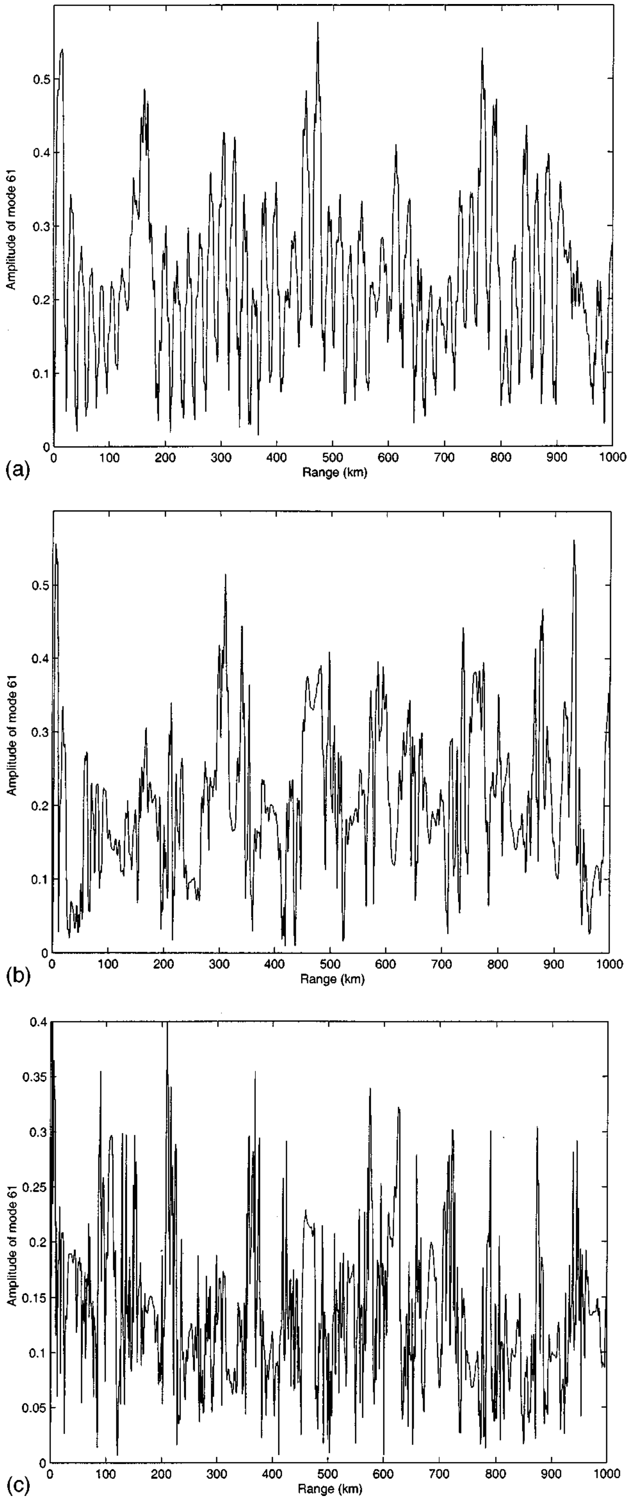

FIG. 2. The amplitude of the 61st mode vs range in model 1 (a), model 2 (b), and model 3 (c).

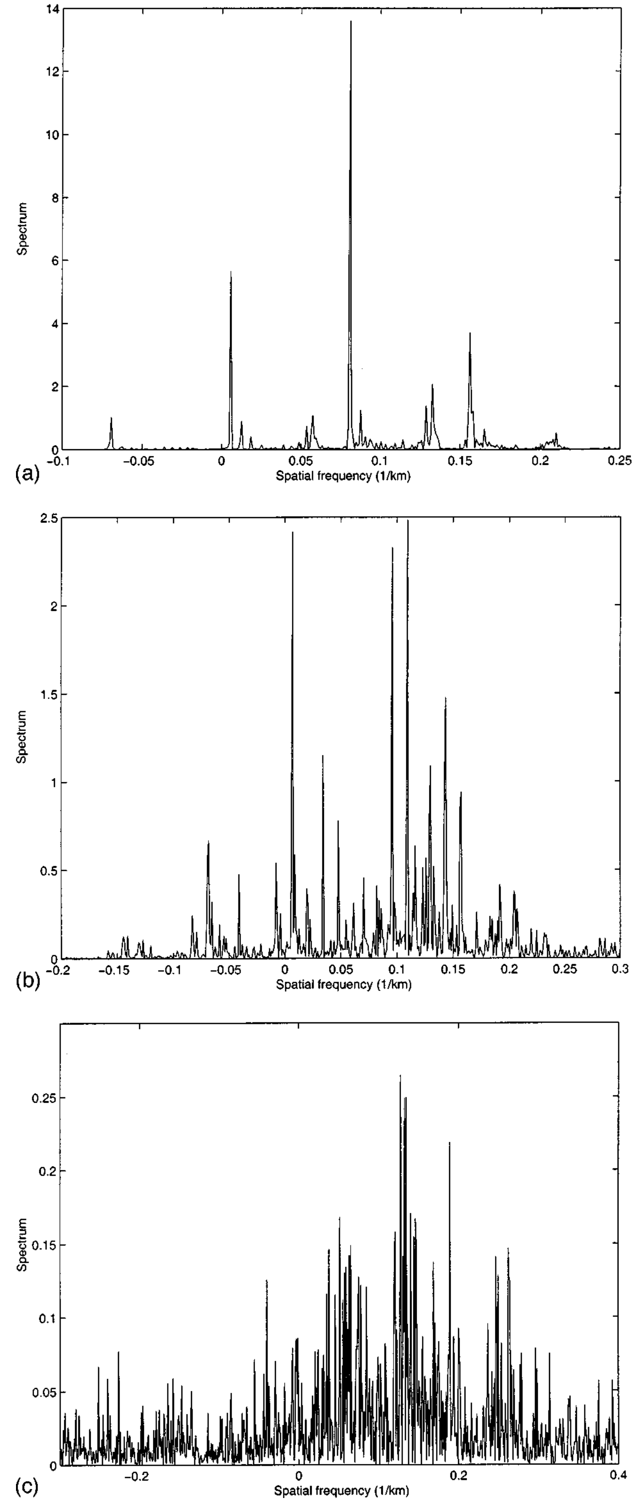

FIG. 3. The Fourier-spectrum of the 61st mode complex amplitude in model 1 (a), model 2 (b), and model 3 (c).

vestigation of an irregular modal structure. All our results have been obtained in the framework of the quasiclassical approximation. In the context of this approach the mode amplitude at the given range is presented as a sum of contributions from several rays, i.e., the mode amplitude is expressed through solutions of ray equations. Under condition of ray chaos the number of rays contributing to a particular mode should grow exponentially with range leading to complicated variations of modal structure with range.

So, this approach not only clarifies ray-mode relations in a range-dependent environment but also seems to be a convenient tool for establishing a bridge between ray chaos and wave chaos.

Considerable attention has been given to discussing ray-medium nonlinear resonance, which is a very important factor in the mechanism of the emergence of ray chaos. An analog to this phenomenon for modes, which we call the mode-medium resonance, has been formulated. It has turned

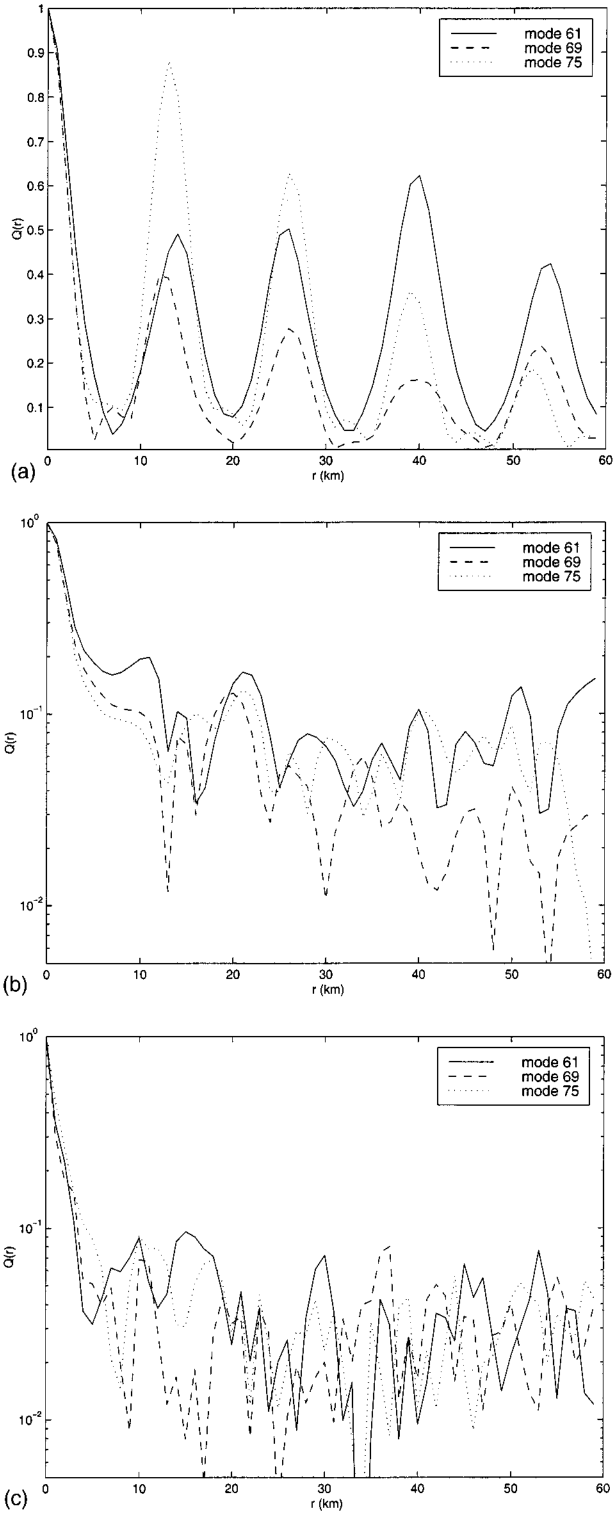

FIG. 4. Absolute value of the autocorrelation functions of three modes in model 1 (a), model 2 (b), and model 3 (c).

out that the analytical description of the mode-medium resonance is as simple as that of its ray prototype.

We argue that overlapping of different mode-medium resonances leads to an irregular behavior of mode amplitudes in analogy to the well-known fact that the overlapping of ray-medium resonances causes ray chaos.

We have presented the results of numerical calculations that illustrate the mode-medium resonance and confirm qualitatively that, under the condition of overlapping of resonances, mode amplitude range dependence becomes irregular.

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APPENDIX

In this appendix we give a brief derivation of Eq. (5) for the reader's convenience. (For more details, see [2,7,31]). To make the applicability conditions of these equations clearer we start here not from 2D parabolic equation (3) presenting an approximation to the initial wave equation, but from the 3D wave equation itself taken in the form

$$\Delta u - \frac{1}{C^2(\mathbf{r})}\frac{\partial^2 u}{\partial t^2} = 0 \tag{A1}$$

with field amplitude $u = u(\mathbf{r}) = u(x,y,z)$ and local wave velocity $C(\mathbf{r}) = C(x,y,z)$. In Eq. (A1) we made the first assumption that C does not depend on time. For the underwater acoustics, that means fairly slow changes in time of the ocean parameters on the wave packet characteristic length, which is typically the case (see the detailed estimation in [22]).

In the short wave approximation, when the wavelength λ is much smaller than the nonuniformity length $l(\lambda \ll l)$, the field u can be found in the form

$$u(\mathbf{r},t) = A(\mathbf{r})\exp[ikS(\mathbf{r}) - i\omega t], \tag{A2}$$

where k and ω are some dimensional constants and $S(\mathbf{r})$ is eikonal. In the first approximation, A and S satisfy the equations

$$(\nabla S)^2 = n^2(\mathbf{r}),$$

$$\nabla(A^2 \nabla S) = 0,$$

$$n(\mathbf{r}) = C_0/C(\mathbf{r}), \quad C_0 = \omega/k, \tag{A3}$$

where $n(\mathbf{r})$ is the refraction index. Let us introduce a generalized momentum

$$\mathbf{p} = \nabla S \tag{A4}$$

and rewrite the first equation in Eq. (A3) (the so-called eikonal equation) in the form

$$\mathcal{H} = \mathcal{H}(\mathbf{r},\mathbf{p}) = \tfrac{1}{2}[\mathbf{p}^2 - n^2(\mathbf{r})] = 0. \tag{A5}$$

Then it follows from Eqs. (A4), (A5) [31,32] that

$$\frac{d\mathbf{r}}{d\tau} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \quad \frac{\partial \mathbf{p}}{\partial \tau} = -\frac{\partial \mathcal{H}}{\partial \mathbf{r}} \tag{A6}$$

with element of a trajectory length

$$dl \equiv (d\mathbf{r}^2)^{1/2} = n(\mathbf{r})d\tau. \tag{A7}$$

We have Hamiltonian equations (A6), which describe ray dynamics for arbitrary dependence $n(\mathbf{r})$ and with effective ''time'' defined by

$$d\tau = dl/n(\mathbf{r}) = |d\mathbf{r}|/n(\mathbf{r}). \tag{A8}$$

The so-called parabolic approximation is applied for waves that propagate at small angles relative to some direction, say x, i.e., ray deviation from x is assumed to be small,

$$p_\perp \ll p_\parallel. \tag{A9}$$

In addition to Eq. (A9), deviations of n from unity should be small

$$n^2 = 1 + \delta n^2(x,y,z),$$

$$\delta n^2(x,y,z) = C_0^2/C^2(x,y,z) - 1 \ll 1. \tag{A10}$$

It follows from Eq. (A5) under conditions (A9), (A10) that

$$H \equiv -p_\parallel = -1 + \tfrac{1}{2}(p_\perp^2 - \delta n^2), \tag{A11}$$

where we neglect higher-order terms. In the same approximation we can consider

$$d\tau = dx. \tag{A12}$$

Then, using Eqs. (A11) and (A12) we can rewrite Eqs. (A6) in the form

$$\frac{d\mathbf{r}_\perp}{dx} = \mathbf{p}_\perp = -\frac{\partial H}{\partial \mathbf{p}_\perp},$$

$$\frac{d\mathbf{p}_\perp}{dx} = \frac{1}{2}\frac{\partial n^2}{\partial \mathbf{r}_\perp} = -\frac{\partial H}{\partial \mathbf{r}_\perp}. \tag{A13}$$

We arrived at the Hamiltonian form of equations with $\mathbf{r}_\perp = (y,z)$, $\mathbf{p}_\perp = (p_y, p_z)$ and x as a ''time.''

Neglecting the constant in Eq. (A11) and using notations

$$-\delta n^2 = U(y,z) + \varepsilon V(x,y,z) \tag{A14}$$

we obtain Eqs. (4) and (5) when there is no dependence on y. The range dependence of the refraction index is reflected in the second term of Eq. (A14). Both terms U and εV can be of the same order but they should satisfy the conditions (A10).

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